The emittance characterizes the whole beam quality, assuming linear behavior as described by second order differential equation.

It is defined within the phase space as: $\varepsilon_x = \frac{1}{\pi}\int_A dxdx'$

The measurement is based on determination of:

either profile width $\boldsymbol{\sigma_x}$ and angular width $\boldsymbol{\sigma_x'}$ at one location
or $\boldsymbol{\sigma_x}$ at different locations and linear transformations.

Different devices are used at transfer lines:

- Lower energies $\boldsymbol{E_{kin}}$ < 100 MeV/u: slit-grid device, pepper-pot
  (suited in case of non-linear forces).
- All beams: Quadrupole variation, 'three grid' method using linear transformations
  (**not** well suited in the presence of non-linear forces)

Synchrotron: lattice functions results in stability criterion

$\Rightarrow$ beam width delivers emittance: $\varepsilon_x = \frac{1}{\beta_x(s)}\left[\sigma_x^2 - \left(D(s)\frac{\Delta p}{p}\right)\right]$ and $\varepsilon_y = \frac{\sigma_y^2}{\beta_y(s)}$

Definition of transverse Emittance

The emittance characterizes the whole beam quality: $\varepsilon_x = \frac{1}{\pi}\int_A dx dx'$

Ansatz:

Beam matrix at one location: $\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} = \varepsilon \cdot \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$ with $\vec{x} = \begin{pmatrix} x \\ x' \end{pmatrix}$

It describes a 2-dim probability distr.

The value of emittance is:

$$\varepsilon_x = \sqrt{\det \boldsymbol{\sigma}} = \sqrt{\sigma_{11}\sigma_{22} - \sigma_{12}^2}$$

For the profile and angular measurement:

$$x_\sigma = \sqrt{\sigma_{11}} = \sqrt{\varepsilon\beta} \quad \text{and}$$

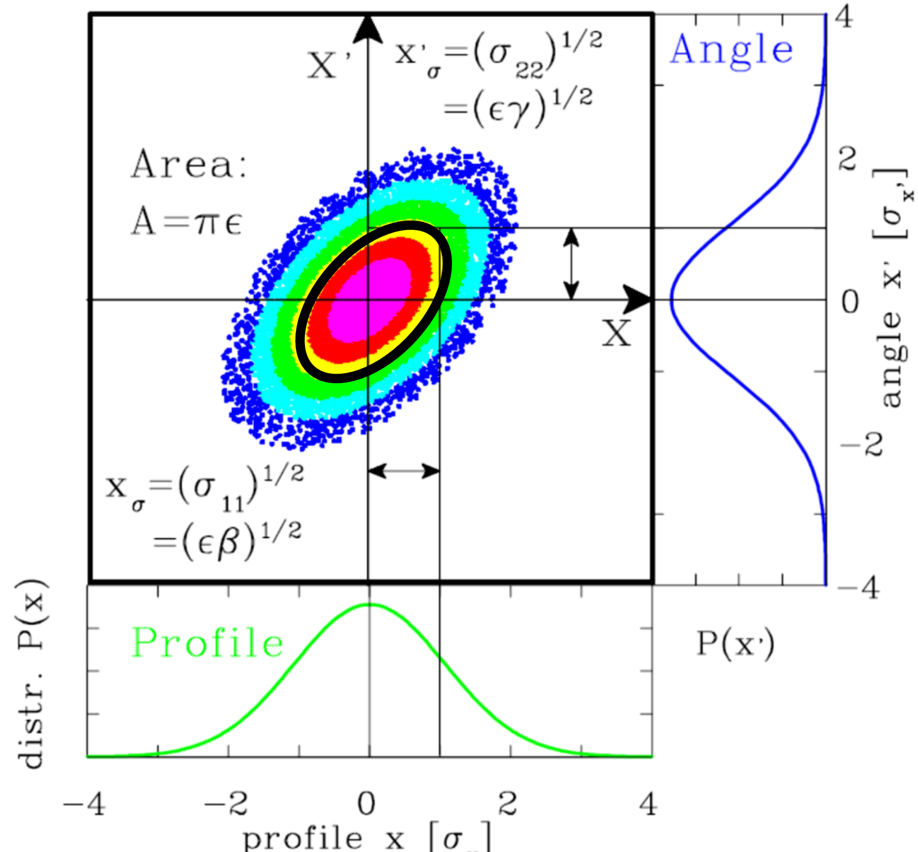
$$x'_\sigma = \sqrt{\sigma_{22}} = \sqrt{\varepsilon\gamma}$$

Geometrical interpretation:

All points $\boldsymbol{x}$ fulfilling $\boldsymbol{x}^{\mathbf{t}} \cdot \boldsymbol{\sigma}^{\mathbf{-1}} \cdot \boldsymbol{x} = \mathbf{1}$

are located on a **ellipse**

$$\boldsymbol{\sigma_{22}x^2 - 2\sigma_{12}xx' + \sigma_{11}x'^2} = \det \boldsymbol{\sigma} = \boldsymbol{\varepsilon_x^2}$$



The Emittance for Gaussian Beams

The density function for a 2-dim Gaussian distribution is:

$$\rho(x,x') = \frac{1}{2\pi\epsilon} \exp\left[-\frac{1}{2}\,\vec{x}^T\,\sigma^{-1}\,\vec{x}\,\right]$$

$$= \frac{1}{2\pi\epsilon} \exp\left[\frac{-1}{2\det\sigma}\left(\sigma_{22}x^2 - 2\sigma_{12}xx' + \sigma_{11}x'^2\right)\right]$$

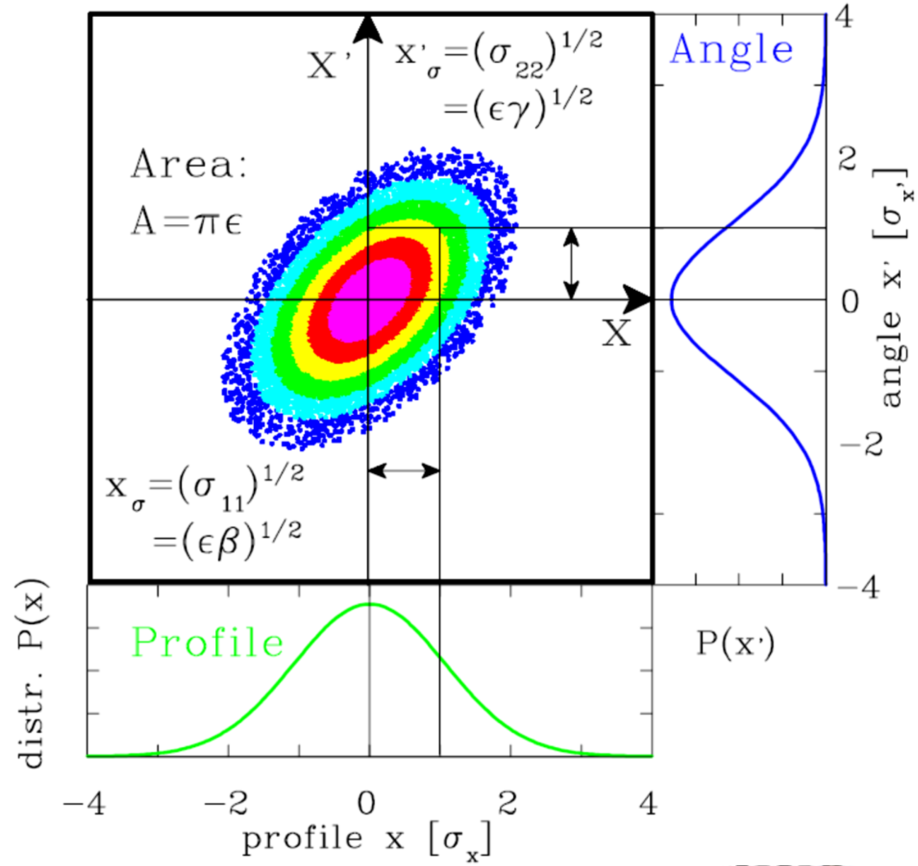
It describes an ellipse with the characteristics profile and angle Gaussian distribution of width

$$x_\sigma \equiv \sqrt{\langle x^2 \rangle} = \sqrt{\sigma_{11}} \quad \text{and}$$

$$x'_\sigma \equiv \sqrt{\langle x'^2 \rangle} = \sqrt{\sigma_{22}}$$

and the correlation or covariance

$$\mathrm{cov} \equiv \sqrt{\langle xx' \rangle} = \sqrt{\sigma_{12}}$$



For $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ it is $\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

assuming $\det(\mathbf{A}) = ad\text{-}bc \neq 0 \Leftrightarrow$ matrix invertible

The Emittance for Gaussian and non-Gaussian Beams

The beam distribution can be non-Gaussian, e.g. at:

- beams behind ion source
- space charged dominated beams at LINAC & synchrotron
- cooled beams in storage rings

Variances

Covariance i.e. correlation

General description of emittance

using terms of 2-dim distribution:

$$\varepsilon_{rms} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$$

It describes the value for 1 stand. derivation

$$\langle x \rangle \equiv \mu = \frac{\iint x \cdot \rho(x,x')\,dxdx'}{\iint \rho(x,x')\,dxdx'} \qquad \langle x' \rangle \equiv \mu' = \frac{\iint x' \cdot \rho(x,x')\,dxdx'}{\iint \rho(x,x')\,dxdx'}$$

$$\langle x^n \rangle = \frac{\iint (x-\mu)^n \cdot \rho(x,x')\,dxdx'}{\iint \rho(x,x')\,dxdx'} \qquad \langle x'^n \rangle = \frac{\iint (x'-\mu')^n \cdot \rho(x,x')\,dxdx'}{\iint \rho(x,x')\,dxdx'}$$

$$\text{covariance}: \ \langle xx' \rangle = \frac{\iint (x-\mu)(x'-\mu') \cdot \rho(x,x')\,dxdx'}{\iint \rho(x,x')\,dxdx'}$$

