

JUAS-2014

**Mathematical Foundations
of Electromagnetic Fields**

Stephan Russenschuck



- Introduction and overview
- Mappings, Vector-space, the framework of EM-fields
- The Maxwell equations in integral and local form
- Iron dominated magnets

- Vector analysis, Harmonic fields
- Magnetic field measurements

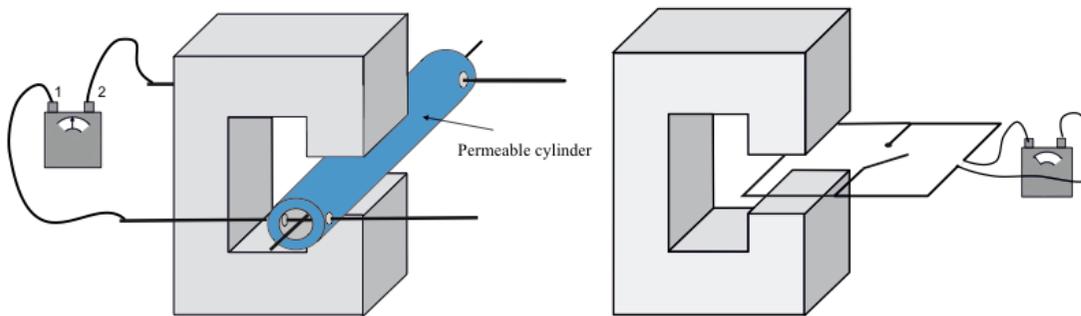
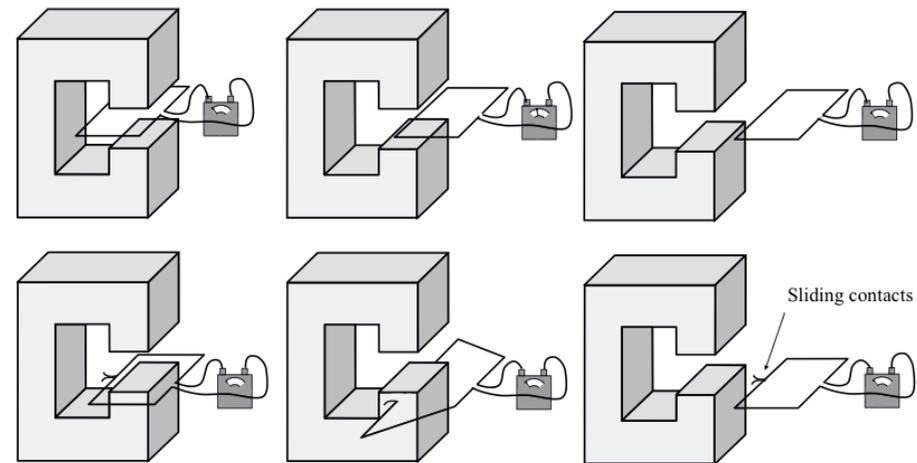
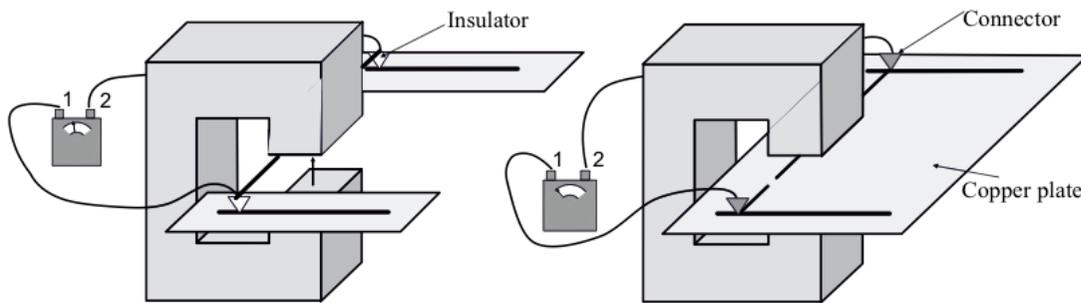
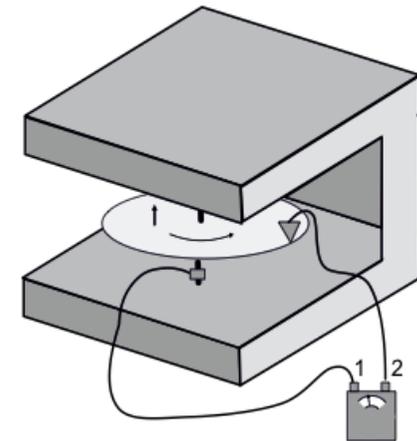
- Field of line-currents, Biot-Savart
- Coil fields of superconducting magnets, field harmonics

- Principles of numerical field computation

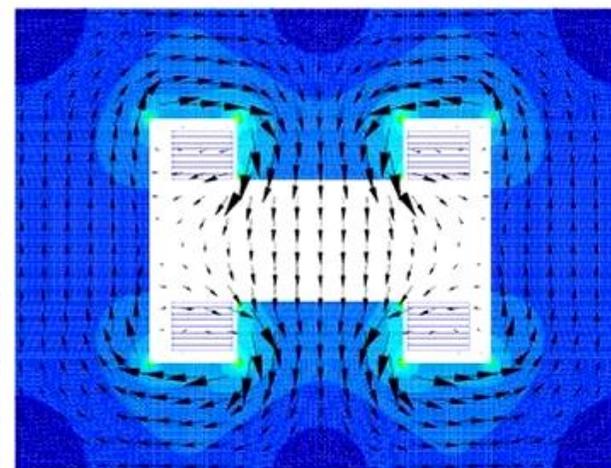
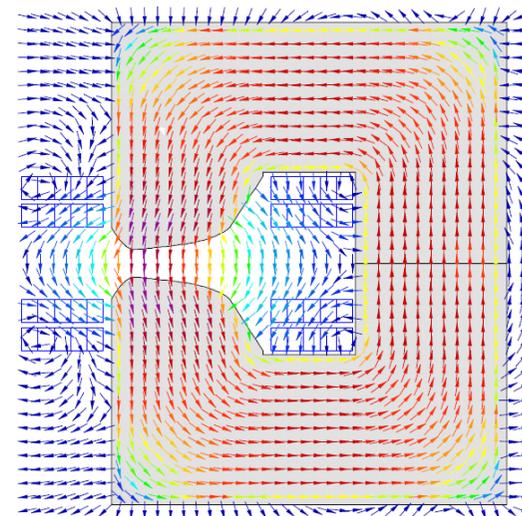
A Motivation: Faraday Paradoxes

$$U = -\frac{d\Phi}{dt} = -\frac{d}{dt} \int_{\mathcal{A}} \vec{B} \cdot d\vec{a}$$

$$= -\int_{\mathcal{A}} \frac{\partial}{\partial t} \vec{B} \cdot d\vec{a} + \int_{\partial\mathcal{A}} (\vec{v} \times \vec{B}) \cdot d\vec{r},$$

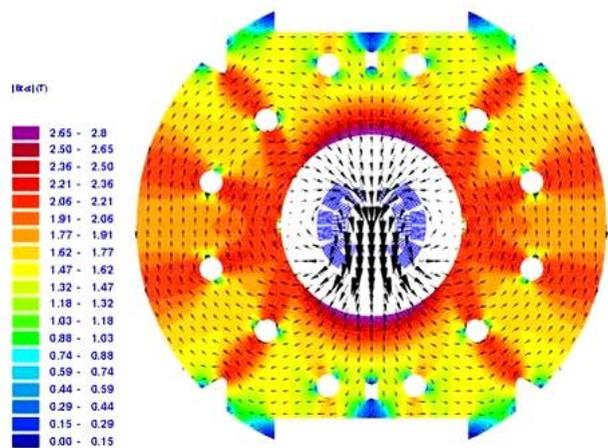


Iron Dominated Magnets

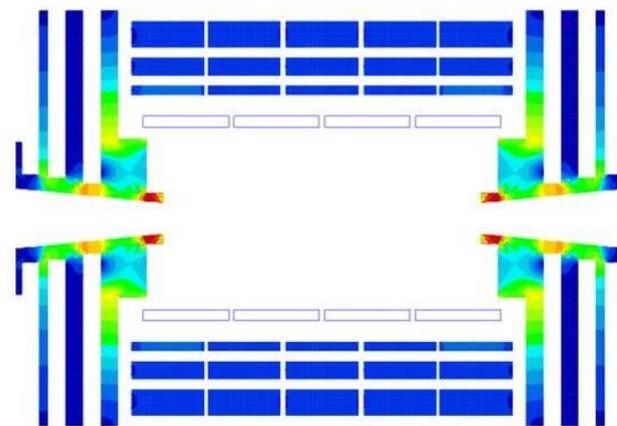
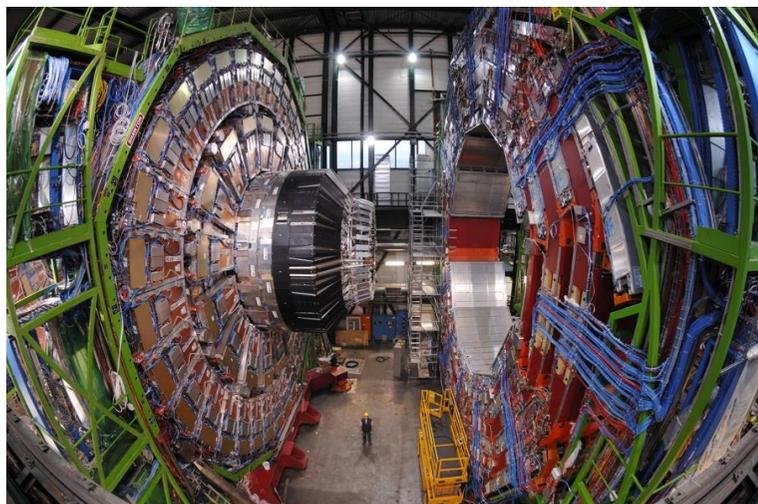


$N \cdot I = 24000 \text{ A}$ $B_1 = 0.3 \text{ T}$ $B_s = 0.065 \text{ T}$ Fill.fac. 0.98

Coil Dominated Magnets



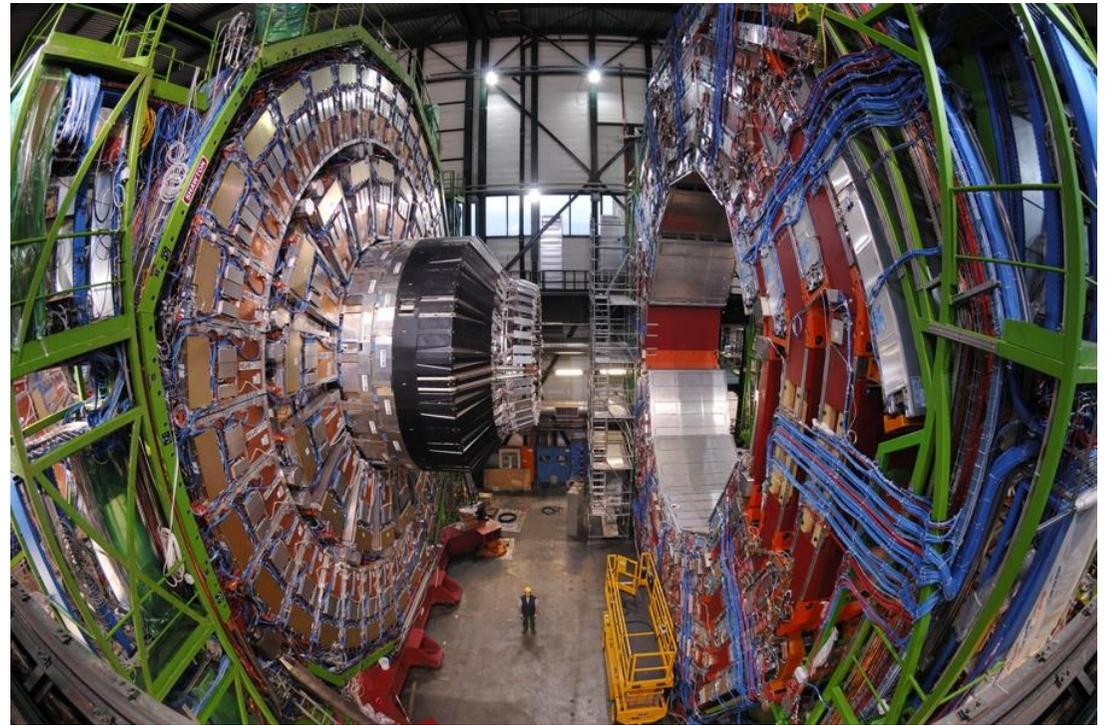
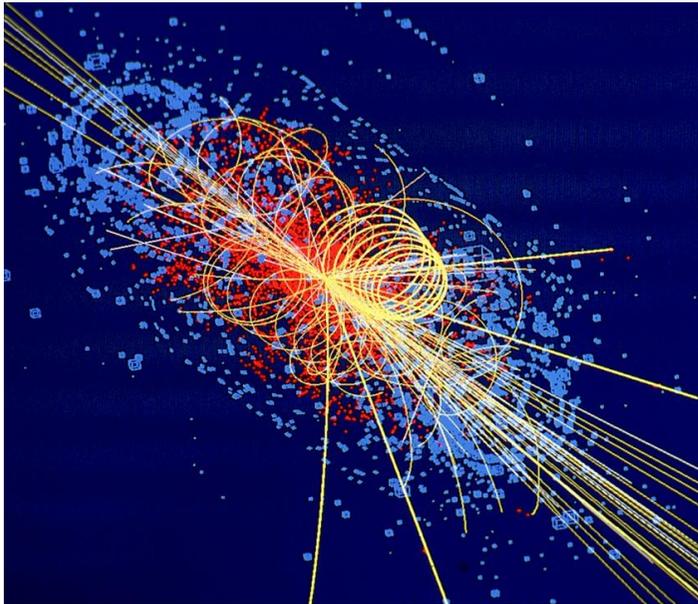
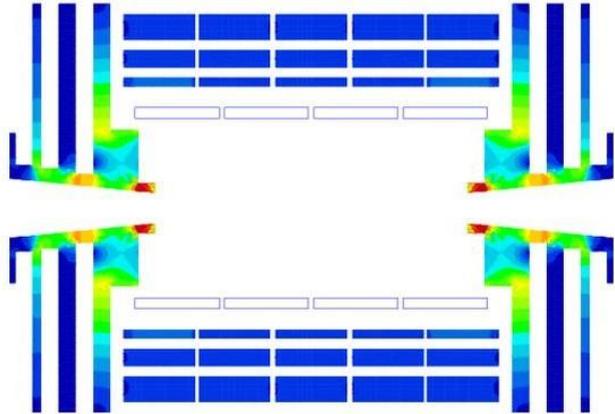
$B = 8.33 \text{ T}$ $B_s = 7.77 \text{ T}$



$B = 4 \text{ T}$

$B_s = 3.69 \text{ T}$

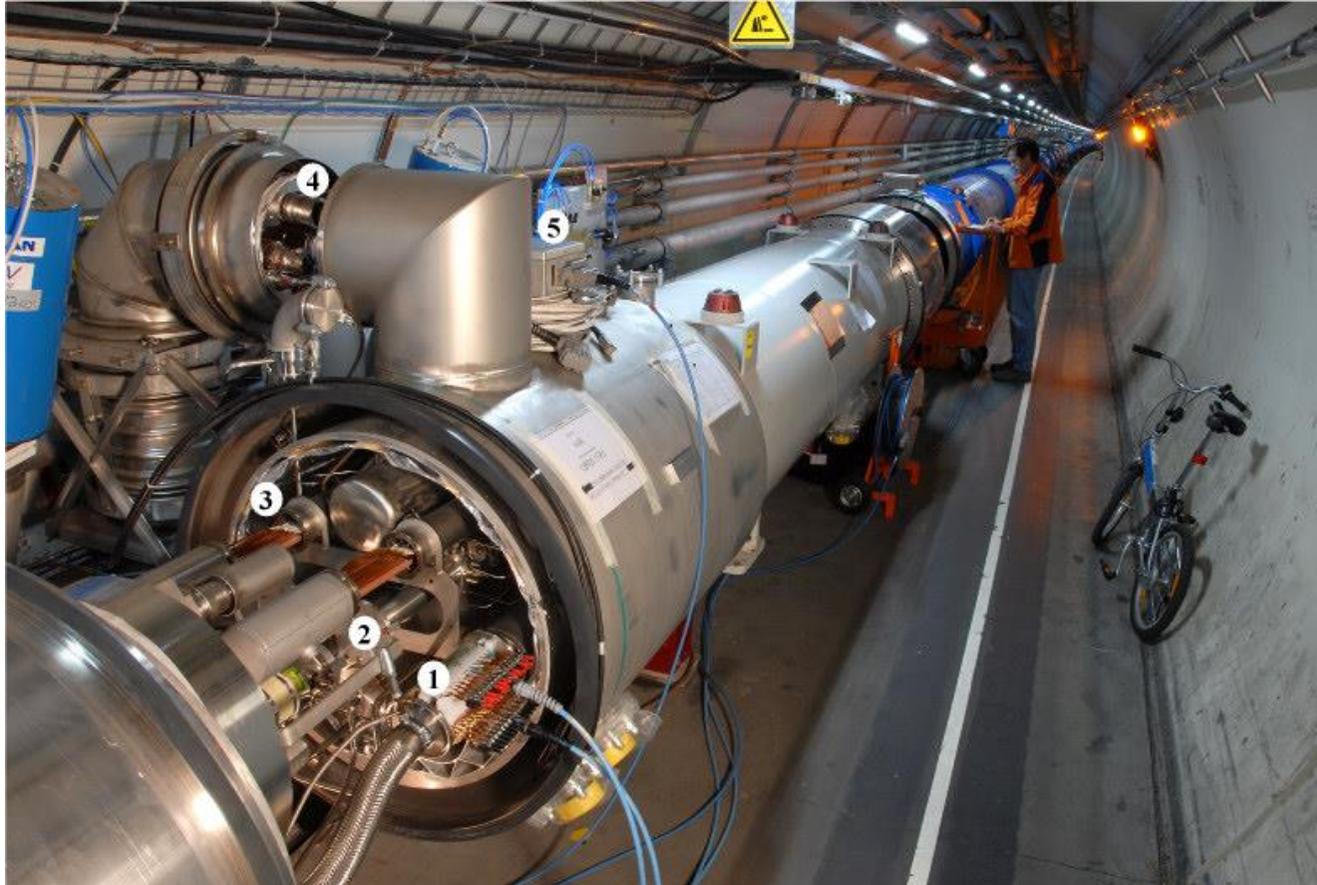
CMS (Class 1 Magnets)



$$S = R(1 - \cos \frac{\alpha}{2}) \approx \frac{R\alpha^2}{8} = \frac{QBL^2}{8p}$$

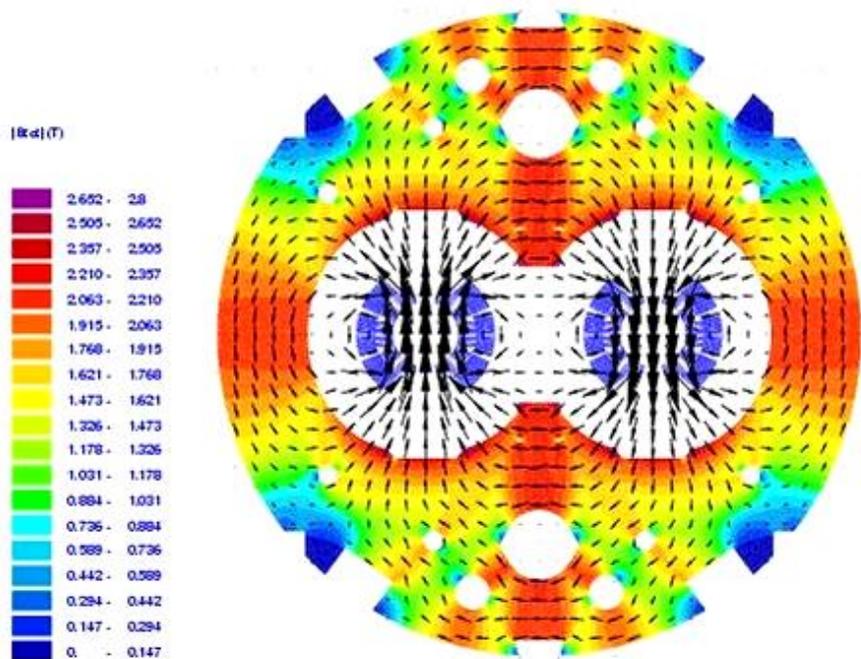
String of LHC Magnets in the Tunnel (Class 2 Magnets)

$$\{p\}_{\text{GeV}/c} \approx 0.3\{Q\}_e\{R\}_m\{B_0\}_T$$



High field and high current density

LHC Two-in-one Dipole

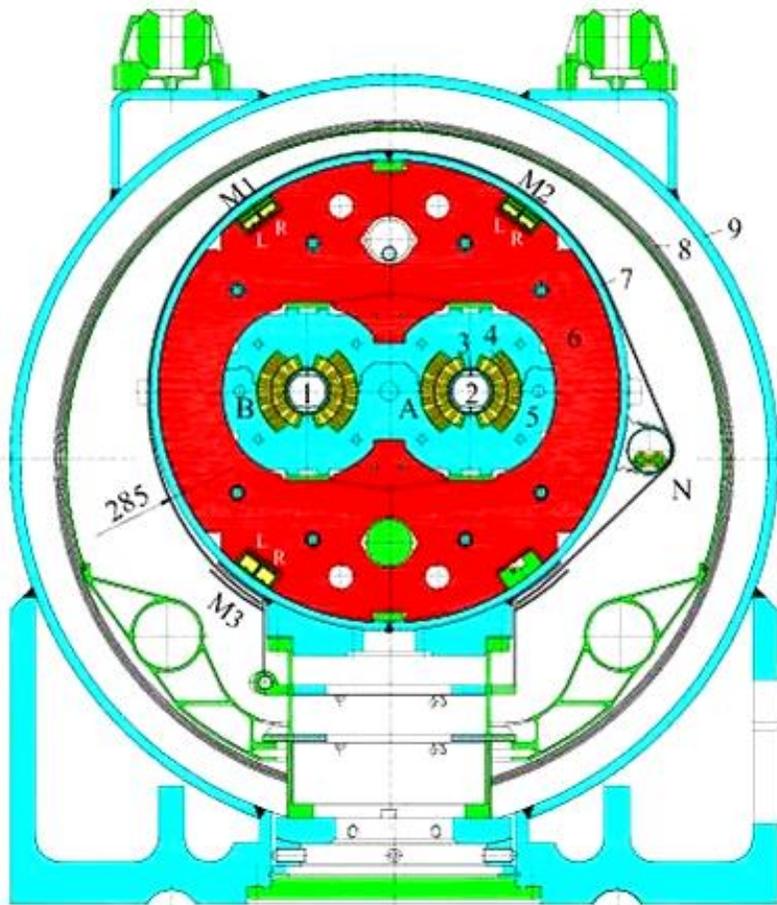


$$N \cdot I = 2 \times 944000 \text{ A} \quad B_1 = 8.32 \text{ T} \quad B_s = 7.44 \text{ T}$$

Storage of cold-masses



Cross-section of Cryodipole



Cryostat integration at CERN



→ Conventional magnets

- Important ohmic losses require water cooling
- Field is defined by the iron pole shape (max 1.5 T)
- Easy electrical and beam-vacuum interconnections
- Voltage drop over one coil of the MBW magnets = 22 V

→ Superconducting magnets

- Field is defined by the coil layout
- Maximum field limited to 10 T (NbTi), 12 T (Nb₃Sn)
- Enormous electromagnetic forces (400 tons/m in MB for LHC)
- Quench protection system required
- Cryogenic installation (1.8 K)
- Electrical interconnections in cryo-lines
- Voltage drop on LHC magnet string (154 MB) 155 V

→ Conventional magnets

- Ideal pole shape known from potential theory
- One-dimensional (analytical) field computation for main field
- Commercial FEM software can be used as a black box (hysteresis modeling)

→ Superconducting magnets

- Decoupling of coil and yoke optimization
- Accuracy of the field solution
- Modeling of the coils
- Filament magnetization
- Quench simulations

A Multiphysics Problem

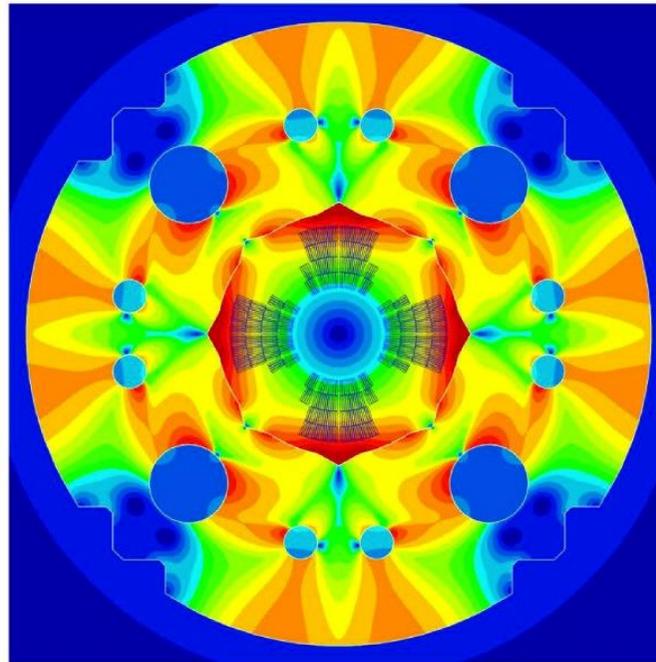
- Beam physics
- Material science: Superconducting cable, Steel, Insulation
- Mechanics and large-scale mechanical engineering
- Vacuum technology
- Cryogenics (Superfluid helium)
- Metrology and alignment
- Field measurements
- Electrical engineering (Power supplies, leads, buswork, quench detection and magnet protection)
- Analytical and numerical field computation

Stephan Russenschuck

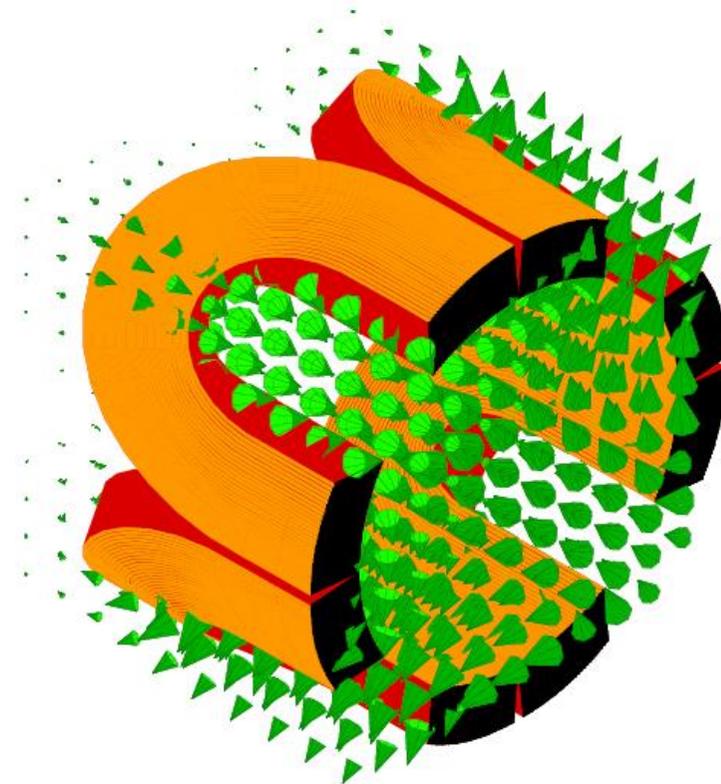
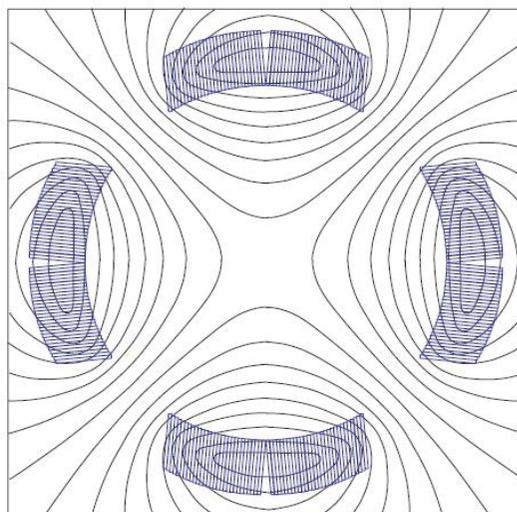
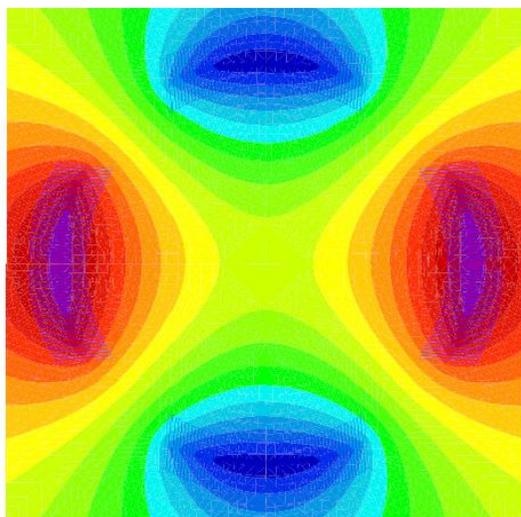
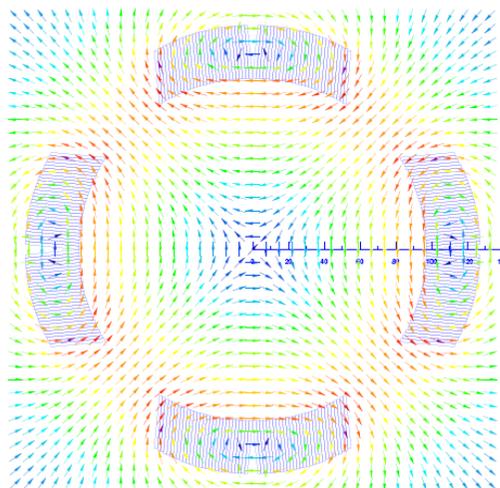
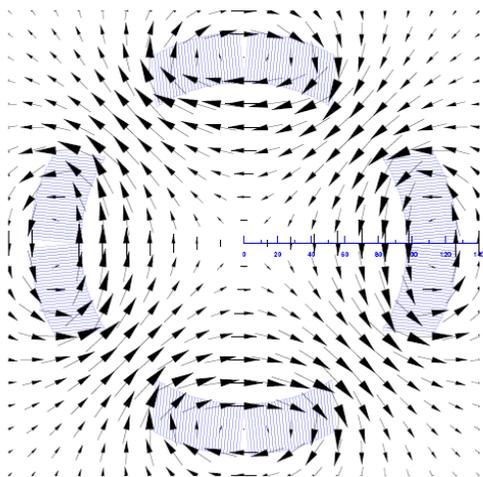
Wiley-VCH

Field Computation for Accelerator Magnets

Analytical and Numerical Methods for Electromagnetic
Design and Optimization



Different Renderings of the Same Vector Field



$$f : X \rightarrow W : x \mapsto f(x),$$

which reads: $f(x)$ is the element of W that f assigns to $x \in X$.

f	:	$[a, b] \rightarrow \mathbb{R}$:	$x \mapsto f(x)$	Real function
$+$:	$\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$:	$(x, y) \mapsto x + y$	Addition
f	:	$\Omega \rightarrow \mathbb{R}$:	$\mathcal{P} \mapsto f(\mathcal{P})$	Scalar field
$\mathbf{x} \cdot \mathbf{y}$:	$\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$:	$(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} \cdot \mathbf{y}$	Scalar product
\mathcal{S}	:	$I \rightarrow E_3$:	$t \mapsto \mathcal{S}(t)$	Space curve
\mathbf{f}	:	$\Omega \rightarrow \mathbb{R}^3$:	$\mathcal{P} \mapsto \mathbf{f}(\mathcal{P})$	Vector field.

The symbol \times in the mapping $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ denotes the Cartesian product¹ and is defined by the set of all ordered pairs (x, y) for $x, y \in \mathbb{R}$. Mappings f with $f : W \times W \rightarrow W$ are called *binary operations* in W , for example, multiplication $\cdot : W \times W \rightarrow W : (x, y) \mapsto xy$ and addition as in the table above.