For discrete distribution:

$$\langle x \rangle = \frac{\sum_{i,j} \rho(i,j) \cdot x_i x'_j}{\sum_{i,j} \rho(i,j)}$$

and correspondingly for all other moments

The beam distribution can be non-Gaussian, e.g. at:

- beams behind ion source
- space charged dominated beams at LINAC & synchrotron
- cooled beams in storage rings

General description of emittance

using terms of 2-dim distribution:

It describes the value for 1 stand. derivation

Variances **Covariance i.e. correlation**

$$\varepsilon_{rms} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$$

For Gaussian beams only:

$\boldsymbol{\varepsilon_{rms}}$ $\leftrightarrow$ interpreted as area containing a fraction ***f*** of ions: $\varepsilon(f) = -2\pi\varepsilon_{rms} \cdot \ln(1-f)$

factor to ϵ_{rms}	$1 \cdot \epsilon_{rms}$	$\pi \cdot \epsilon_{rms}$	$2\pi \cdot \epsilon_{rms}$	$4\pi \cdot \epsilon_{rms}$	$6\pi \cdot \epsilon_{rms}$	$8\pi \cdot \epsilon_{rms}$
faction of beam f [%]	15	39	63	86	95	98

Care: no common definition of emittance concerning the fraction ***f***

Outline:

- Definition and some properties of transverse **emittance**
- Slit-Grid device: scanning method

  scanning slit → beam position & grid → angular distribution
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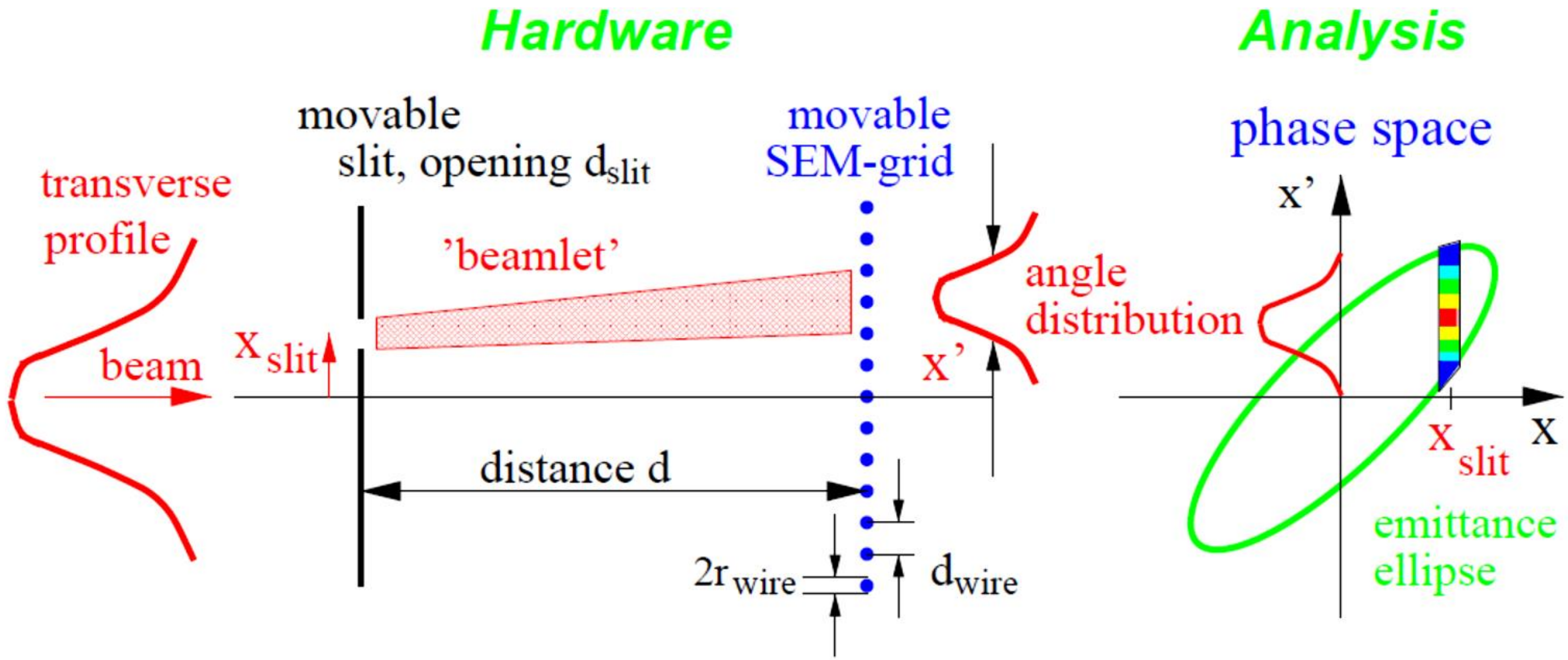
The Slit-Grid Measurement Device

Slit-Grid: Direct determination of position and angle distribution.

Used for protons/heavy ions with E_{kin} < 100 MeV/u ⇒ range R < 1 cm.



Slit: position ***P(x)*** with typical width: 0.1 to 0.5 mm

Distance: 10 cm to 1 m (depending on beam velocity)

SEM-Grid: angle distribution ***P(x')***

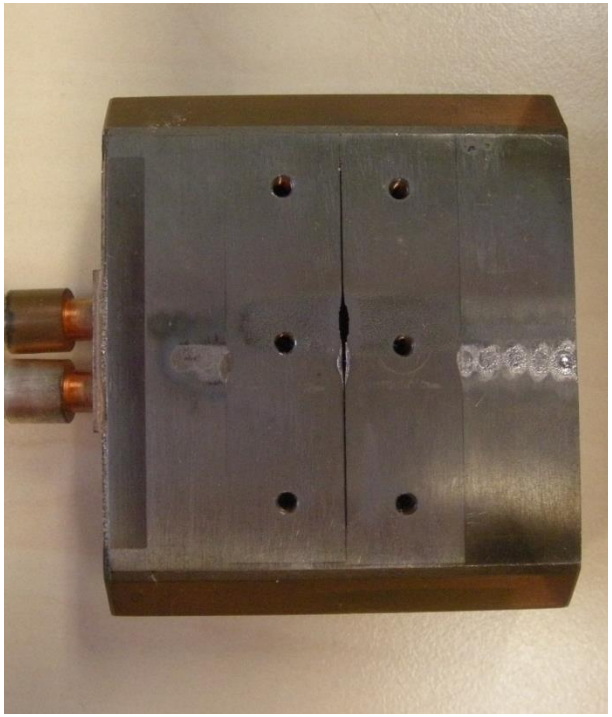
Slit & SEM-Grid

Slit with e.g. 0.1 mm thickness

→ Transmission only from ***Δx***.

Example: Slit of width 0.1 mm (defect)

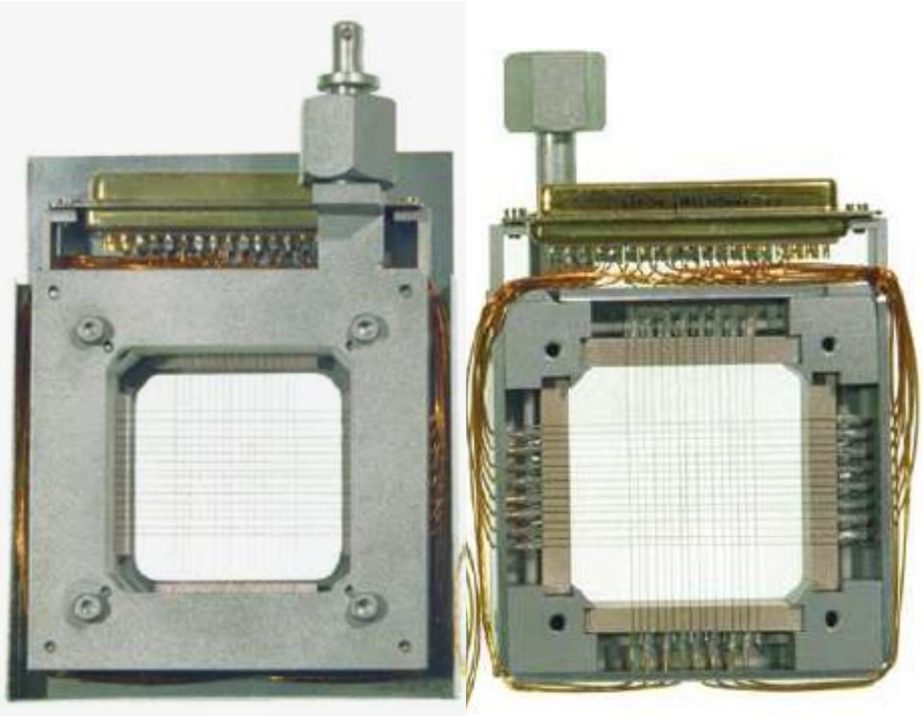
Moved by stepping motor, 0.1 mm resolution



Beam surface interaction: e^- emission

→ measurement of current.