Vector Space (Linear Space) Axioms

$(V, +, \cdot)$, shorthand V , is a *vector space* over \mathbb{F} if the following axioms are fulfilled:

1. For any vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V$: $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$.
2. There is a zero vector $\mathbf{0}$ for which $\mathbf{a} + \mathbf{0} = \mathbf{a}$ for any vector \mathbf{a} .
3. For each vector $\mathbf{a} \in V$ there is a vector $-\mathbf{a}$ in V for which $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$.
4. For any vectors $\mathbf{a}, \mathbf{b} \in V$: $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$.
5. For any scalar $\lambda \in \mathbb{F}$ and any vectors $\mathbf{a}, \mathbf{b} \in V$: $\lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}$.
6. For any scalars $\lambda, \mu \in \mathbb{F}$ and any vector $\mathbf{a} \in V$: $(\lambda + \mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}$.
7. For any scalars $\lambda, \mu \in \mathbb{F}$ and any vector $\mathbf{a} \in V$: $(\lambda\mu)\mathbf{a} = \lambda(\mu\mathbf{a})$.
8. For the unit scalar $1 \in \mathbb{F}$ and any vector $\mathbf{a} \in V$: $1\mathbf{a} = \mathbf{a}$.

1-4 = Group, 5-8 additional structure of linear space

Remark: No mention on dimension; could be infinite

Examples for Linear Spaces

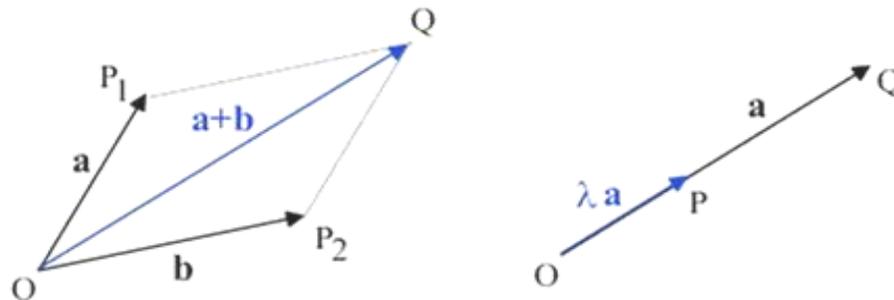
1. Functional space $C^m(X)$ of all m -times continuously differentiable functions from any nonempty set $X \in \mathbb{R}^n$ into the field of real numbers \mathbb{R} with rules of addition $(f + g)(x) = f(x) + g(x)$ and scalar multiplication $(\lambda f)(x) = \lambda f(x)$.
2. Matrix space M of all $m \times n$ matrices $[A] = (a_{ij}); i = 1, \dots, m; j = 1, \dots, n$, over a field \mathbb{F} with addition $[A] + [B] = [C]$ defined by $c_{ij} = a_{ij} + b_{ij}$ and scalar multiplication $\lambda[B] = [C]$ defined by $c_{ij} = \lambda b_{ij}$.
3. Tuple space $\mathbb{R}^n = \{(a_1, a_2, \dots, a_n) \mid a_1, \dots, a_n \in \mathbb{R}\}$, where vector addition and scalar multiplication are defined by

$$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n), \quad (2.19)$$

$$\lambda(a_1, a_2, \dots, a_n) = (\lambda a_1, \lambda a_2, \dots, \lambda a_n), \quad (2.20)$$

and where the zero vector is defined by $\mathbf{0} := (0, 0, \dots, 0)$.

4. Icon space



Linear Independence and the Basis Isomorphism

The vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ in V are said to be linearly independent if $\sum_{i=1}^n \lambda_i \mathbf{x}_i = 0$ holds only for the trivial solution where all the coefficients $\lambda_i = 0, i = 1, \dots, n$.

The vector space is n -dimensional if there exist n linearly independent vectors, and when $n + 1$ vectors are always linearly dependent such that

$$\mu \mathbf{x} + \lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 + \dots + \lambda_n \mathbf{x}_n = 0, \quad \text{with } \mu \neq 0.$$

Because of $\mu \neq 0$ we can express the $(n + 1)$ st vector through the n others in the form $\mathbf{x} = -\frac{1}{\mu} \sum_{i=1}^n \lambda_i \mathbf{x}_i$. The set $S = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ of vectors is called a *basis* or *frame* of V_n and shall be denoted $\{\mathbf{g}_1, \mathbf{g}_2, \dots, \mathbf{g}_n\}$. The basis is a minimal spanning set of V_n .

As a consequence of the linear independence of the basis vectors, there is exactly one way of expressing an element $\mathbf{x} \in V_n$ by

$$\mathbf{x} = \sum_{i=1}^n x^i \mathbf{g}_i$$

Physical definition of a vector:
A quantity having **direction** and magnitude

1. $\mathcal{P} + \mathbf{x} \in A$ if $\mathcal{P} \in A$ and $\mathbf{x} \in V$.
2. $(\mathcal{P} + \mathbf{x}) + \mathbf{y} = \mathcal{P} + (\mathbf{x} + \mathbf{y})$ for $\mathcal{P} \in A$ and $\mathbf{x}, \mathbf{y} \in V$.
3. There is a unique $\mathbf{x} \in V$ such that $\mathcal{P}_1 = \mathcal{P}_2 + \mathbf{x}$ for $\mathcal{P}_1, \mathcal{P}_2 \in A$.

Affine transformations preserve Barycenters, this is what you do in your brain when looking at an exhibition catalogue and the original in the museum.

$$\mathcal{P} \in A_n \xrightarrow{\text{Origin}} \mathbf{r} \in V_n \xrightarrow{\text{Basis}} (x^1, \dots, x^n) \in \mathbb{R}^n .$$

Physical definition of a vector:
A quantity having direction and **magnitude**

We must now introduce the *inner product space* $(V, \langle \cdot, \cdot \rangle)$, shorthand V_n , a real vector space with an inner product $\langle \mathbf{a}, \mathbf{b} \rangle : V_n \times V_n \rightarrow \mathbb{R}$ that obeys bilinearity, symmetry, and positive definiteness:

1. $\langle \mathbf{a} + \mathbf{b}, \mathbf{c} \rangle = \langle \mathbf{a}, \mathbf{c} \rangle + \langle \mathbf{b}, \mathbf{c} \rangle$ and $\langle \mathbf{a}, \lambda \mathbf{b} + \mu \mathbf{c} \rangle = \lambda \langle \mathbf{a}, \mathbf{b} \rangle + \mu \langle \mathbf{a}, \mathbf{c} \rangle$.
2. $\langle \mathbf{a}, \mathbf{b} \rangle = \langle \mathbf{b}, \mathbf{a} \rangle$.
3. $\langle \mathbf{a}, \mathbf{a} \rangle > 0$ and $\langle \mathbf{a}, \mathbf{a} \rangle = 0$ if and only if $\mathbf{a} = \mathbf{0}$.

Examples: Euclidean space

$$\mathbf{a} \cdot \mathbf{b} := a^1 b^1 + a^2 b^2 + a^3 b^3$$

Minkowski space

$$\langle \mathbf{a}, \mathbf{b} \rangle := a^1 b^1 + a^2 b^2 + a^3 b^3 - (c)^2 a^4 b^4,$$

Functional space

$$\langle f, g \rangle := \int_{-1}^1 f(x)g(x)dx.$$

Norm and Distance

Note: There are norms that are not induced by the scalar product, e.g. the Manhattan norm
There are distance concepts not induced by a norm, e.g., the French railroad metric

Length (Norm induced by the scalar product) $\| \mathbf{a} \| = \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle},$

Angle $\cos \alpha(\mathbf{a}, \mathbf{b}) := \frac{\langle \mathbf{a}, \mathbf{b} \rangle}{\| \mathbf{a} \| \| \mathbf{b} \|}, \quad 0 \leq \alpha \leq \pi.$

Cauchy Schwarz inequality $|\langle \mathbf{a}, \mathbf{b} \rangle| \leq \| \mathbf{a} \| \| \mathbf{b} \|,$

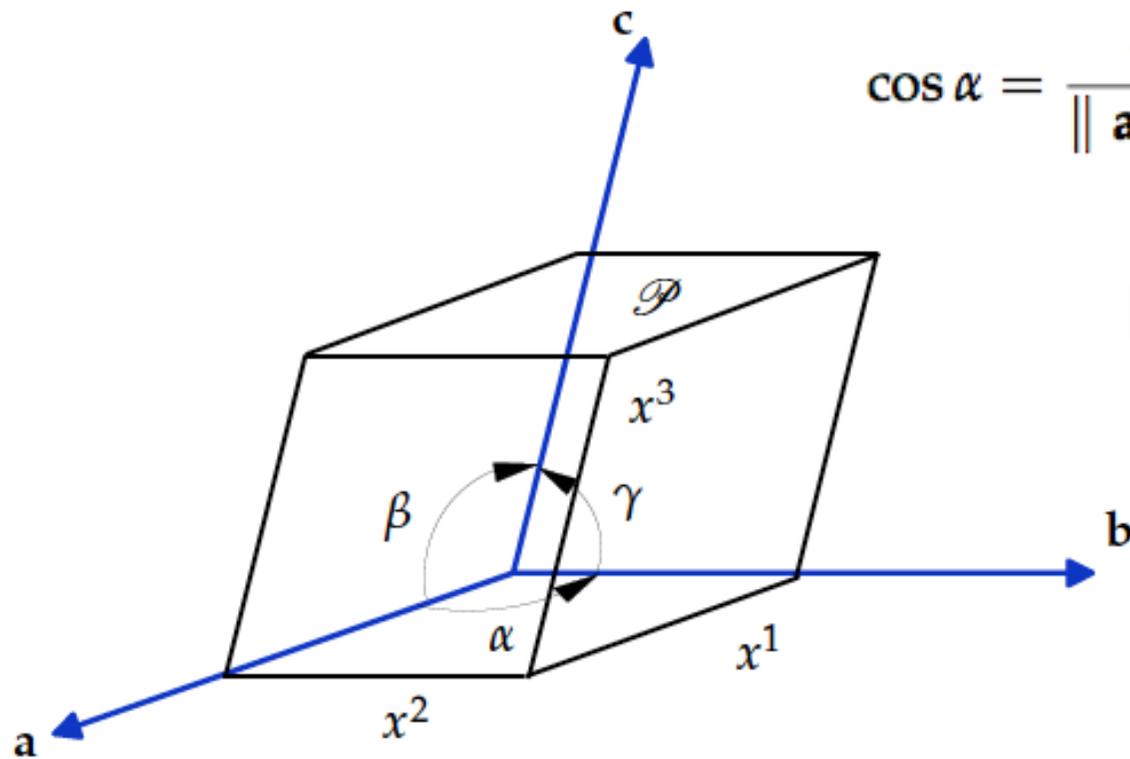
Distance in associated affine space $d(\mathcal{P}_1, \mathcal{P}_2) := \| \mathbf{r}_{\mathcal{P}_1} - \mathbf{r}_{\mathcal{P}_2} \|$

If a basis is present:

$$\begin{aligned}\langle \mathbf{a}, \mathbf{b} \rangle &= a^1 b^1 \langle \mathbf{g}_1, \mathbf{g}_1 \rangle + a^1 b^2 \langle \mathbf{g}_1, \mathbf{g}_2 \rangle + a^1 b^3 \langle \mathbf{g}_1, \mathbf{g}_3 \rangle \\ &\quad + a^2 b^1 \langle \mathbf{g}_2, \mathbf{g}_1 \rangle + a^2 b^2 \langle \mathbf{g}_2, \mathbf{g}_2 \rangle + a^2 b^3 \langle \mathbf{g}_2, \mathbf{g}_3 \rangle \\ &\quad + a^3 b^1 \langle \mathbf{g}_3, \mathbf{g}_1 \rangle + a^3 b^2 \langle \mathbf{g}_3, \mathbf{g}_2 \rangle + a^3 b^3 \langle \mathbf{g}_3, \mathbf{g}_3 \rangle,\end{aligned}$$

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=1}^3 \sum_{j=1}^3 a^i b^j \langle \mathbf{g}_i, \mathbf{g}_j \rangle \equiv a^i b^j \langle \mathbf{g}_i, \mathbf{g}_j \rangle =: a^i b^j g_{ij},$$

Why so Complicated



$$\cos \alpha = \frac{\langle \mathbf{a}, \mathbf{b} \rangle}{\|\mathbf{a}\| \|\mathbf{b}\|} = \frac{a^i b^j g_{ij}}{\sqrt{a^p a^q g_{pq}} \sqrt{b^r b^s g_{rs}}}.$$

$$\|\mathbf{a}\| = \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle} = \sqrt{a^i a^j g_{ij}},$$

$$[G] = \begin{pmatrix} 1 & \cos \alpha & \cos \beta \\ \cos \alpha & 1 & \cos \gamma \\ \cos \beta & \cos \gamma & 1 \end{pmatrix}$$

Orthogonality

$$\langle \mathbf{g}_i, \mathbf{g}_j \rangle = g_{ij} = \delta_{ij} = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases} .$$

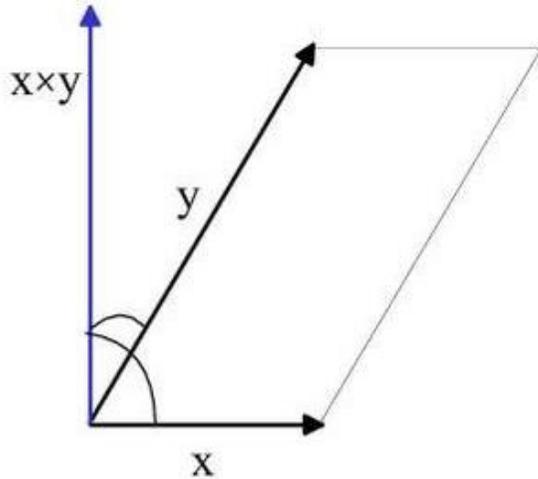
Distance in associated affine space (Pythagoras)

$$d(\mathcal{P}_1, \mathcal{P}_2) = \sqrt{\sum_{i=1}^n (x^i(\mathcal{P}_1) - x^i(\mathcal{P}_2))^2} .$$

The inner product then takes the usual Euclidean form

$$\langle \mathbf{a}, \mathbf{b} \rangle = \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} = a^1 b^1 + a^2 b^2 + a^3 b^3 + \dots + a^n b^n .$$

Vector Product



For an orthogonal basis (always possible to find with Erhard Schmidt orthogonalization)

$$\begin{aligned}\vec{e}_1 \times \vec{e}_2 &= -\vec{e}_2 \times \vec{e}_1 = \vec{e}_3 \\ \vec{e}_2 \times \vec{e}_3 &= -\vec{e}_3 \times \vec{e}_2 = \vec{e}_1 \\ \vec{e}_3 \times \vec{e}_1 &= -\vec{e}_1 \times \vec{e}_3 = \vec{e}_2\end{aligned}$$

$$\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2) \vec{e}_1 + (a_3 b_1 - a_1 b_3) \vec{e}_2 + (a_1 b_2 - a_2 b_1) \vec{e}_3$$

Requires orientation; later

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$(\lambda \mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (\lambda \mathbf{b}) = \lambda \mathbf{a} \cdot \mathbf{b},$$

$$(\lambda \mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (\lambda \mathbf{b}) = \lambda \mathbf{a} \times \mathbf{b},$$

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c},$$

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c},$$

$$(\mathbf{a} + \mathbf{b}) \times (\mathbf{c} + \mathbf{d}) = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c}) + (\mathbf{a} \times \mathbf{d}) + (\mathbf{b} \times \mathbf{d}),$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}),$$

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}),$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}),$$

$$\mathbf{x}(\mathbf{abc}) = \mathbf{a}(\mathbf{xbc}) + \mathbf{b}(\mathbf{axc}) + \mathbf{c}(\mathbf{abx}),$$

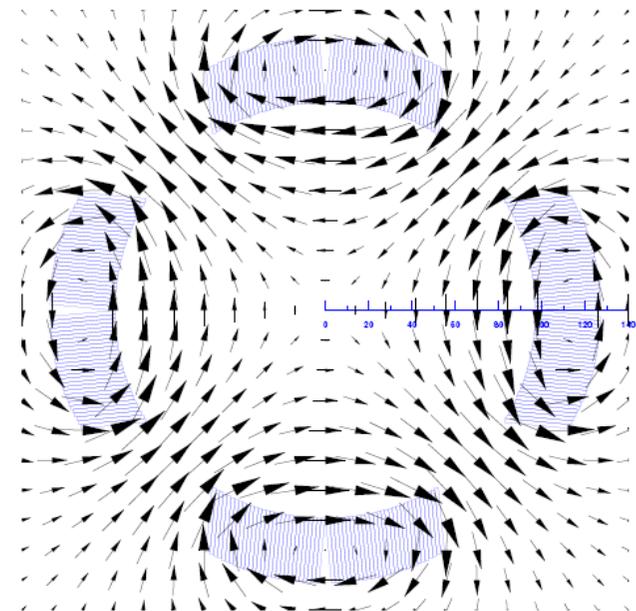
$$\mathbf{x}(\mathbf{abc}) = (\mathbf{a} \cdot \mathbf{x})(\mathbf{b} \times \mathbf{c}) + (\mathbf{b} \cdot \mathbf{x})(\mathbf{c} \times \mathbf{a}) + (\mathbf{c} \cdot \mathbf{x})(\mathbf{a} \times \mathbf{b}),$$

$$(\mathbf{abc})^2 = (\mathbf{a} \times \mathbf{b})(\mathbf{b} \times \mathbf{c})(\mathbf{c} \times \mathbf{a}).$$