Example: 15 wire spaced by 1.5 mm:



Each wire is equipped with one I/U converter

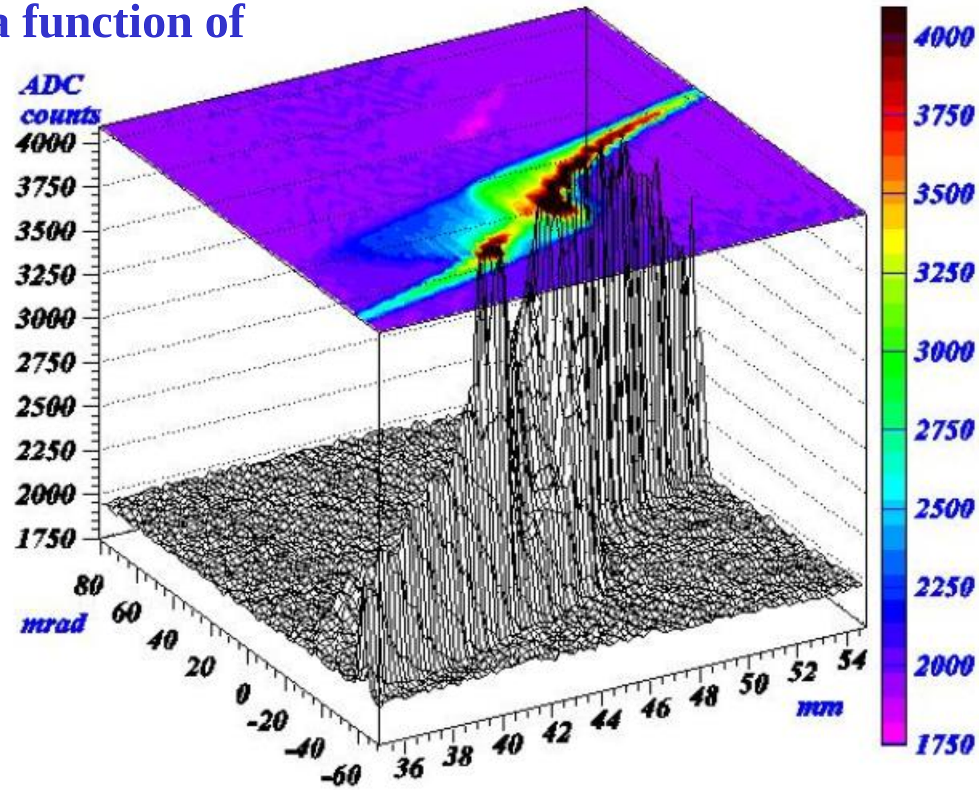
different ranges settings by $\boldsymbol{R_i}$

→ very large dynamic range up to 10^6.

Display of Measurement Results

The distribution of the ions is depicted as a function of
- Position [mm]
- Angle [mrad]

The distribution can be visualized by
- Mountain plot
- Contour plot



Calc. of 2nd moments $\langle x^2 \rangle$, $\langle x'^2 \rangle$ & $\langle xx' \rangle$

Emittance value ε_{rms} from

$$\varepsilon_{rms} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$$

$\Rightarrow$ **Problems:**

- Finite **binning** results in limited resolution
- **Background** $\rightarrow$ large influence on $\langle x^2 \rangle$, $\langle x'^2 \rangle$ and $\langle xx' \rangle$

Or fit of distribution i.e. ellipse to data

$\Rightarrow$ **Effective emittance only**

Beam: Ar^{4+}, 60 KeV, 15 µA

at Spiral2 Phoenix ECR source.

Plot from P. Ausset, DIPAC 2009

The Resolution of a Slit-Grid Device

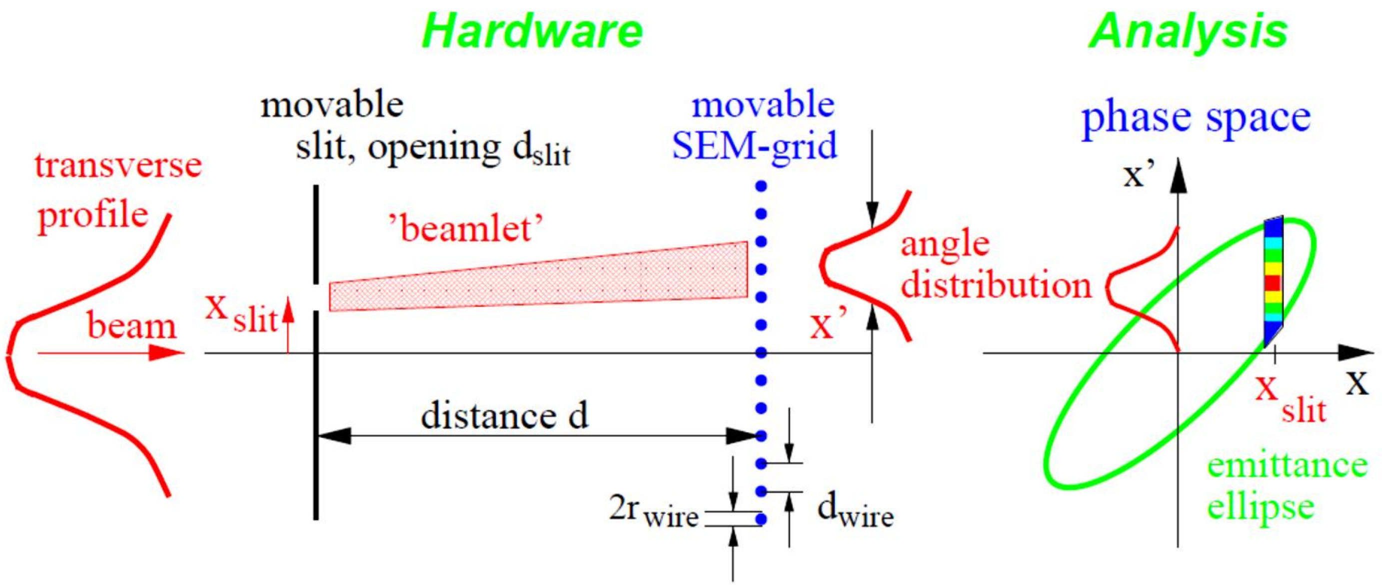
The width of the slit d_{slit} gives the resolution in space $\Delta x = d_{slit}$.

The angle resolution is $\Delta x' = (d_{slit} + 2r_{wire})/d$

$\Rightarrow$ discretization element $\Delta x \cdot \Delta x'$.

By scanning the SEM-grid the angle resolution can be improved.

Problems for small beam sizes or parallel beams.

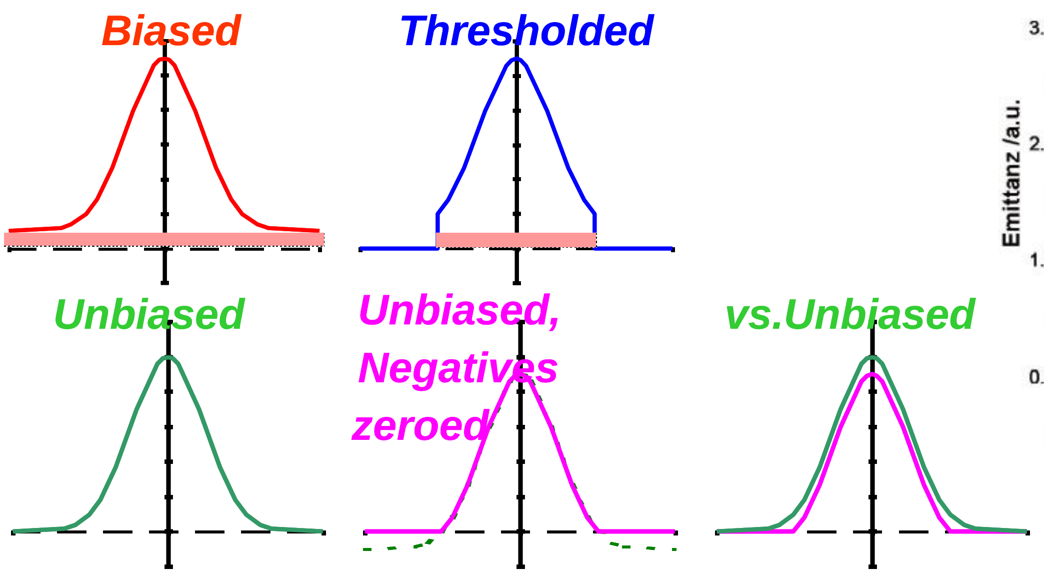


For pulsed LINACs: Only one measurement each pulse → long measuring time required.

The Noise Influence for Emittance Determination

A real measurement of beamlets contains:

- **Noise i.e. fluctuation of the output**
- **Bias i.e. electrical offset from amplifier**

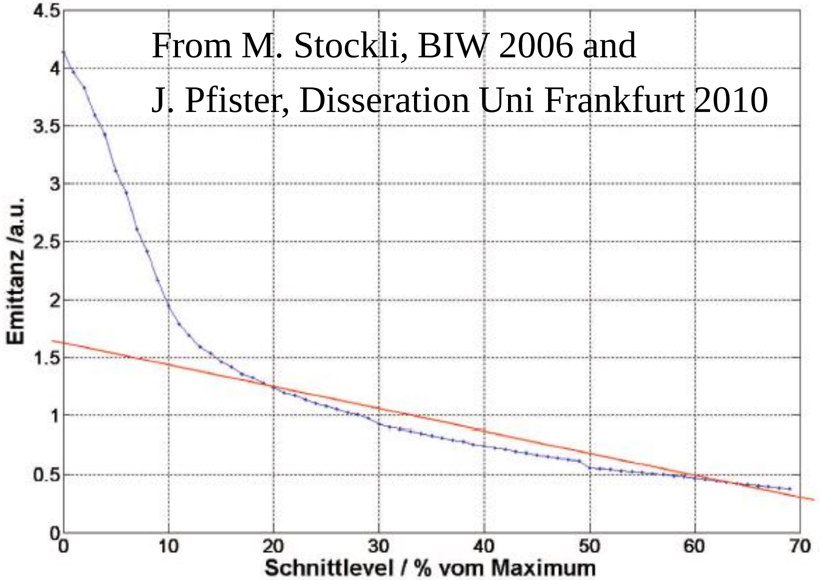


→ Strong influence of noise reduction to numerical values of **<*x*>, <*x'*^{2}>** and **<*xx'*>** and on ε_{rms}