Framework of our Vectorfields E_3

- E_3 has the structure of the affine point space
- It carries the vector (linear) space structure of its associated vector space
- It is equipped with a metric that gives rise to distance and angles

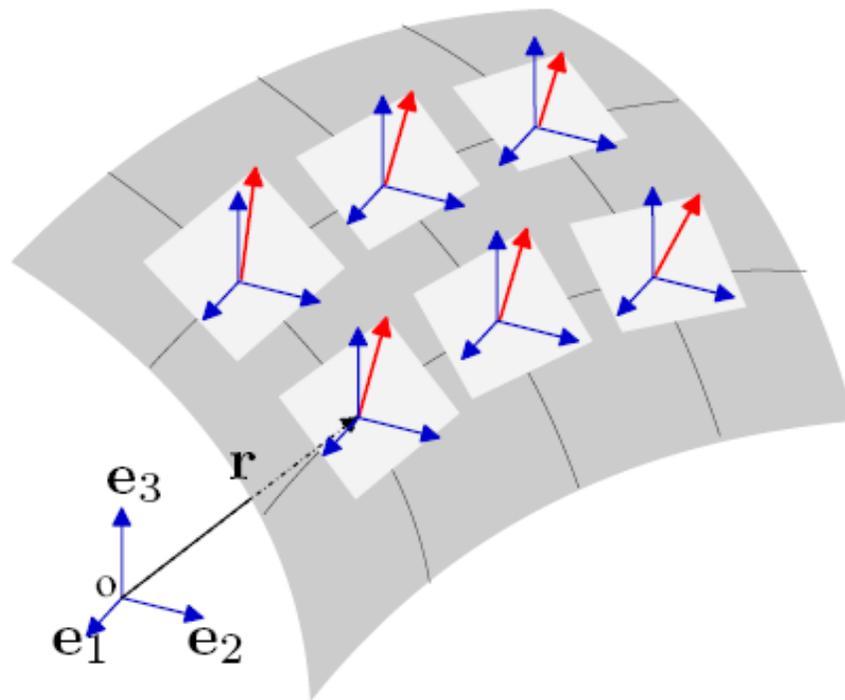
- If an origin and basis is selected, the projection of the position vector on the basis yields the coordinates (in \mathbb{R}^3)
- The canonical basis (e_1, e_2, e_3) can be made to a basis field by translation
- The components of the field at some point are then the projection on this basis field



Vector and Scalar Fields

$$\mathbf{a} : \Omega \rightarrow \mathbb{R}^3 : \mathbf{r} \mapsto \mathbf{a}(\mathbf{r}) : \mathbf{a}(\mathbf{r}) = (a^1(\mathbf{r}), a^2(\mathbf{r}), a^3(\mathbf{r}))$$

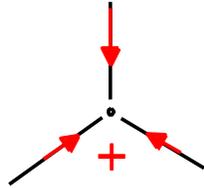
$$\Omega \subset \mathbb{R}^3$$



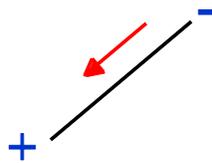
$$\phi : \Omega \rightarrow \mathbb{R} : \phi \mapsto \phi(\mathbf{r})$$

Orientation of Space Elements

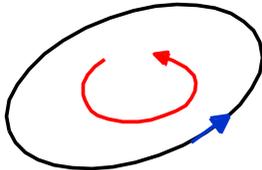
Inner orientation



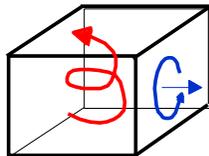
Point:
A positive point is oriented as a sink



Line/Edge:
Selecting a vector pointing in forward direction

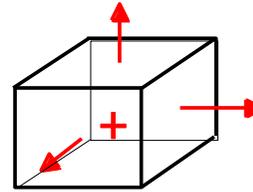


Surface:
Sense of rotation

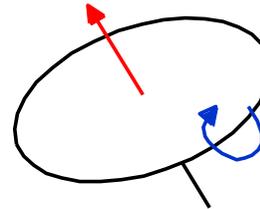


Volume:
Sense of a screw

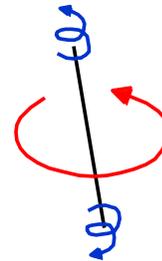
Outer orientation



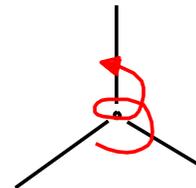
Volume:
Choice of outward normals



Surface:
Crossing direction of a line



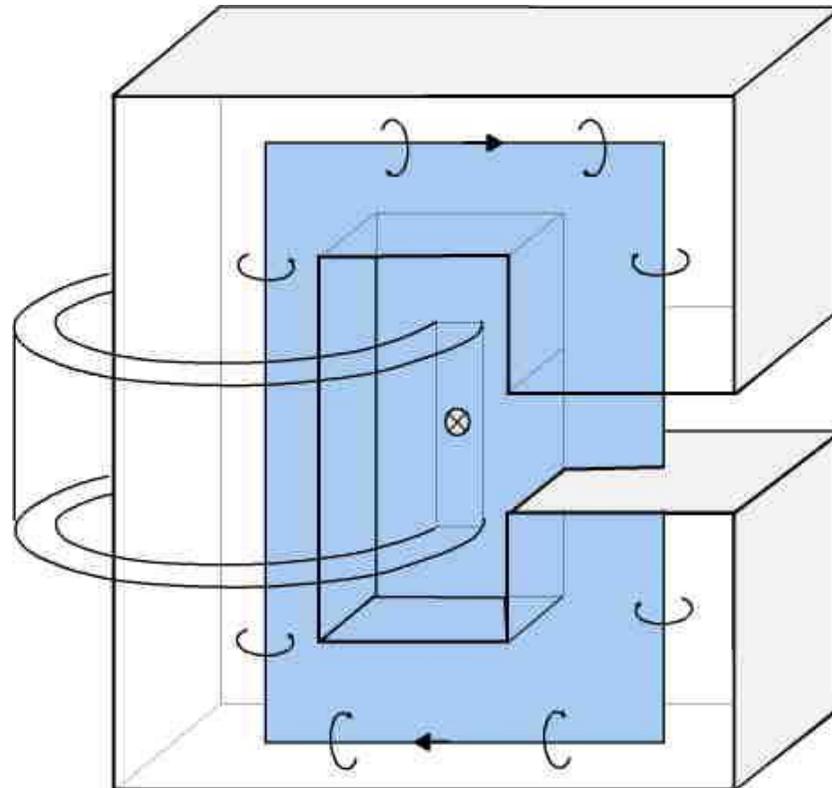
Line/Edge:
Direction of circulation of a surface around this line



Point:
The inner orientation of the volume containing the point

Inner and Outer Oriented Surfaces

$$\int_{\partial a} \mathbf{H} \cdot d\mathbf{s} = \int_a \mathbf{J} \cdot d\mathbf{a}$$

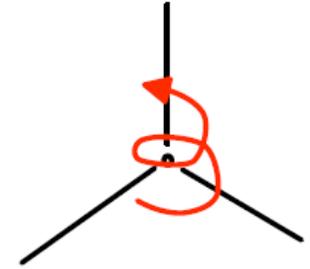
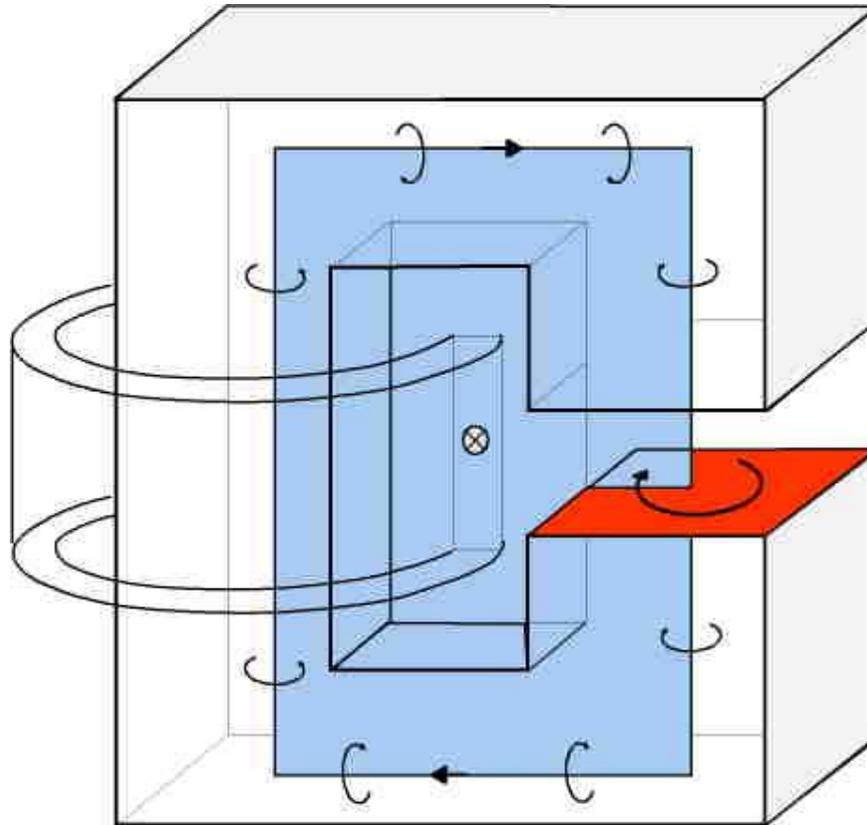


Outer oriented
by the current –
from inside out

$$\Phi(a) = \int_a \mathbf{B} \cdot d\mathbf{a}$$

Inner and Outer Oriented Surfaces

$$\int_{\partial a} \mathbf{H} \cdot d\mathbf{s} = \int_a \mathbf{J} \cdot d\mathbf{a}$$

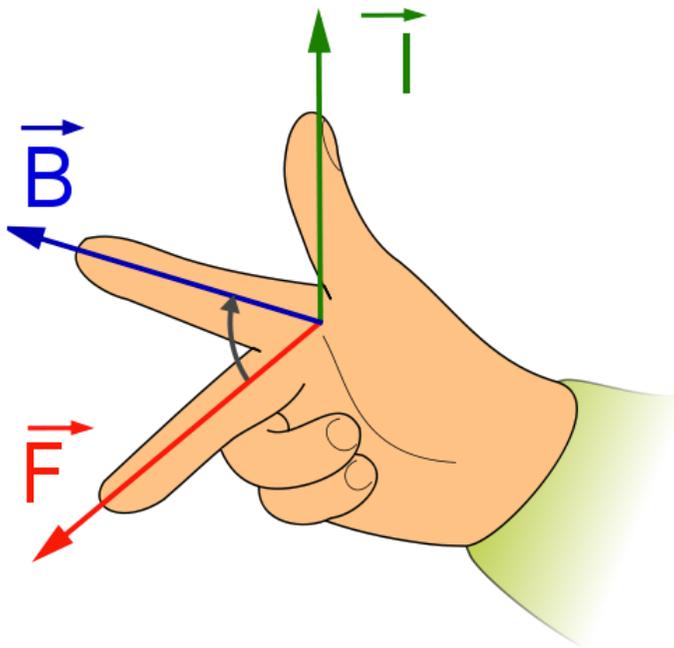


Inner oriented
because flux is a
measure for the
voltage that can be
generated on the
rim

Embedding into oriented ambient
space (Origin, coordinates)

$$\Phi(a) = \int_a \mathbf{B} \cdot d\mathbf{a}$$

The Right-Hand Rule or “Magnetic Discussion”



Bruno Touschek (1921-1978)

Mathematical Foundations of Magnet Design

Maxwell Equations

Integral Form

Local Form

Global Form

Laplace's Equation

The Curl-Curl Equation

Harmonic Fields

Green's Functions

Weak-Forms

Kichhoff's Theorem

1D Calculation of NC Magnets

Field Quality in Accelerator Magnets

The Field of Line-currents Coil-Dominated Magnets

FEM

BEM

DEM



Different Incarnations of Maxwell's Equations

$$\int_{\partial\mathcal{A}} \mathbf{H} \cdot d\mathbf{r} = \int_{\mathcal{A}} \mathbf{J} \cdot d\mathbf{a} + \frac{d}{dt} \int_{\mathcal{A}} \mathbf{D} \cdot d\mathbf{a},$$

$$\int_{\partial\mathcal{A}} \mathbf{E} \cdot d\mathbf{r} = -\frac{d}{dt} \int_{\mathcal{A}} \mathbf{B} \cdot d\mathbf{a},$$

$$\int_{\partial\mathcal{V}} \mathbf{B} \cdot d\mathbf{a} = 0,$$

$$\int_{\partial\mathcal{V}} \mathbf{D} \cdot d\mathbf{a} = \int_{\mathcal{V}} \rho dV.$$

$$V_m(\partial\mathcal{A}) = I(\mathcal{A}) + \frac{d}{dt} \Psi(\mathcal{A}),$$

$$U(\partial\mathcal{A}) = -\frac{d}{dt} \Phi(\mathcal{A}),$$

$$\Phi(\partial\mathcal{V}) = 0,$$

$$\Psi(\partial\mathcal{V}) = Q(\mathcal{V}).$$

$$\text{curl } \mathbf{H} = \mathbf{J} + \frac{\partial}{\partial t} \mathbf{D},$$

$$\text{curl } \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{B},$$

$$\text{div } \mathbf{B} = 0,$$

$$\text{div } \mathbf{D} = \rho.$$

Maxwell's Equations in Global Form

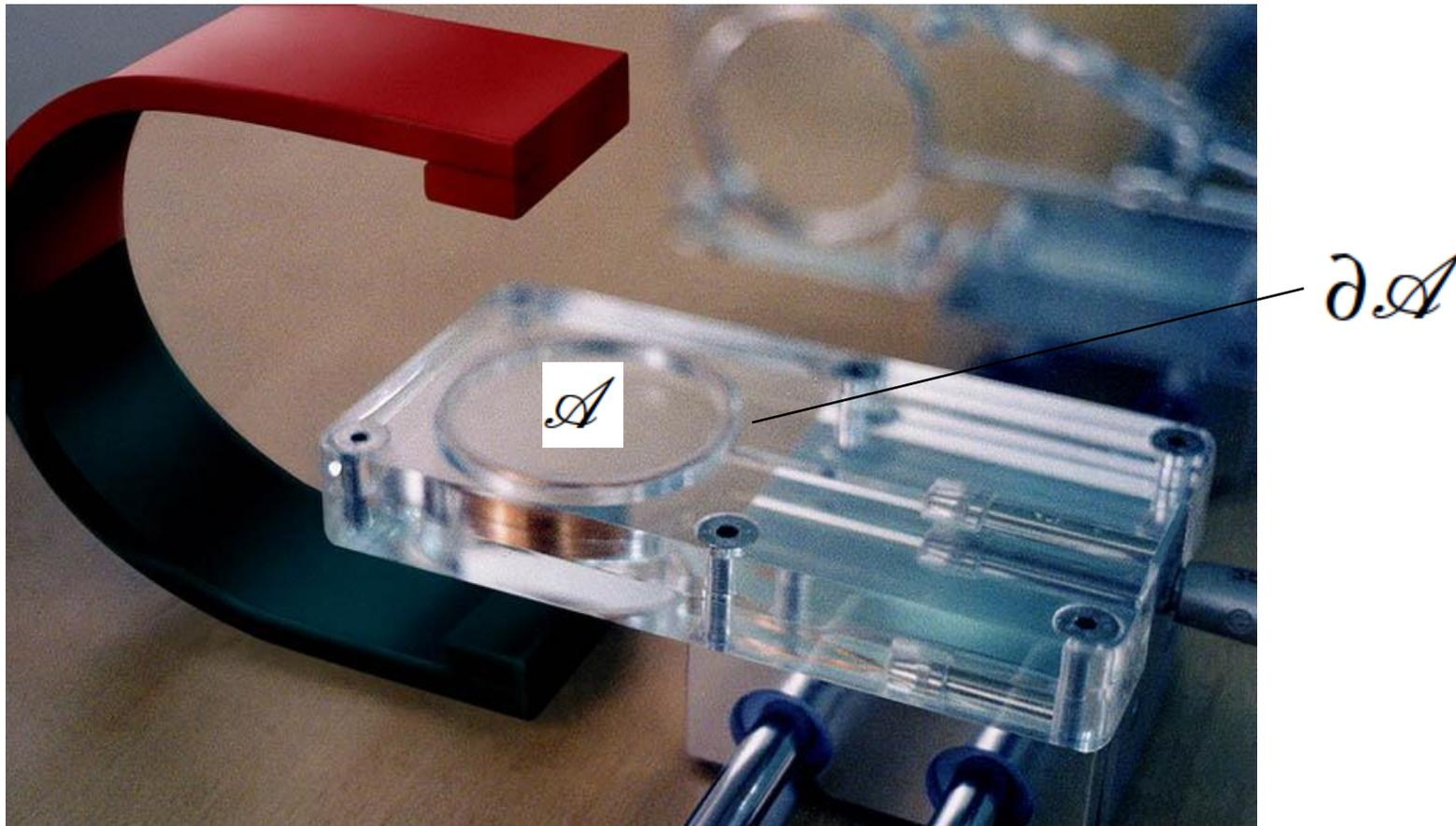
Ampere $V_m(\partial a) = I(a) + \frac{d}{dt}\psi(a)$