⇒Algorithm & cut-level must be given for evaluation

General: Typical error $\Delta\varepsilon/\varepsilon > 10\ \%$

Example: Dependence of ε_{rms} on threshold value



$$\langle x'^2 \rangle = \frac{\int x'^2 \cdot \rho(x,x')\,dx\,dx'}{\int \rho(x,x')\,dx\,dx'} \quad \text{for continous values}$$

$$= \frac{\sum_{i,j} x'^2_{ij} \cdot P(x_{ij}, x'_{ij})}{\sum_{i,j} P(x_{ij}, x'_{ij})} \quad \text{for discrete values}$$

$$\varepsilon_{rms} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$$

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  scanning slit → beam position & grid → angular distribution
- **Pepper-pot device: single shot device**

  hole-plate → beam position & screen → angular distribution
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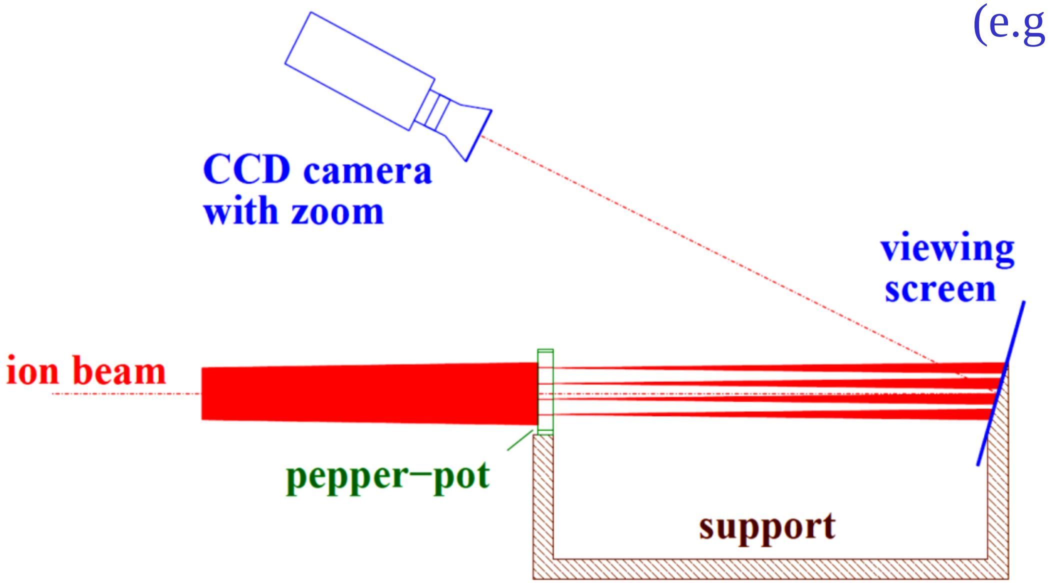
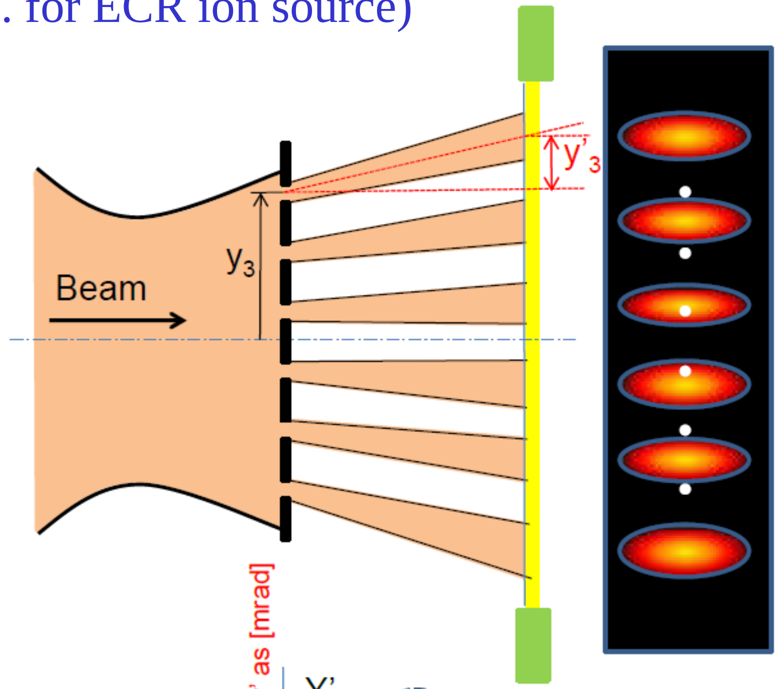
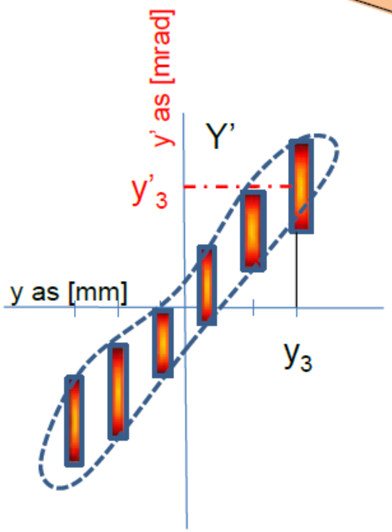
- For pulsed LINAC: Measurement within one pulse is an advantage
- If horizontal and vertical direction coupled → 2-dim evaluation **required** (e.g. for ECR ion source)







Good **spatial** resolution if many holes are illuminated.

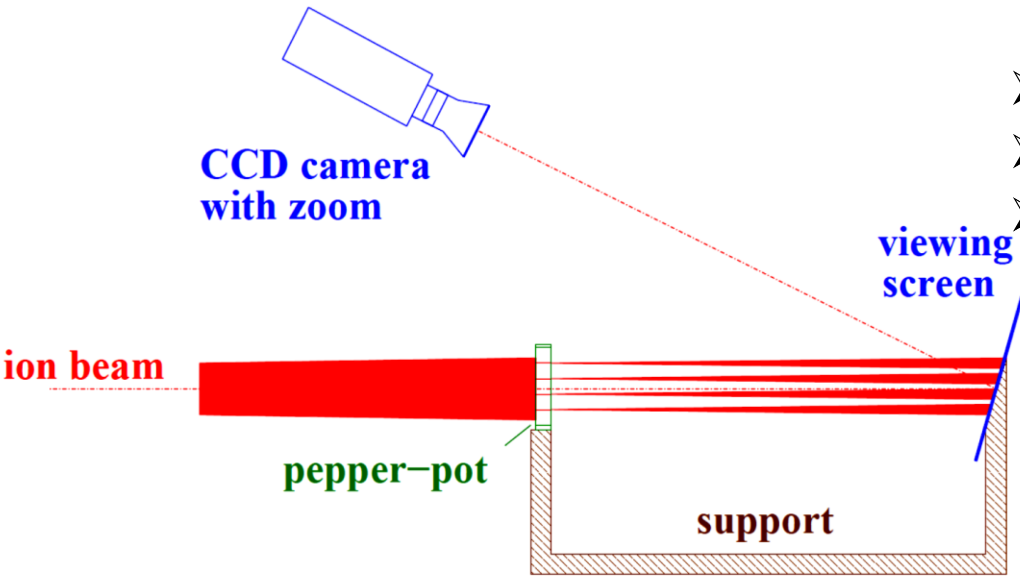
Good **angle** resolution ***only*** if spots do not overlap.

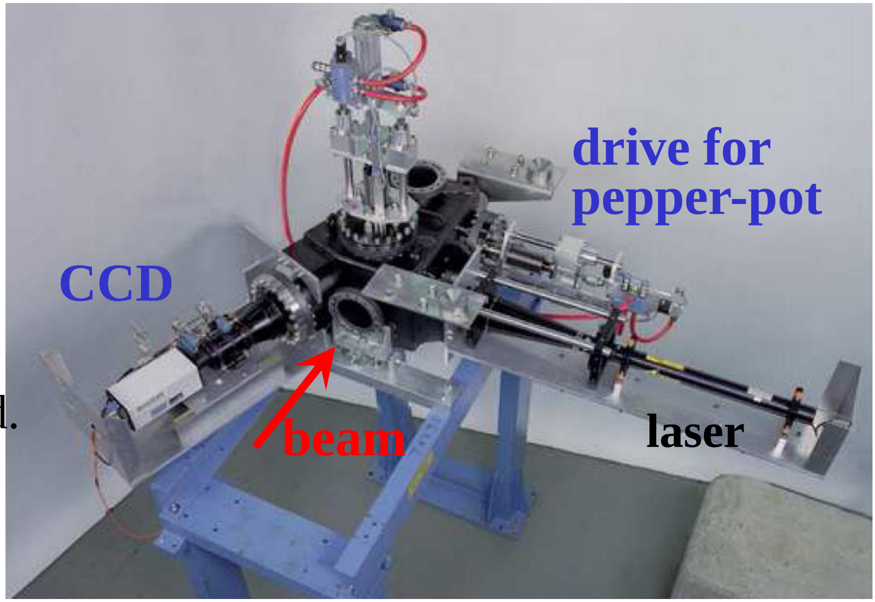
Partly from H.R. Kremers et al., ECRIS 2010

The Pepperpot Emittance Device at GSI UNILAC

Example GSI-LINAC 0.12 to 11 MeV/u:

- ***Pepper-pot***: 15 × 15 holes with Ø 0.1mm on a 50 × 50 mm^2 copper plate
- ***Distance***: pepper-pot-screen: 25 cm
- ***Screen:*** Al_2O_3, Ø 50 mm
- ***Data acquisition***: high resolution CCD



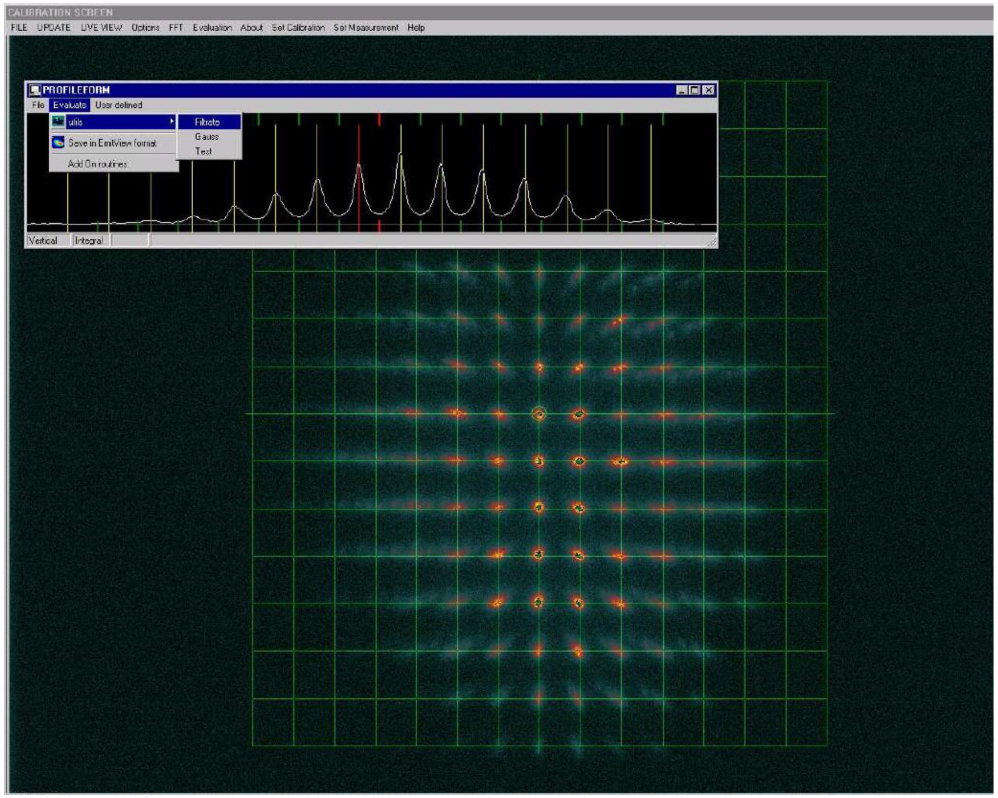


Good **spatial** resolution if many holes are illuminated.

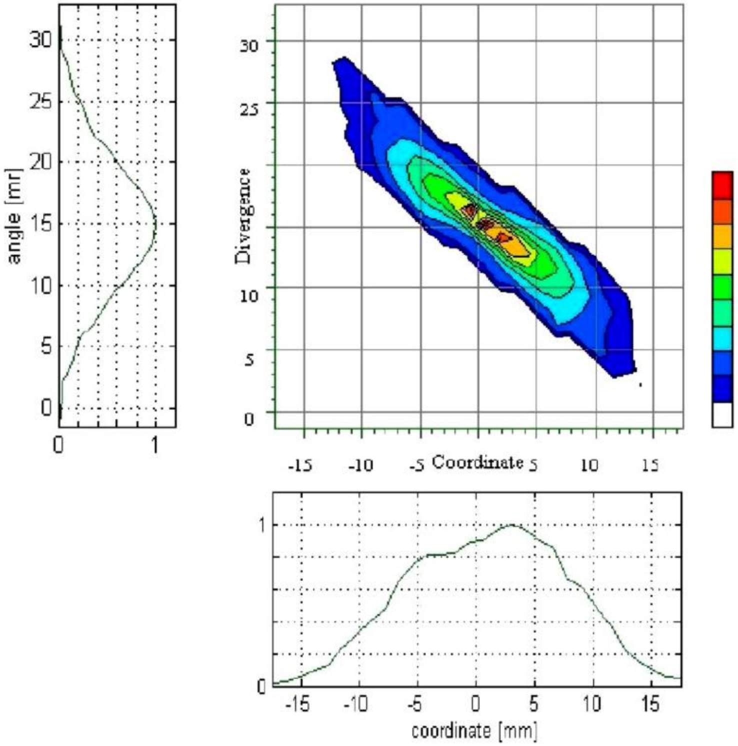
Good **angle** resolution ***only*** if spots do not overlap.

Result of a Pepperpot Emittance Measurement

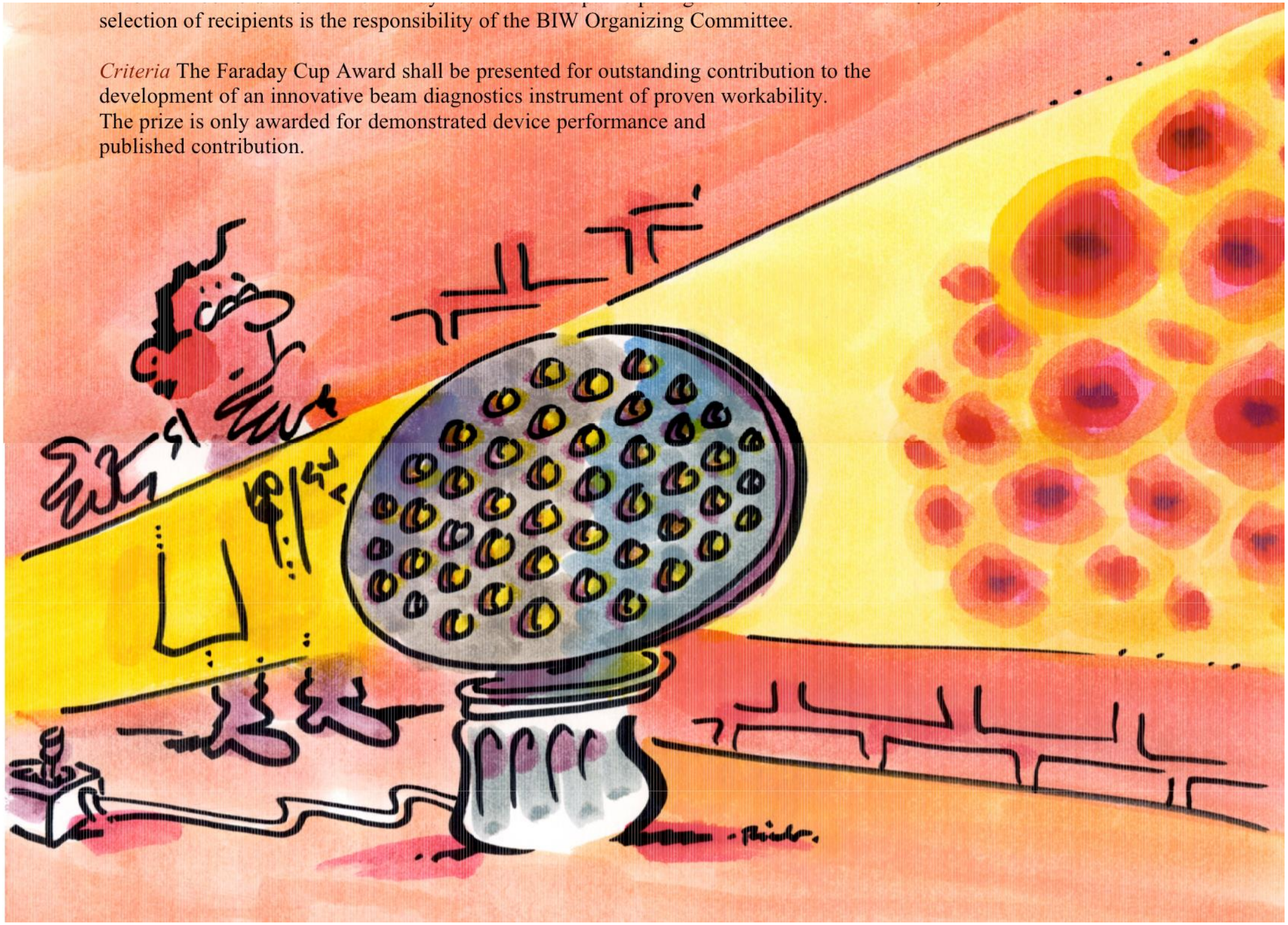
***Example*:** Ar $^{1+}$ ion beam at 1.4 MeV/u,
screen image from single shot at GSI:



Data analysis:
Projection on
horizontal and vertical plane
→ analog to slit-grid.