Faraday $U(\partial a) = -\frac{d}{dt}\Phi(a)$

Flux conservation $\Phi(\partial V) = 0$

Gauss $\psi(\partial V) = Q(V)$

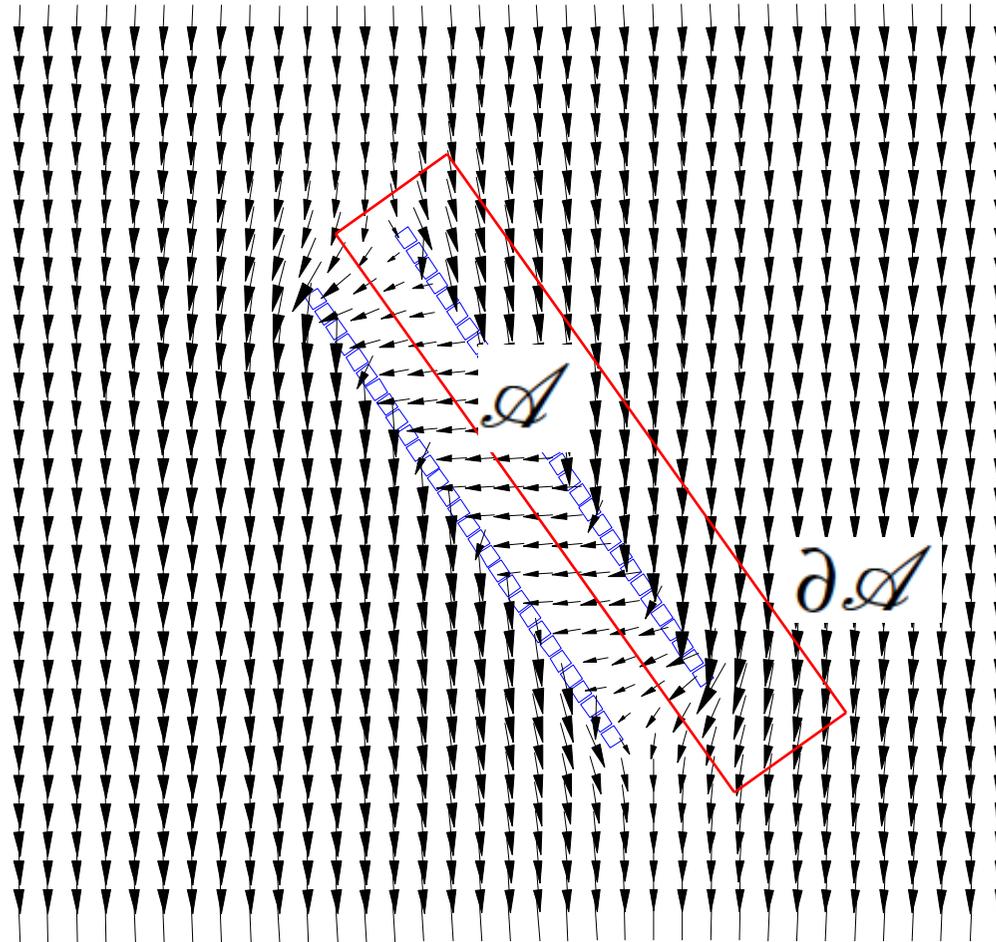
Faraday's Law (Inner Oriented Surface, Voltage along its Rim)



$$U(\partial\mathcal{A}) = -\frac{d}{dt}\Phi(\mathcal{A})$$

The potential to induce a voltage

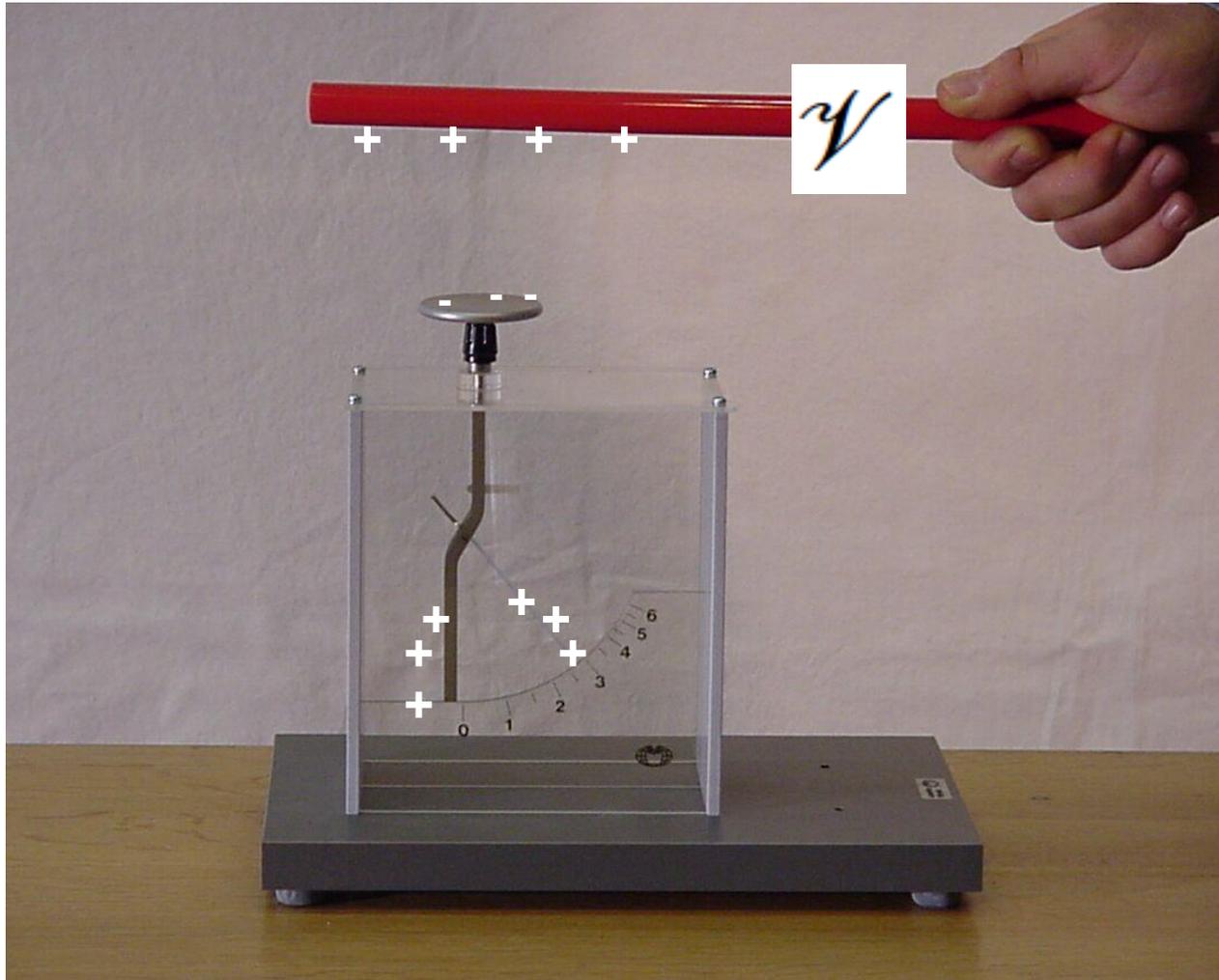
Ampere's Law (Outer Oriented Surface; Current crossing)



$$V_m(\partial \mathcal{A}) = I(\mathcal{A}).$$

The current needed to cancel the longitudinal field component

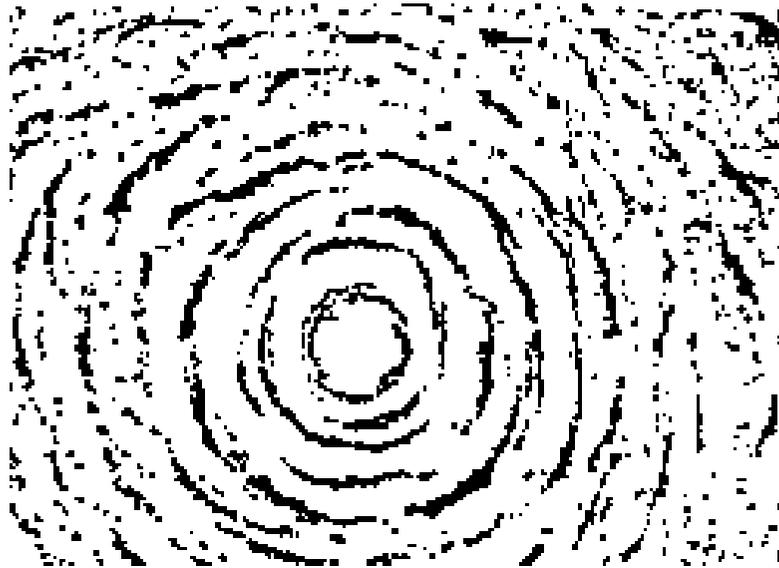
Gauss Law (Outer Oriented Volume; Electric Charge that can be influenced)



$$\Psi(\partial\mathcal{V}) = Q(\mathcal{V})$$

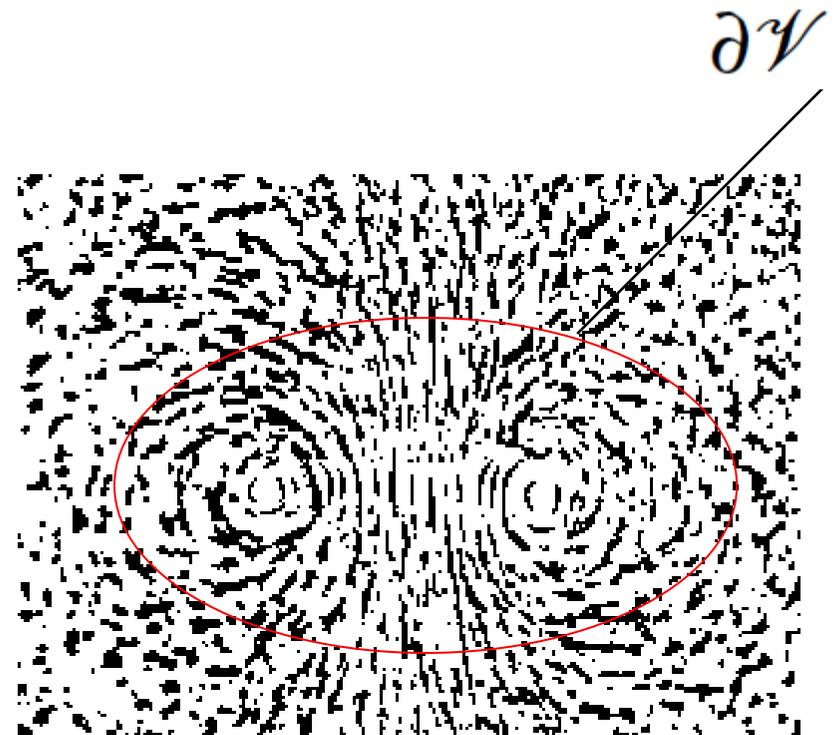
The capacity to induce charge

Magnetic Flux Conservation Law (Inner Oriented Volume)

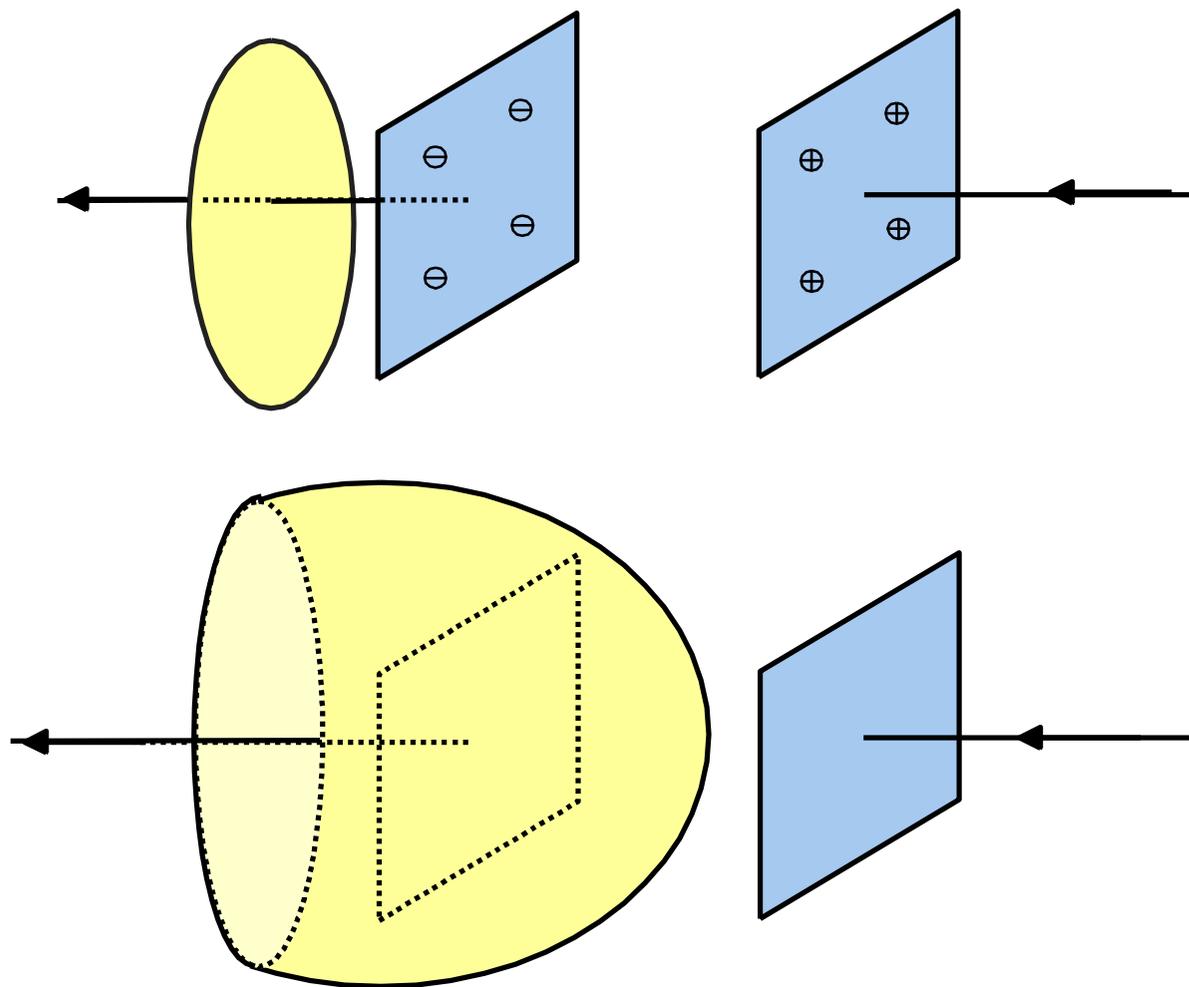


$$\Phi(\partial\mathcal{V}) = 0$$

Conservation of flux



Maxwell's Extension



Ampere

$$V_m(\partial\mathcal{A}) = I(\mathcal{A}) + \frac{d}{dt}\Psi(\mathcal{A})$$

Rate of change of charge

Electromagnetic Fields

Global quantity	SI unit	Relation	SI unit	Field
MMF	1 A	$V_m(\mathcal{L}) = \int_{\mathcal{L}} \mathbf{H} \cdot d\mathbf{r}$	1 A m^{-1}	Magnetic field
Electric voltage	1 V	$U(\mathcal{L}) = \int_{\mathcal{L}} \mathbf{E} \cdot d\mathbf{r}$	1 V m^{-1}	Electric field
Magnetic flux	1 V s	$\Phi(\mathcal{A}) = \int_{\mathcal{A}} \mathbf{B} \cdot d\mathbf{a}$	1 V s m^{-2}	Magnetic flux density
Electric flux	1 A s	$\Psi(\mathcal{A}) = \int_{\mathcal{A}} \mathbf{D} \cdot d\mathbf{a}$	1 A s m^{-2}	Electric flux density
Electric current	1 A	$I(\mathcal{A}) = \int_{\mathcal{A}} \mathbf{J} \cdot d\mathbf{a}$	1 A m^{-2}	Electric current density
Electric charge	1 A s	$Q(\mathcal{V}) = \int_{\mathcal{V}} \rho \cdot dV$	1 A s m^{-3}	Electric charge density