The Artist View of a Pepperpot Emittance Device



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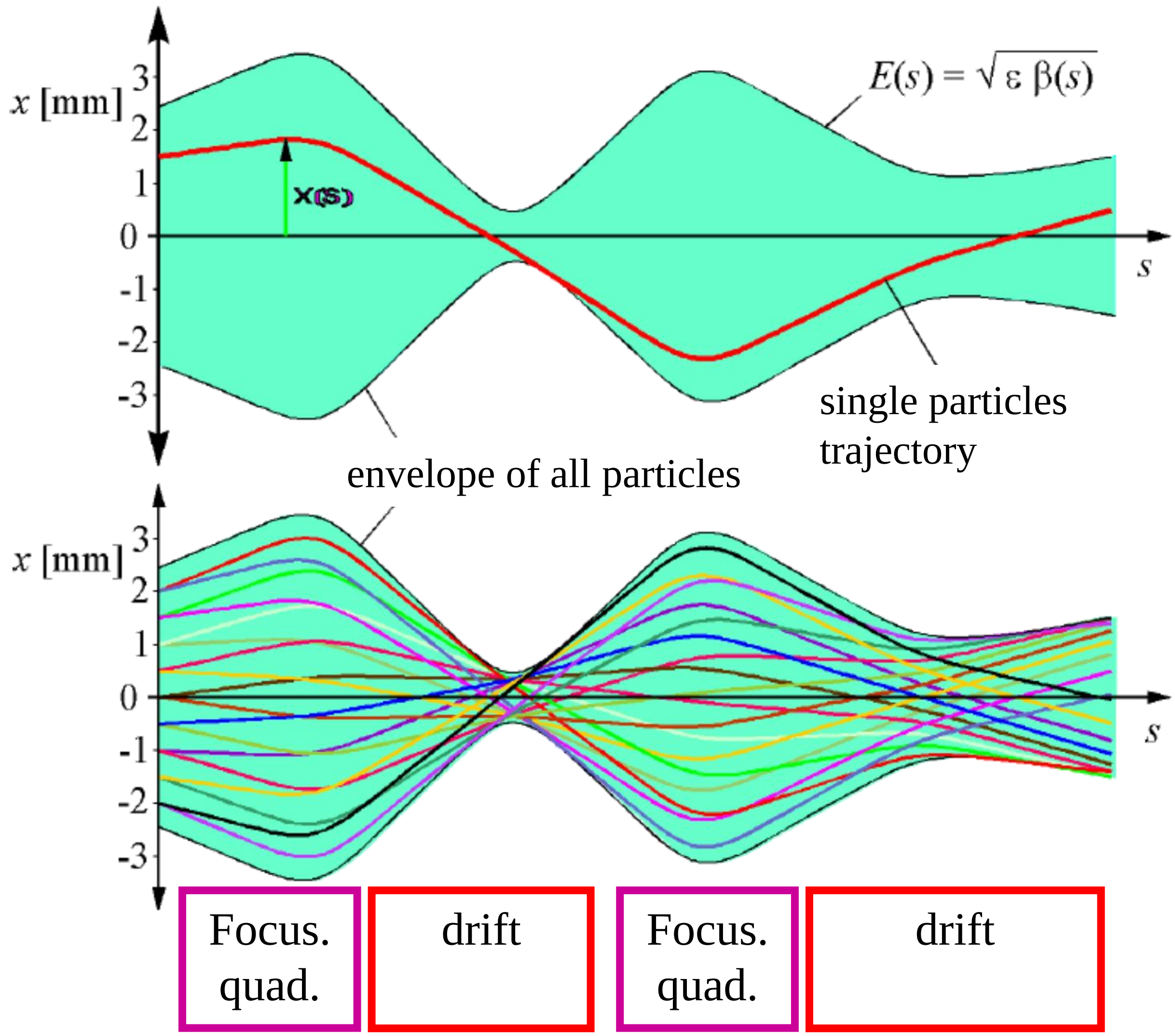
  hole-plate → beam position & screen → angular distribution
- Quadrupole strength variation and position measurement

  emittance from several profile measurement and beam optical calculation
- Summary

Particle Trajectory and Characterization of many Particles



- Single particle trajectories are forming a beam
- They have a distribution of start positions and angles

$\Rightarrow$ Characteristic quantity is the **beam envelope**

- **Goal:**

Transformation of envelope

Behavior of whole ensemble

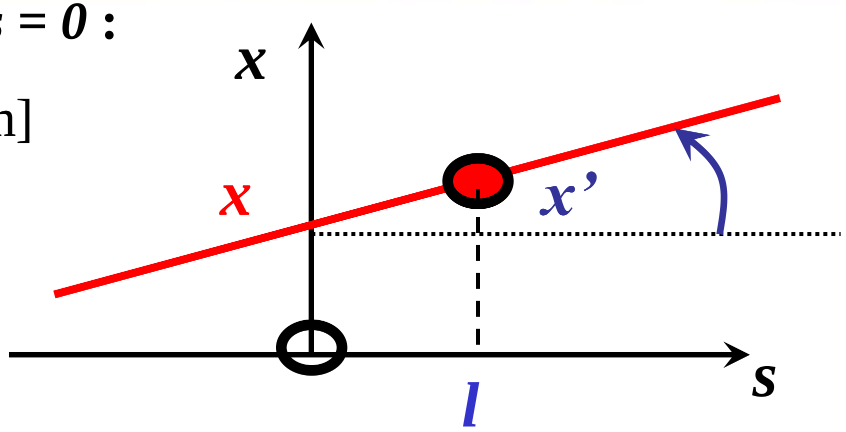
Plot: Wille

Definition of Offset and Divergence

Horizontal and vertical coordinates at *s = 0* :

- x **:** Offset from reference orbit in [mm]
- x'**:** Angle of trajectory in unit [mrad]

  $x' = dx / ds$



Assumption: par-axial beams:

- x is small compared to ρ_0
- Small angle with $p_x / p_s \ll 1$

Longitudinal coordinate:

- Longitudinal orbit difference: $l = -v_0 \cdot (t - t_0)$ in unit [mm]
- Momentum deviation: $\delta = (p - p_0) / p_0$ sometimes in unit [mrad] = [‰]

For **continuous** beam: l has no meaning $\Rightarrow$ set $l \equiv 0$!

Reference particle: no horizontal and vertical offset $x \equiv y \equiv 0$ and $l \equiv 0$ for all s

Definition of Coordinates

The basic vector is 6 dimensional:

$$\vec{x}(s) = \begin{pmatrix} x \\ x' \\ y \\ y' \\ l \\ \delta \end{pmatrix} = \begin{pmatrix} \text{hori. spatial deviation} \\ \text{horizontal divergence} \\ \text{vert. spatial deviation} \\ \text{vertical divergence} \\ \text{longitudinal deviation} \\ \text{momentum deviation} \end{pmatrix} = \begin{pmatrix} [\text{mm}] \\ [\text{mrad}] \\ [\text{mm}] \\ [\text{mrad}] \\ [\text{mm}] \\ [\text{‰}] \end{pmatrix}$$

The transformation from a location s_0 to s_1 is given by the Transfer Matrix R

$$\vec{x}(s_1) = \mathrm{R}(s) \cdot \vec{x}(s_0) = \begin{pmatrix} R_{11} & R_{12} & R_{13} & R_{14} & R_{15} & R_{16} \\ R_{21} & R_{22} & R_{23} & R_{24} & R_{25} & R_{26} \\ R_{31} & R_{32} & R_{33} & R_{34} & R_{35} & R_{36} \\ R_{41} & R_{42} & R_{43} & R_{44} & R_{45} & R_{46} \\ R_{51} & R_{52} & R_{53} & R_{54} & R_{55} & R_{56} \\ R_{61} & R_{62} & R_{63} & R_{64} & R_{65} & R_{66} \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x_0' \\ y_0 \\ y_0' \\ l_0 \\ \delta_0 \end{pmatrix}$$

Remark: At ring accelerator a comparable (i.e. a bit different) matrix is called **M**

Some Properties of the Transfer Matrix

- The transformation can be done successive: with

with $\mathbf{R}_1 = \mathbf{R}(\, s_0 \rightarrow s_1 \,), \ldots, \mathbf{R}_n = \mathbf{R}(\, s_{n-1} \rightarrow s_n \,)$

It is $\mathbf{R} = \mathbf{R}_n \cdot \mathbf{R}_{n-1} \cdot \ldots \cdot \mathbf{R}_1$

- The elements describe the coupling between the components

$R_{11} = (x\,|\,x)$, $R_{12} = (x\,|\,x')$, $R_{13} = (x\,|\,y)$, $R_{14} = (x\,|\,y')$, $R_{15} = (x\,|\,l)$, $R_{16} = (x\,|\,\delta)$

$R_{21} = (x'\,|\,x)$, $R_{22} = (x'\,|\,x')$ …..

- If all forces are symmetric along the reference orbit than the horizontal and vertical plane are decoupled:
$\Rightarrow$ sub-matrix is sufficient

$$\mathrm{R} = \begin{pmatrix} (x\,|\,x) & (x\,|\,x') & 0 & 0 & 0 & (x\,|\,\delta) \\ (x'|\,x) & (x'|\,x') & 0 & 0 & 0 & (x'|\,\delta) \\ 0 & 0 & (y\,|\,y) & (y\,|\,y') & 0 & 0 \\ 0 & 0 & (y'|\,y) & (y'|\,y') & 0 & 0 \\ (l\,|\,x) & (l\,|\,x') & 0 & 0 & 1 & (l\,|\,\delta) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- It is **det(R) = 1** (Liouville's Theorem) $\Rightarrow$ **R** is invertible
- For un-bunched beams: delete row 5 and column 5

Liouville's Theorem:

The phase space density can not changes with conservative e.g. linear forces.

The beam distribution at one location s_0 is described by the beam matrix $\boldsymbol{\sigma}(s_0)$

This beam matrix is transported from location s_0 to s_1 via the transfer matrix

$$\boxed{\boldsymbol{\sigma}(s_1) = \mathbf{R} \cdot \boldsymbol{\sigma}(s_0) \cdot \mathbf{R}^T}$$

6-dim beam matrix with decoupled horizontal and vertical plane:

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 & 0 & \sigma_{15} & \sigma_{16} \\ \sigma_{12} & \sigma_{22} & 0 & 0 & \sigma_{25} & \sigma_{26} \\ 0 & 0 & \sigma_{33} & \sigma_{34} & 0 & 0 \\ 0 & 0 & \sigma_{34} & \sigma_{44} & 0 & 0 \\ \sigma_{15} & \sigma_{25} & 0 & 0 & \sigma_{55} & \sigma_{56} \\ \sigma_{16} & \sigma_{26} & 0 & 0 & \sigma_{56} & \sigma_{66} \end{pmatrix}$$

The beam width concerning the three coordinates are:

$$x_{rms} = \sqrt{\sigma_{11}}$$

$$y_{rms} = \sqrt{\sigma_{33}}$$

$$l_{rms} = \sqrt{\sigma_{55}}$$

Some Examples for linear Transformations

Without dispersion one can use the 2-dim sub-space (x, x').