Maxwell's Equations in Integral Form

$$\int_{\partial\mathcal{A}} \mathbf{H} \cdot d\mathbf{r} = \int_{\mathcal{A}} \mathbf{J} \cdot d\mathbf{a} + \frac{d}{dt} \int_{\mathcal{A}} \mathbf{D} \cdot d\mathbf{a},$$

$$\int_{\partial\mathcal{A}} \mathbf{E} \cdot d\mathbf{r} = -\frac{d}{dt} \int_{\mathcal{A}} \mathbf{B} \cdot d\mathbf{a},$$

$$\int_{\partial\mathcal{V}} \mathbf{B} \cdot d\mathbf{a} = 0,$$

$$\int_{\partial\mathcal{V}} \mathbf{D} \cdot d\mathbf{a} = \int_{\mathcal{V}} \rho dV.$$

$$V_m(\partial\mathcal{A}) = I(\mathcal{A}) + \frac{d}{dt} \Psi(\mathcal{A}),$$

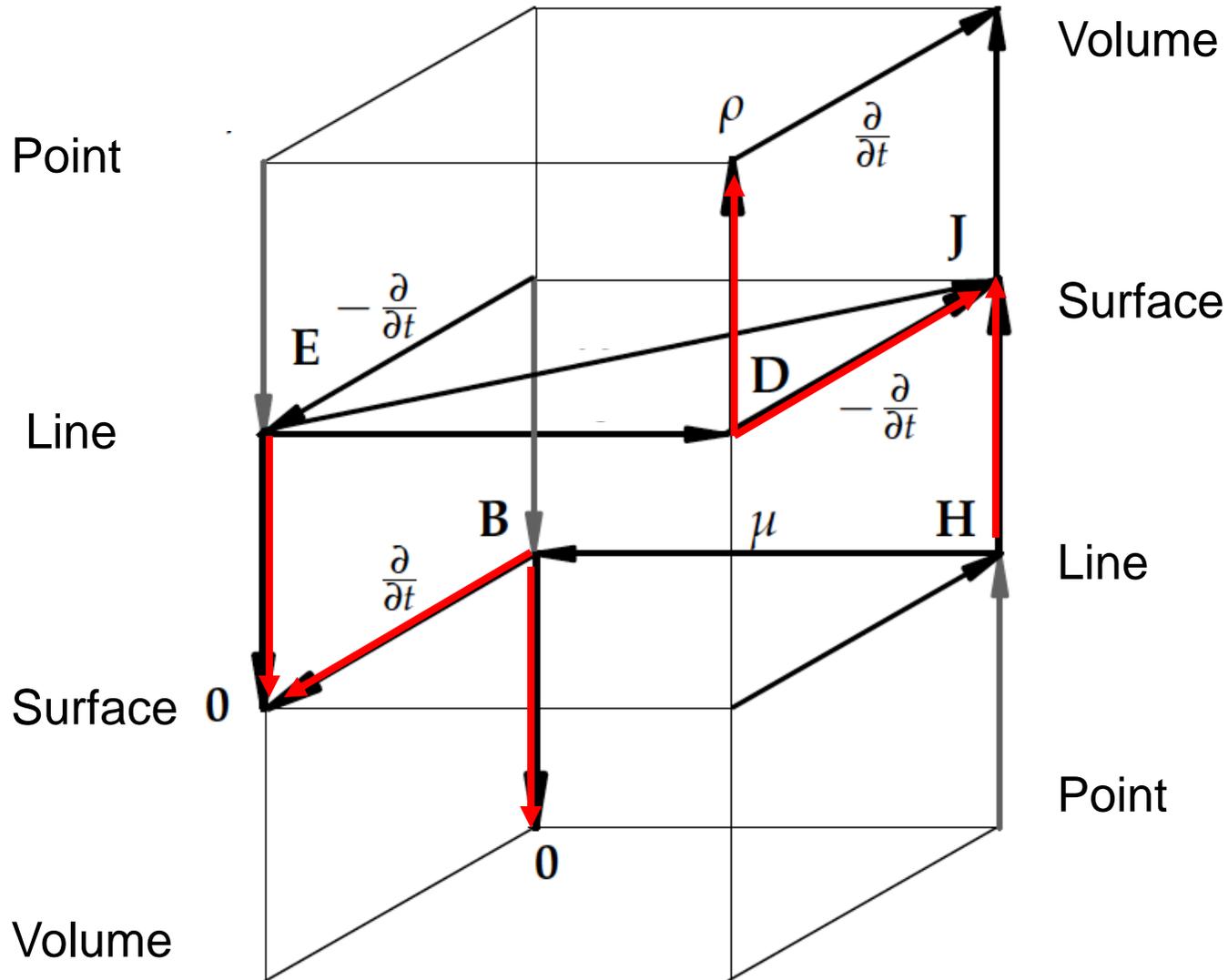
$$U(\partial\mathcal{A}) = -\frac{d}{dt} \Phi(\mathcal{A}),$$

$$\Phi(\partial\mathcal{V}) = 0,$$

$$\Psi(\partial\mathcal{V}) = Q(\mathcal{V}).$$

8 Equations
16 Unknowns

Maxwell's House



Constitutive Equations

$$\mathbf{B} = \mu \mathbf{H}, \quad \mathbf{D} = \varepsilon \mathbf{E}, \quad \mathbf{J} = \varkappa \mathbf{E},$$

Permeability: $[\mu] = 1 \text{ V s A}^{-1} \text{ m}^{-1} = 1 \text{ H m}^{-1},$

Permittivity: $[\varepsilon] = 1 \text{ A s V}^{-1} \text{ m}^{-1},$

Conductivity: $[\varkappa] = 1 \text{ A V}^{-1} \text{ m}^{-1} = 1 \Omega^{-1} \text{ m}^{-1}.$

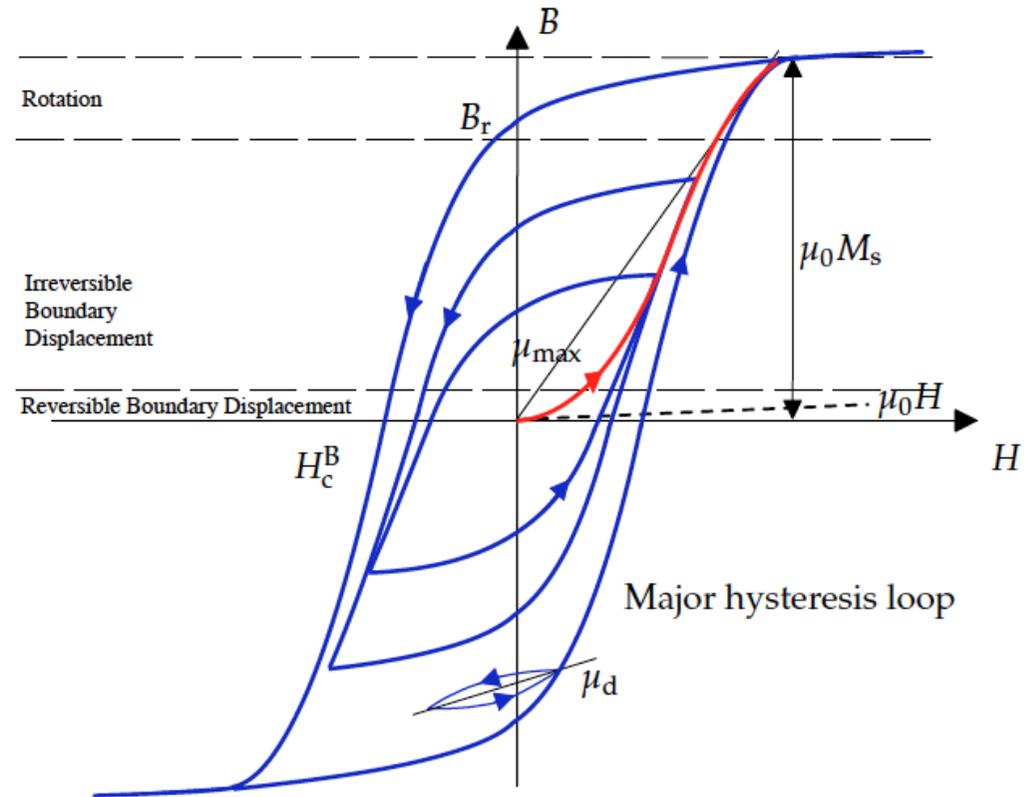
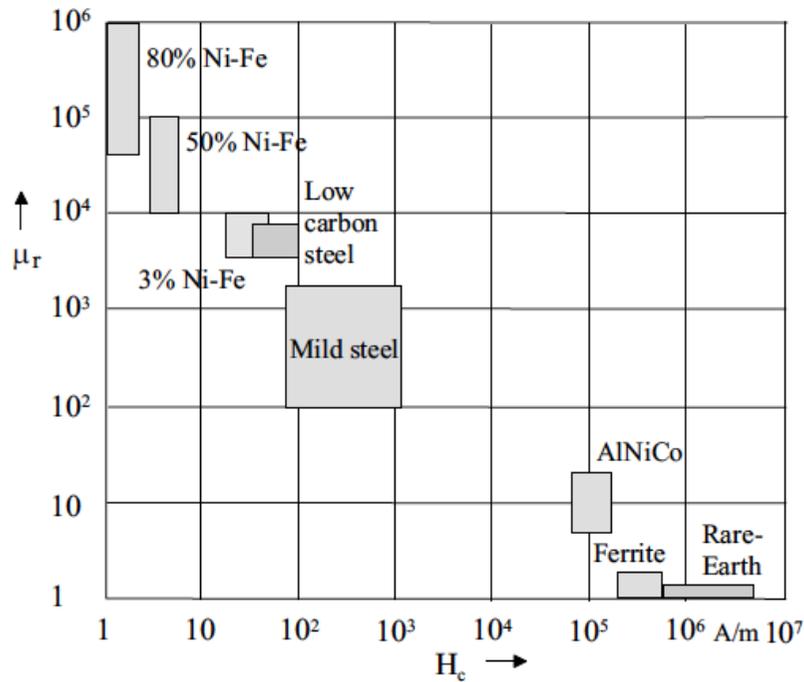
Linear (field independent, homogeneous (position independent),
lossless, isotropic (direction independent, stationary)

$$\mu = \mu_r \mu_0, \quad \varepsilon = \varepsilon_r \varepsilon_0,$$

$$\mu_0 = 4\pi \times 10^{-7} \text{ H m}^{-1},$$

$$\varepsilon_0 = 8.8542 \dots \times 10^{-12} \text{ F m}^{-1},$$

Hysteresis



$$\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{P}_m(\mathbf{H}) = \mu_0 (\mathbf{H} + \mathbf{M}(\mathbf{H})),$$

B(H) Measurement

$H = NI/2\pi r$ within the specimen, which is

$$\bar{H} = \frac{NI}{2\pi(r_2 - r_1)} \ln\left(\frac{r_2}{r_1}\right).$$

$$U = \frac{d}{dt}\Phi = \frac{d}{dt}\bar{B}a,$$

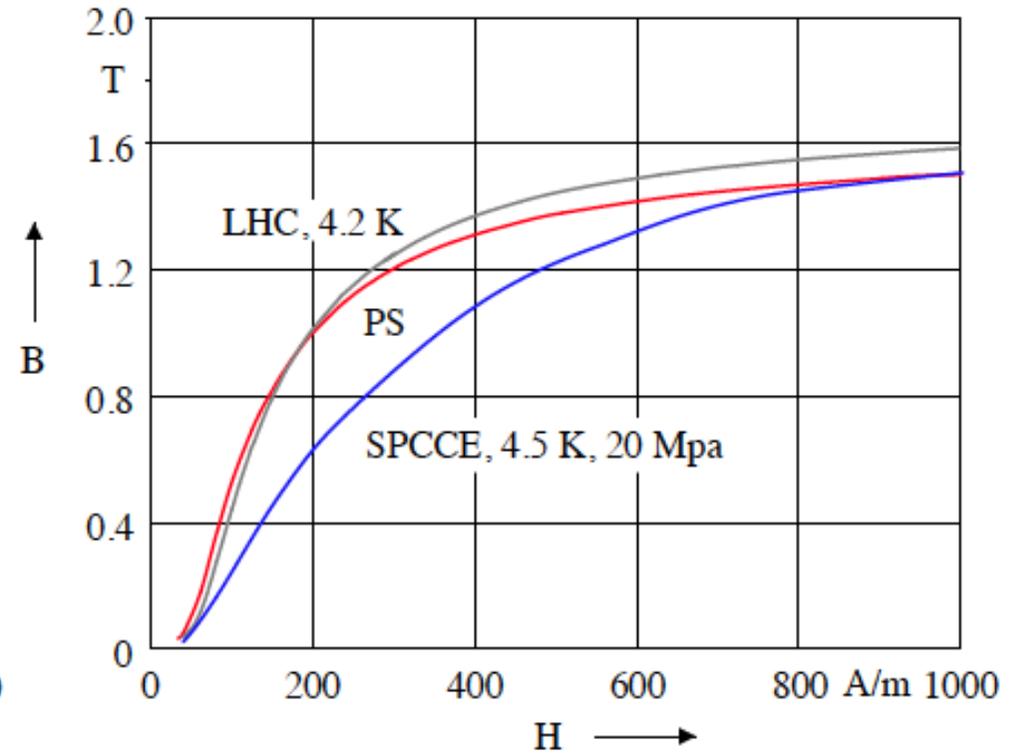
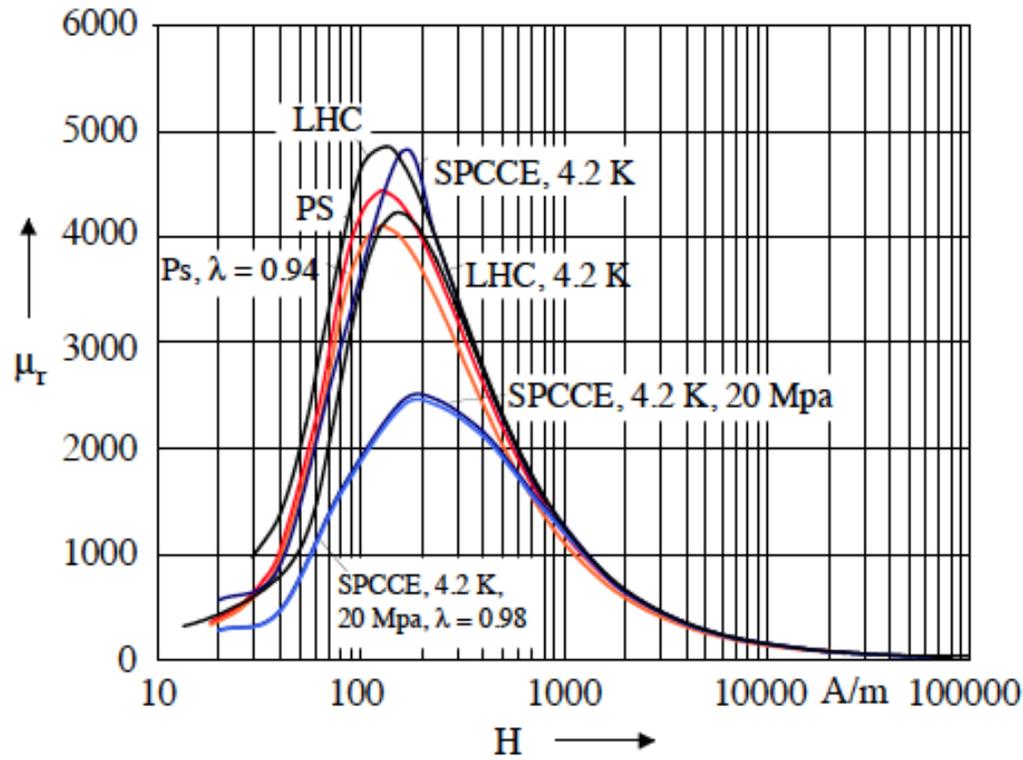
$$\int U dt = \bar{B} a \qquad \mu = \bar{B} / \bar{H}$$