- Drift with length L: $\mathbf{R_{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$
- Horizontal *focusing* with quadrupole constant k and eff. length L:

$$\mathbf{R_{focus}} = \begin{pmatrix} \cos\sqrt{k}L & \frac{1}{\sqrt{k}}\sin\sqrt{k}L \\ -\frac{1}{\sqrt{k}}\sin\sqrt{k}L & \cos\sqrt{k}L \end{pmatrix}$$

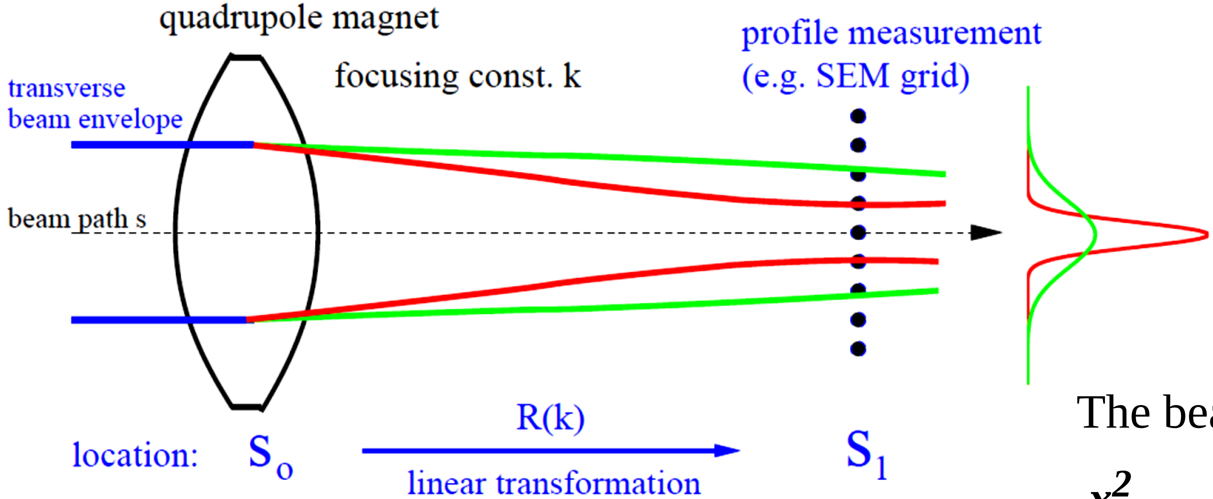
- Horizontal *de-focusing* with quadrupole constant k and eff. length L:

$$\mathbf{R_{defocus}} = \begin{pmatrix} \cosh\sqrt{k}L & \frac{1}{\sqrt{k}}\sinh\sqrt{k}L \\ -\frac{1}{\sqrt{k}}\sinh\sqrt{k}L & \cosh\sqrt{k}L \end{pmatrix}$$

For a (ideal) quadrupole with field gradient $g = B_{pole}/a$, B_{pole} is the field at the pole and a the aperture, the quadrupole constant $k = |g|/(B\rho)_0$ for a magnetic rigidity $(B\rho)_0$.

Emittance Measurement by Quadrupole Variation

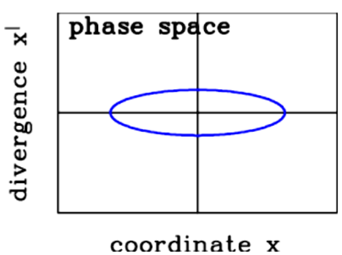
From a profile determination, the emittance can be calculated via linear transformation, if a well known and constant distribution (e.g. Gaussian) is assumed.



The beam width x_{max} and

$x^2_{max} = \sigma_{11}(1, k)$ is measured,

matrix **R(k)** describes the focusing.

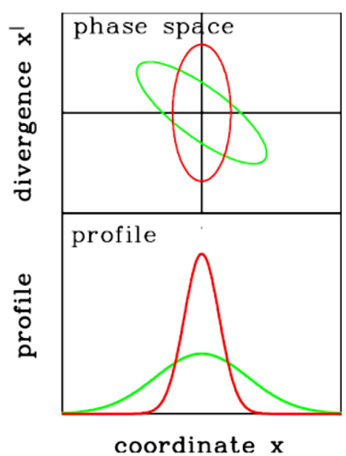


beam matrix:

(Twiss parameters)

$\sigma_{11}(0)$, $\sigma_{12}(0)$, $\sigma_{22}(0)$

to be determined



measurement:

$x^2(k) = \sigma_{11}(1,k)$

Measurement of transverse Emittance

- The beam width x_{max} at s_1 is measured, and therefore $\sigma_{11}(1,k_i) = x^2_{max}(k_i)$.
- Different focusing of the quadrupole $k_1, k_2 ... k_n$ is used: $\Rightarrow \mathbf{R_{focus}}(k_i)$, including the drift, the transfer matrix is changed $\mathbf{R}(k_i) = \mathbf{R_{drift}} \cdot \mathbf{R_{focus}}(k_i)$.
- Task: Calculation of *beam* matrix $\sigma(0)$ at entrance s_0 (size and orientation of ellipse)
- The transformations of the beam matrix are: $\sigma(1,k) = \mathbf{R}(k) \cdot \sigma(0) \cdot \mathbf{R}^{\mathbf{T}}(k)$.

$\Longrightarrow$ Resulting in a redundant system of linear equations for $\sigma_{ij}(0)$:

$$
\begin{aligned}
\sigma_{11}(1,k_1) &= R^2_{11}(k_1)\cdot\sigma_{11}(0) + 2R_{11}(k_1)R_{12}(k_1)\cdot\sigma_{12}(0) + R^2_{12}(k_1)\cdot\sigma_{22}(0) \quad \text{focusing } k_1 \\
&\vdots \\
\sigma_{11}(1,k_n) &= R^2_{11}(k_n)\cdot\sigma_{11}(0) + 2R_{11}(k_n)R_{12}(k_n)\cdot\sigma_{12}(0) + R^2_{12}(k_n)\cdot\sigma_{22}(0) \quad \text{focusing } k_n
\end{aligned}
$$

- To learn something on possible errors, $n > 3$ settings have to be performed.
  A setting with a focus close to the SEM-grid should be included to do a good fit.

- *Assumptions:*
  - Only elliptical shaped emittance can be obtained.
  - No broadening of the emittance e.g. due to space-charge forces.
  - If *not* valid: A self-consistent algorithm has to be used.

Using the 'thin lens approximation' i.e. the quadrupole has a focal length of ***f***:

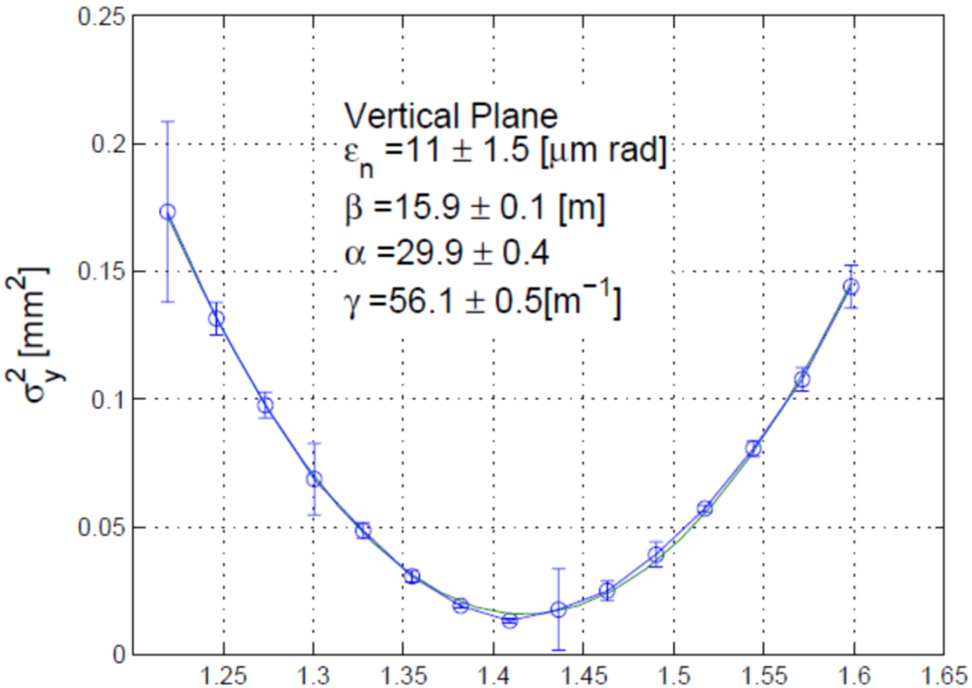
$$\mathrm{R}_{focus}(K) = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ K & 1 \end{pmatrix} \Rightarrow \mathrm{R}(L,K) = \mathrm{R}_{drift}(L)\cdot \mathrm{R}_{focus}(K) = \begin{pmatrix} 1+LK & L \\ K & 1 \end{pmatrix}$$

Example: Square of the beam width at ELETTRA 100 MeV e$^-$ Linac, YAG:Ce:

Measurement of **$\sigma(1,K) = \mathrm{R}(K)\cdot\sigma(0)\cdot\mathrm{R}^T(K)$**

$$\sigma_{11}(1,K) = L^2\sigma_{11}(0)\cdot K^2 + 2\cdot(\ L\sigma_{11}(0) + L^2\sigma_{12}(0)\ )\cdot K + L^2\sigma_{22}(0) + \sigma_{11}(0)$$

$$\equiv a\cdot K^2 \ - \ 2ab\cdot K \ + \ ab^2 + c$$



Focusing strength K [m^{-1}]

G. Penco et al., EPAC'08

The σ-matrix at quadrupole is:

$$\sigma_{11}(0) = \frac{a}{L^2}$$

$$\sigma_{12}(0) = -\frac{a}{L^2}\left(\frac{1}{L} + b\right)$$

$$\sigma_{22}(0) = \frac{1}{L^2}\left(ab^2 + c + \frac{2ab}{L} + \frac{a}{L^2}\right)$$