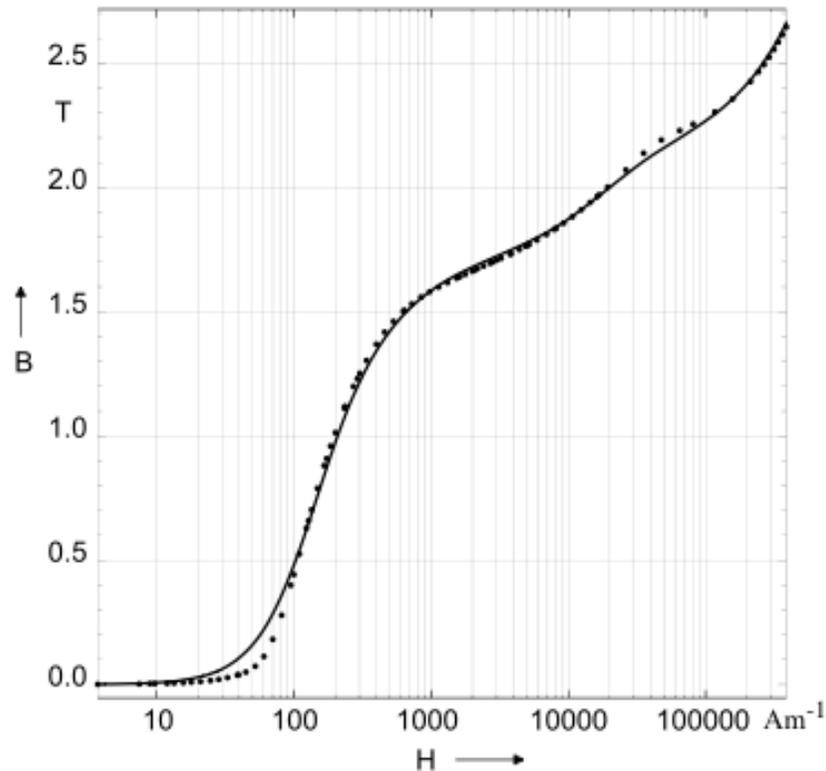
Always check conditions of measurements

Temperature T	Stress	Coercive field H_c^B	Remanence B_r	max μ_r
K	MPa	$A\ m^{-1}$	T	
300	0	68.4	1.07	5900
77	0	79.6	1.12	5600
4.2	0	85.1	1.06	4800
4.2	20	110	0.67	2460

Nonlinear Iron Magnetization



Nonlinear Iron Magnetization



$$L\left(\frac{H}{a}\right) := \coth\left(\frac{H}{a}\right) - \left(\frac{a}{H}\right)$$

$$M(H) = M_a L\left(\frac{H}{a}\right) + M_b \tanh\left(\frac{|H|}{b}\right) L\left(\frac{H}{b}\right)$$

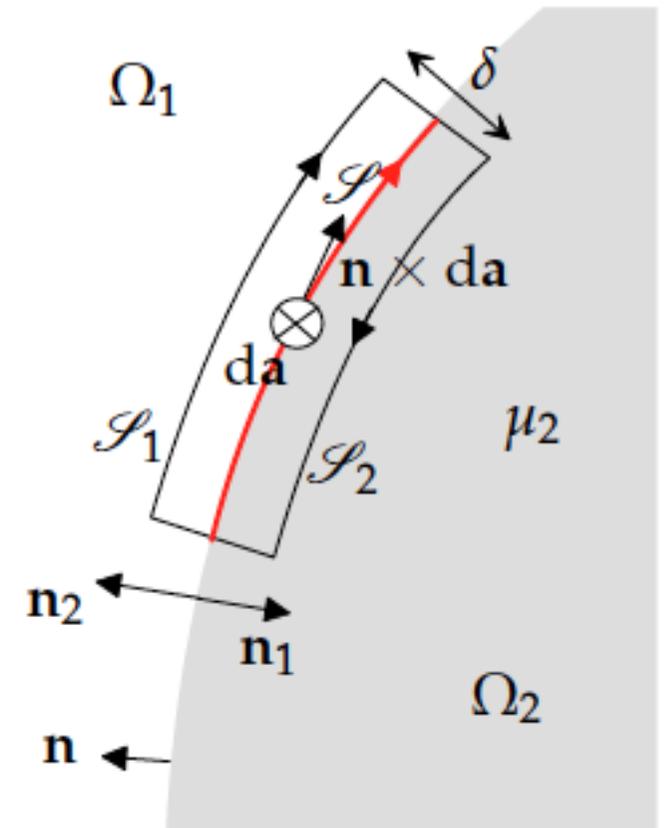
**Always fulfilling the smoothness requirements for M(B)
and Newton-Raphson iterative solvers**

Continuity Conditions (1)

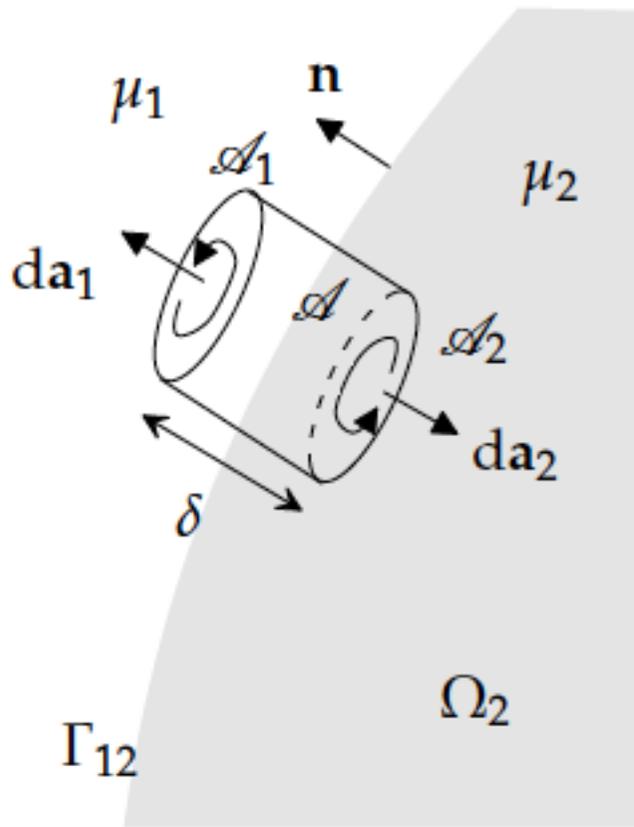
$$\int_{\mathcal{S}_2} \mathbf{H}_2 \cdot d\mathbf{r} + \int_{\mathcal{S}_1} \mathbf{H}_1 \cdot d\mathbf{r} = \int_{\mathcal{S}} (\mathbf{H}_1 - \mathbf{H}_2) \cdot d\mathbf{r} = - \int_{\mathcal{S}} (\mathbf{n} \times \boldsymbol{\alpha}) \cdot d\mathbf{r},$$

where the surface normal vector \mathbf{n} points from Ω_2 to Ω_1

$$H_{t1} = H_{t2} \quad \equiv \quad \mathbf{n} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{0}$$



Continuity Conditions (2)



$$\int_{\partial V} \mathbf{B} \cdot d\mathbf{a} = 0 \quad \delta \rightarrow 0$$

$$\begin{aligned} \int_a \sigma_{\text{mag}} da &= \int_a \mathbf{B}_1 \cdot d\mathbf{a}_1 + \mathbf{B}_2 \cdot d\mathbf{a}_2 \\ &= \int_a (\mathbf{B}_1 - \mathbf{B}_2) \cdot \mathbf{n}_1 da \end{aligned}$$

Holds for any surface a if

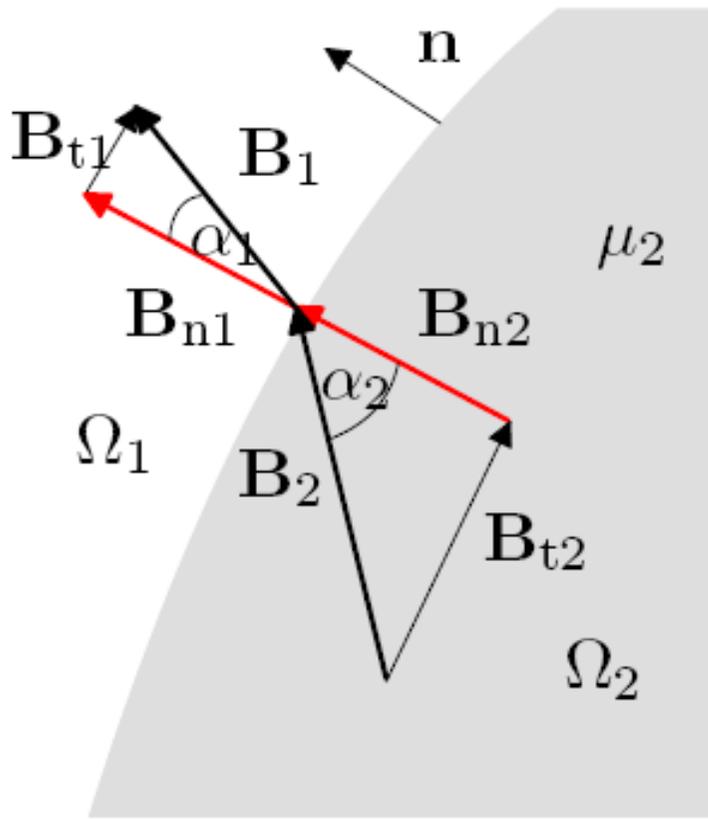
$$\begin{aligned} \sigma_{\text{mag}} &= (\mathbf{B}_1 - \mathbf{B}_2) \cdot \mathbf{n} \\ &= [\mathbf{B} \cdot \mathbf{n}]_{12} \end{aligned}$$

$$B_{n1} = B_{n2} \quad \equiv \quad (\mathbf{B}_1 - \mathbf{B}_2) \cdot \mathbf{n} = 0 \quad \equiv \quad [\mathbf{B} \cdot \mathbf{n}]_{12} = 0$$

Continuity Conditions (3)

No surface currents:

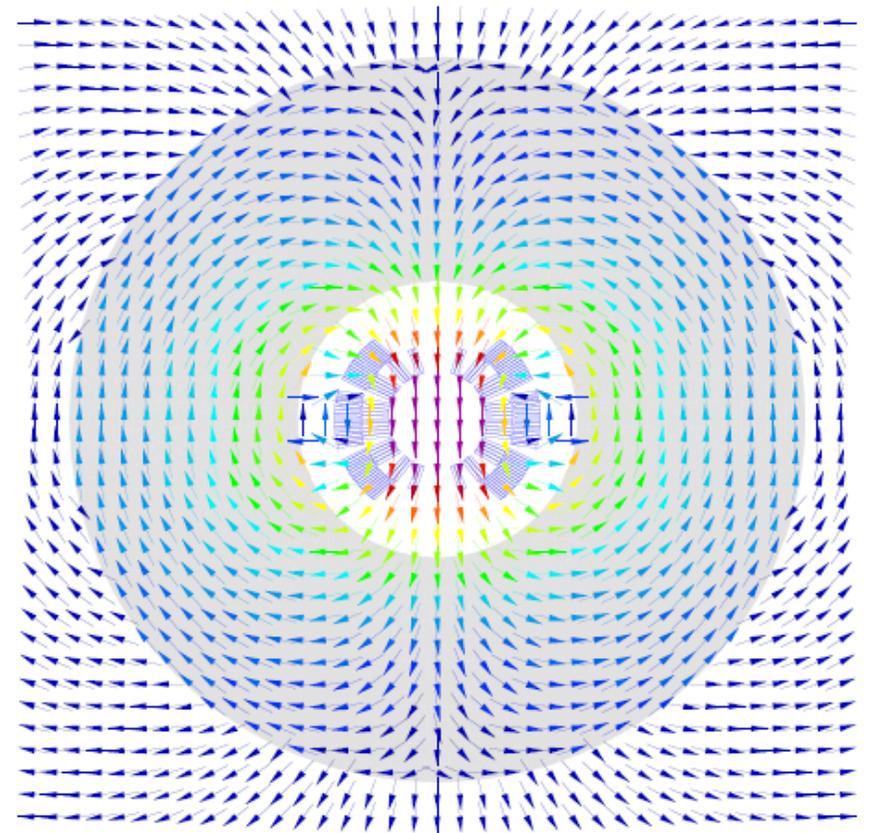
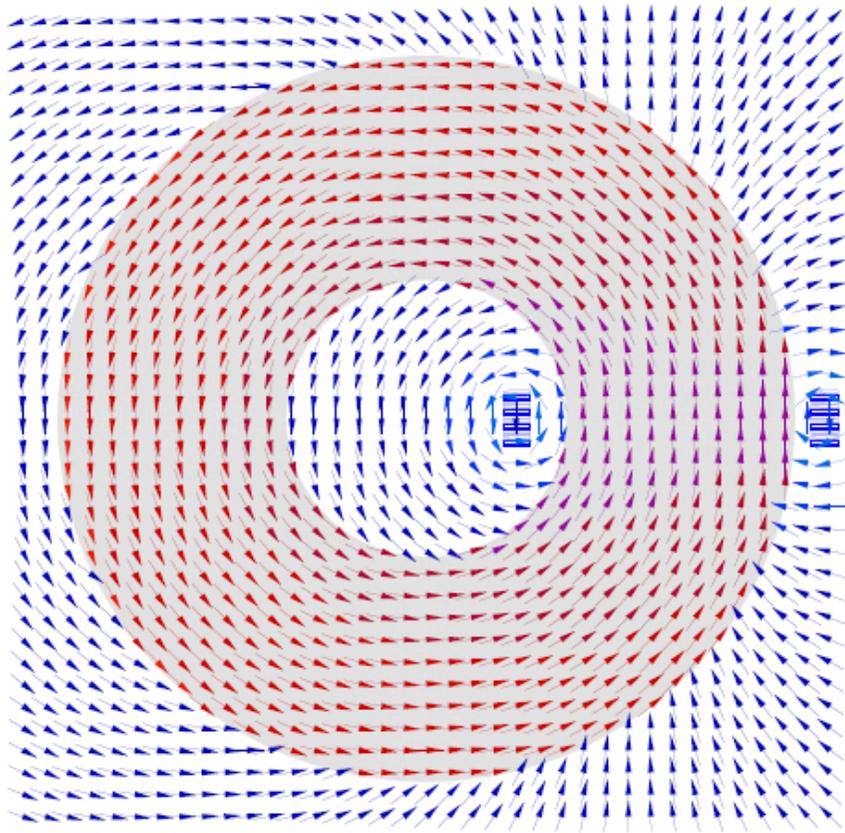
$$\frac{\tan \alpha_1}{\tan \alpha_2} = \frac{\frac{B_{t1}}{B_{n1}}}{\frac{B_{t2}}{B_{n2}}} = \frac{\mu_1 \frac{H_{t1}}{B_{n1}}}{\mu_2 \frac{H_{t2}}{B_{n2}}} = \frac{\mu_1 H_{t1}}{\mu_2 H_{t2}} = \frac{\mu_1}{\mu_2}$$