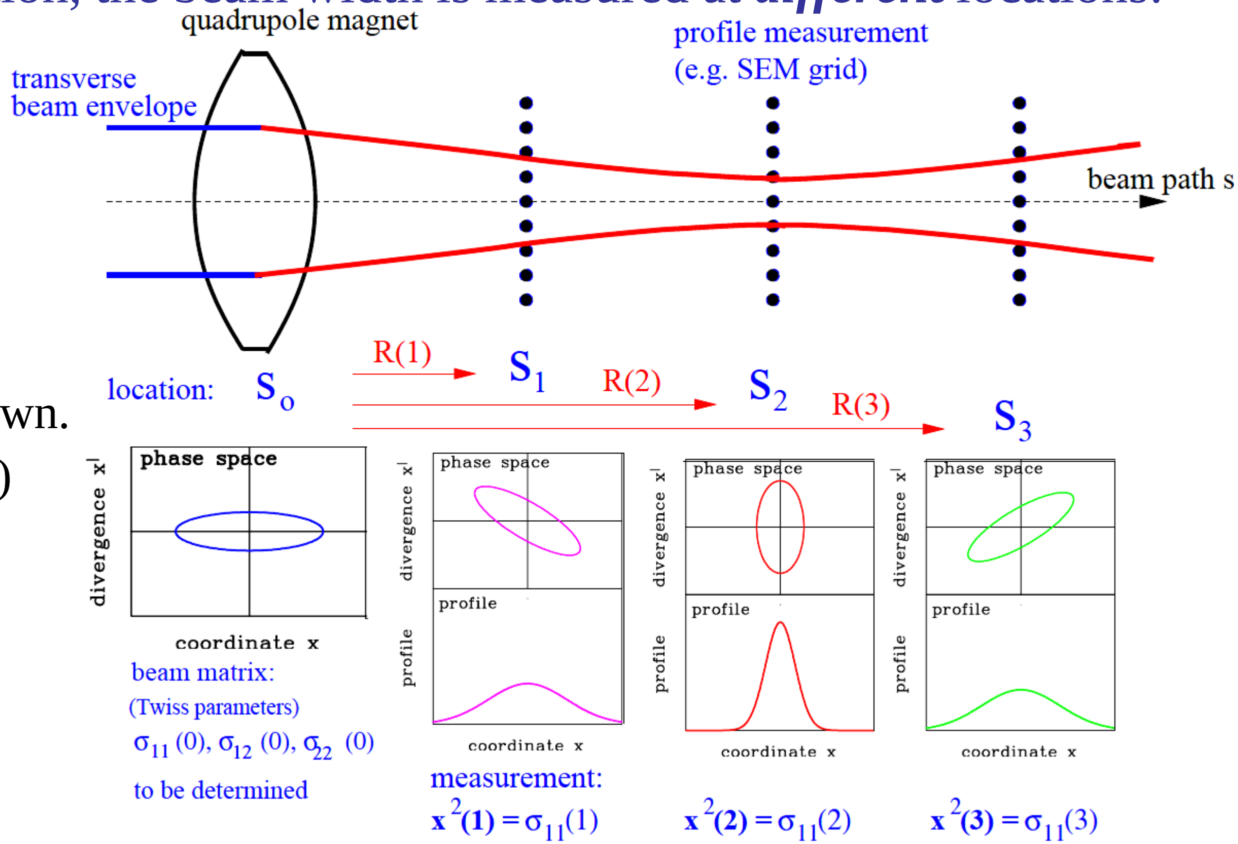
$$\epsilon = \sqrt{\det\sigma(0)} = \sqrt{\sigma_{11}(0)\sigma_{22}(0) - \sigma_{12}^2(0)} = \sqrt{ac}/L^2$$

The 'Three Grid Method' for Emittance Measurement

Instead of quadrupole variation, the beam width is measured at ***different*** locations:

The procedure is:

- Beam width $x(i)$ measured at the locations s_i
  $\Rightarrow$ beam matrix element $x^2(i) = \sigma_{11}(i)$.
- The transfer matrix $\mathbf{R}(i)$ is known. (without dipole a 3 × 3 matrix.)
- The transformations are:
  $\boldsymbol{\sigma}(i) = \mathbf{R}(i)\boldsymbol{\sigma}(0)\mathbf{R}^T(i)$
  $\Rightarrow$ redundant equations:

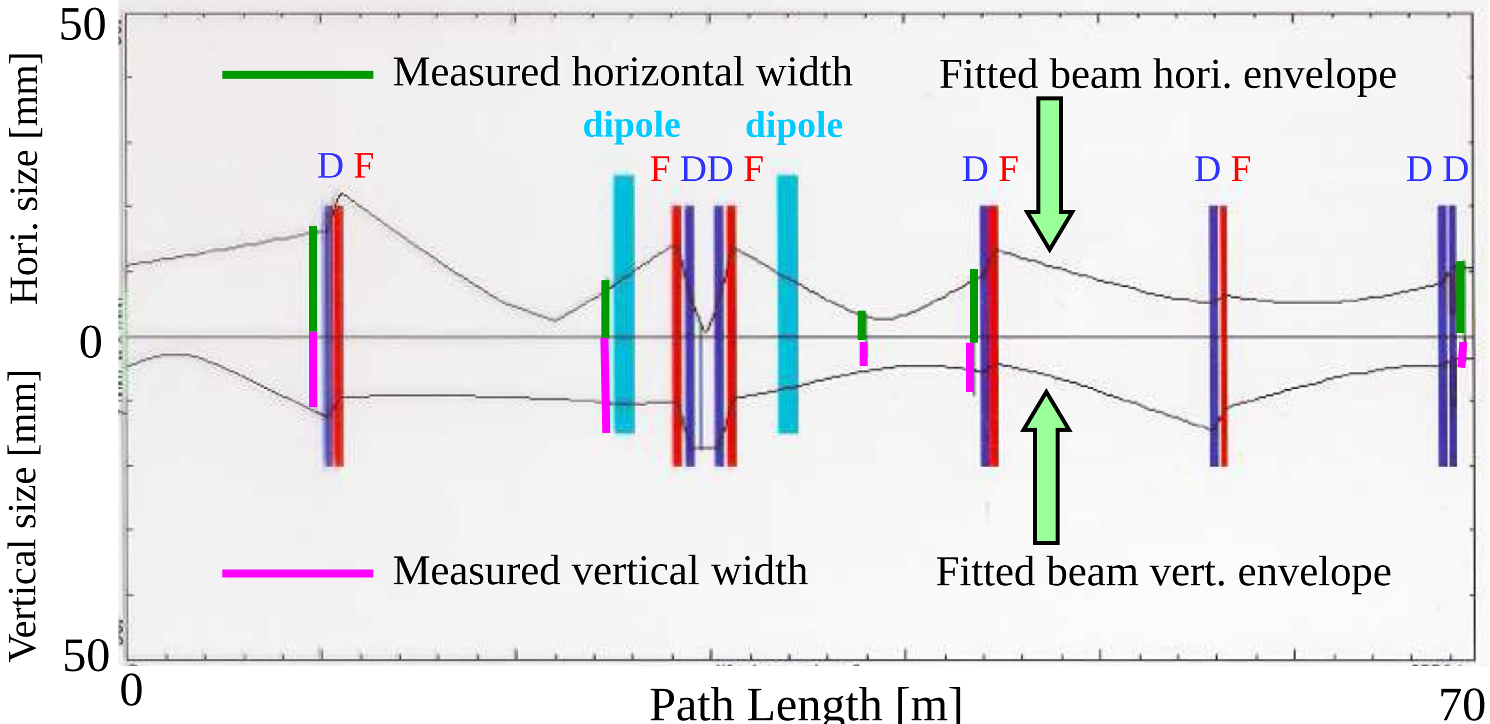


$$\begin{aligned}
\sigma_{11}(1) &= R_{11}^2(1)\cdot\sigma_{11}(0) + 2R_{11}(1)R_{12}(1)\cdot\sigma_{12}(0) + R_{12}^2(1)\cdot\sigma_{22}(0) \qquad \mathbf{R}(1) : s_0 \rightarrow s_1 \\
\sigma_{11}(2) &= R_{11}^2(2)\cdot\sigma_{11}(0) + 2R_{11}(2)R_{12}(2)\cdot\sigma_{12}(0) + R_{12}^2(2)\cdot\sigma_{22}(0) \qquad \mathbf{R}(2) : s_0 \rightarrow s_2 \\
&\vdots \\
\sigma_{11}(n) &= R_{11}^2(n)\cdot\sigma_{11}(0) + 2R_{11}(n)R_{12}(n)\cdot\sigma_{12}(0) + R_{12}^2(n)\cdot\sigma_{22}(0) \qquad \mathbf{R}(n) : s_0 \rightarrow s_n
\end{aligned}$$

Results of a 'Three Grid Method' Measurement

Solution: Solving the linear equations like for quadrupole variation or fitting the profiles with linear optics code (e.g. TRANSPORT, WinAgile, MadX).

Example: The hor. and vert. beam envelope and the beam width at a transfer line:



Assumptions:
- ➢ constant emittance, in particular no space-charge broadening
- ➢ 100 % transmission i.e. no loss due to vacuum pipe scraping
- ➢ no misalignment, i.e. beam center equals center of the quadrupoles.

Summary for transverse Emittance Measurement

Emittance measurements are very important for comparison to theory.

It includes size (value of ε) and orientation in phase space (σ_{ij} or α, β and γ)

(three independent values)

Techniques for transfer lines (synchrotron: width measurement sufficient):

Low energy beams → direct measurement of x- and x'-distribution

- ***Slit-grid:*** movable slit → ***x***-profile, grid → ***x'***-profile
- ***Pepper-pot***: holes → ***x***-profile, scintillation screen → ***x'***-profile

All beams → profile measurement + linear transformation:

- ***Quadrupole variation:*** one location, different setting of a quadrupole
- ***'Three grid method':*** different locations
- ***Assumptions:***
  - well aligned beam, no steering
  - no emittance blow-up due to space charge.