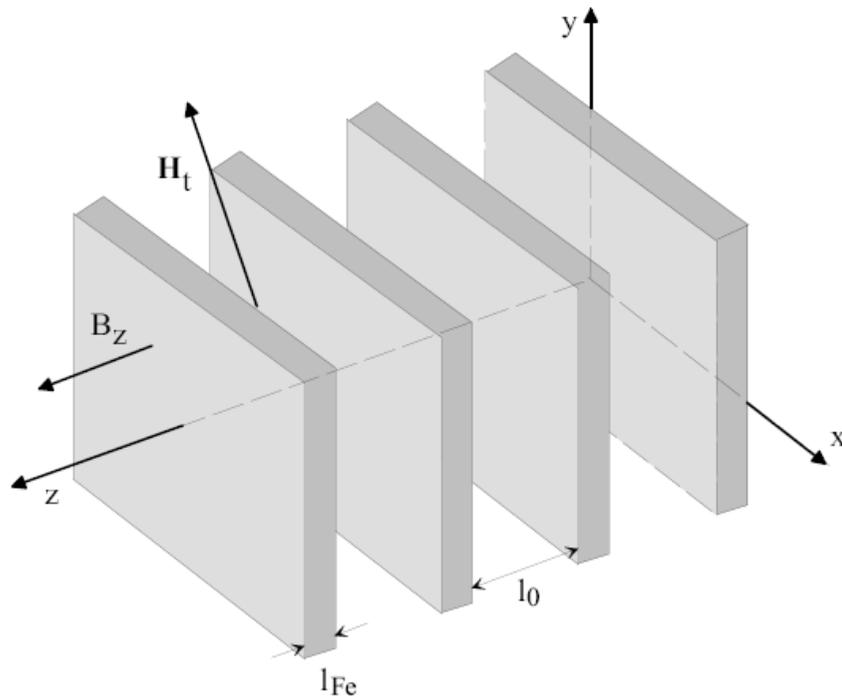
$$\mu_2 \gg \mu_1$$

$$\alpha_1 \approx 0, \quad \text{or} \quad \alpha_2 \approx \pi/2,$$

Continuity at Iron Boundaries



Stacking Factor for Yoke Laminations



$$\mathbf{H}_t^0 = \mathbf{H}_t^{Fe} = \bar{\mathbf{H}}_t$$

$$\bar{\mathbf{B}}_t = \frac{1}{l_{Fe} + l_0} (l_{Fe} \mu \bar{\mathbf{H}}_t + l_0 \mu_0 \bar{\mathbf{H}}_t)$$

$$B_z^0 = B_z^{Fe} = \bar{B}_z$$

$$\bar{H}_z = \frac{1}{l_{Fe} + l_0} \left(l_{Fe} \frac{\bar{B}_z}{\mu} + l_0 \frac{\bar{B}_z}{\mu_0} \right)$$

$$\lambda = \frac{l_{Fe}}{l_{Fe} + l_0}$$

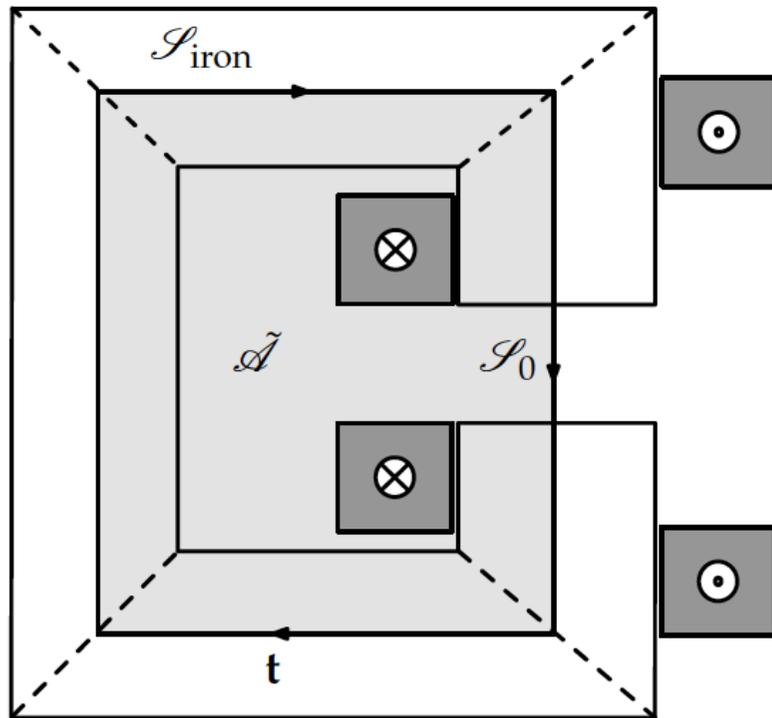
$$\bar{\mu}_t = \lambda \mu + (1 - \lambda) \mu_0$$

$$\bar{\mu}_z = \left(\frac{\lambda}{\mu} + \frac{1 - \lambda}{\mu_0} \right)^{-1}$$

Cash-Back 1

One-dimensional (approximate) calculation of
iron dominated magnets

Main Field in Normal Conducting Dipole



$$\int_{\partial \vec{a}} \mathbf{H} \cdot d\mathbf{r} = \int_{\vec{a}} \mathbf{J} \cdot \mathbf{n} da,$$

$$\int_{\mathcal{S}_{\text{iron}}} \mathbf{H} \cdot d\mathbf{r} + \int_{\mathcal{S}_0} \mathbf{H} \cdot d\mathbf{r} = \int_{\vec{a}_{\text{coil}}} \mathbf{J} \cdot \mathbf{n} da,$$

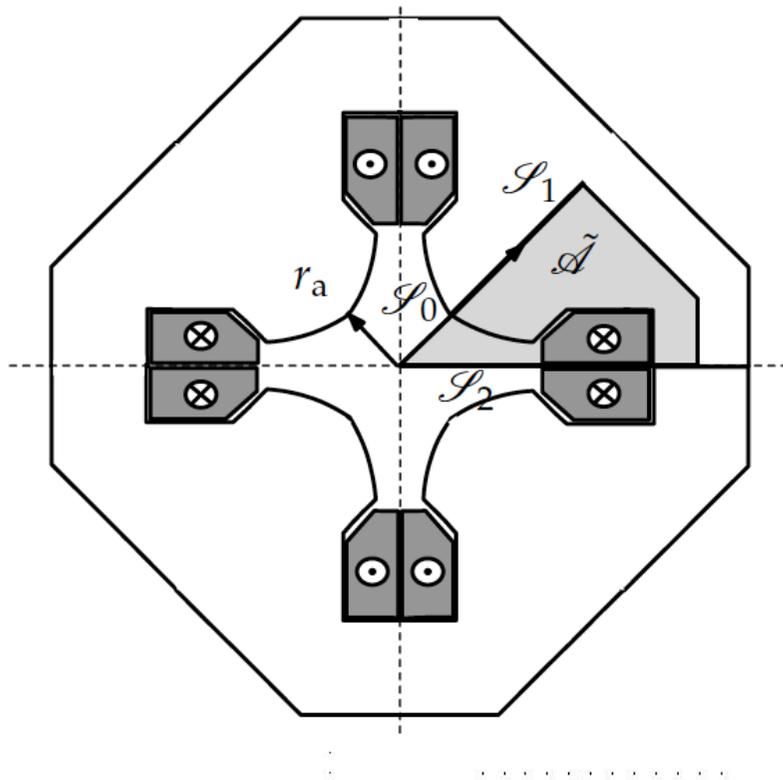
$$H_{\text{iron}} s_{\text{iron}} + H_0 s_0 = N I,$$

$$\frac{1}{\mu_0 \mu_r} B_{\text{iron}} s_{\text{iron}} + \frac{1}{\mu_0} B_0 s_0 = N I,$$

$$B_0 = \frac{\mu_0 N I}{s_0}.$$

Gradient in Normal Conducting Quadrupole

$$\int_{\partial \mathcal{A}} \mathbf{H} \cdot d\mathbf{r} = \int_{\mathcal{I}_0} \mathbf{H}_0 \cdot d\mathbf{r} + \int_{\mathcal{I}_1} \mathbf{H}_1 \cdot d\mathbf{r} + \int_{\mathcal{I}_2} \mathbf{H}_2 \cdot d\mathbf{r} = NI.$$



$$B_x = gy, \quad B_y = gx;$$

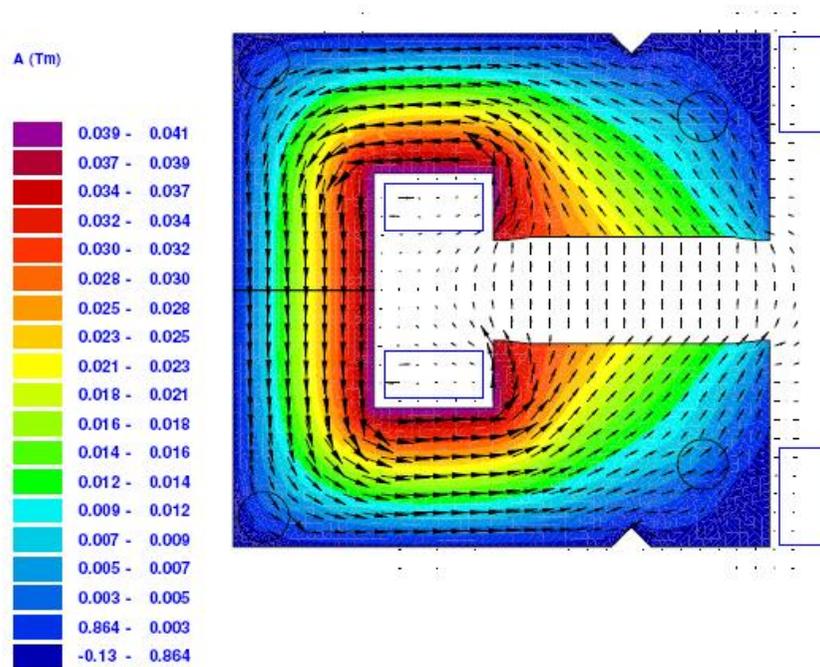
$$H = \frac{g}{\mu_0} \sqrt{x^2 + y^2} = \frac{g}{\mu_0} r.$$

$$\int_0^{r_a} H dr = \frac{g}{\mu_0} \int_0^{r_a} r dr = \frac{g}{\mu_0} \frac{r_a^2}{2} = NI,$$

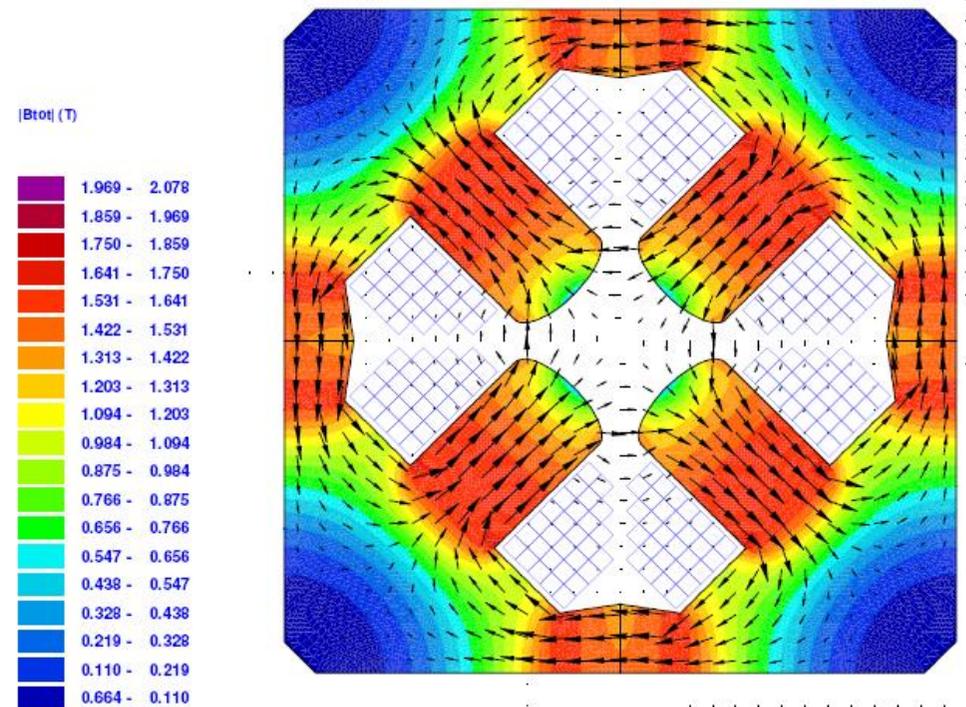
or

$$g = \frac{2\mu_0 NI}{r_a^2}.$$

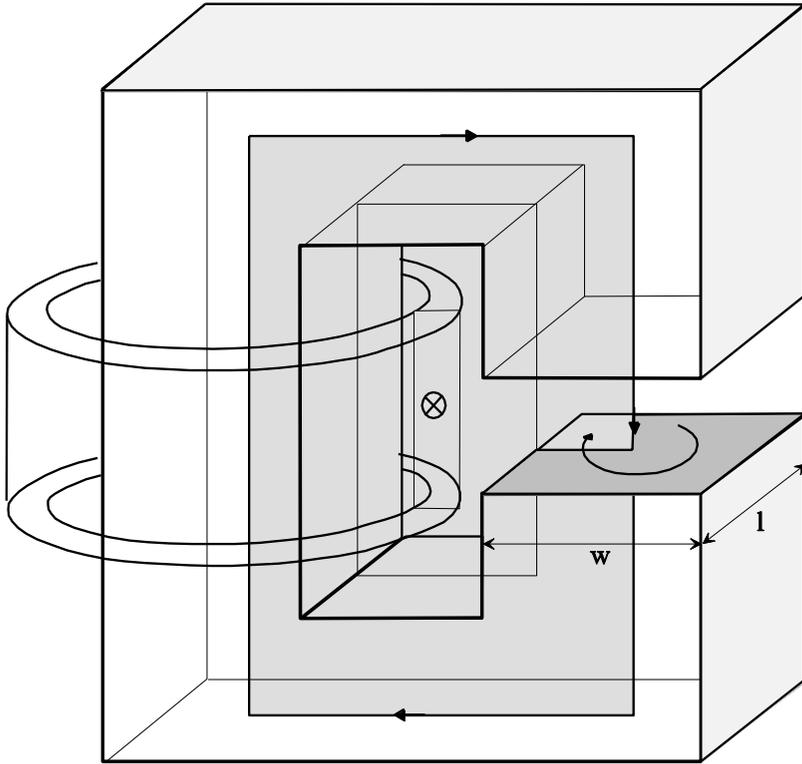
LEP Dipole and Quadrupole



Interrupt: Exercise c-core dipole; comparison to ROXIE simulations



Dipole with Varying Cut-Section



$$\sum_{i=0}^n H_i s_i = N I$$

$$H_i = \frac{B_i}{\mu_i} = \frac{\Phi}{a_i \mu_i}$$

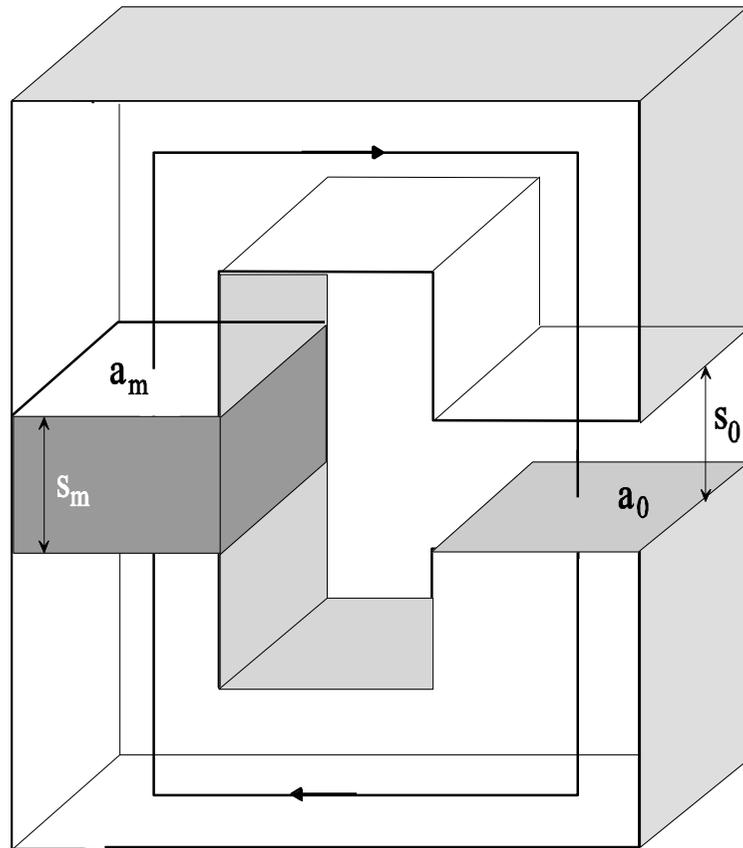
$$\Phi \sum_{i=0}^n \frac{s_i}{a_i \mu_i} = N I = V_m$$

$$\text{Ohm's law: } I \sum_{i=0}^n \frac{s_i}{a_i \kappa_i} = U$$

$$N I = \Phi \sum_{i=0}^n \frac{s_i}{a_i \mu_i} = \Phi \left(\frac{s_0}{a_0 \mu_0} + \sum_{i=1}^n \frac{s_i}{a_i \mu_i} \right)$$

Conclusion: Magnet with large air-gap is stabilized against variations in permeability

Permanent Magnet Excitation



$$H_0 s_0 + H_m s_m = 0$$

$$B_m a_m = B_0 a_0 = \mu_0 H_0 a_0$$

$$H_0 s_0 = -H_m s_m,$$

$$\frac{1}{\mu_0} B_m \frac{a_m}{a_0} s_0 = -H_m s_m,$$

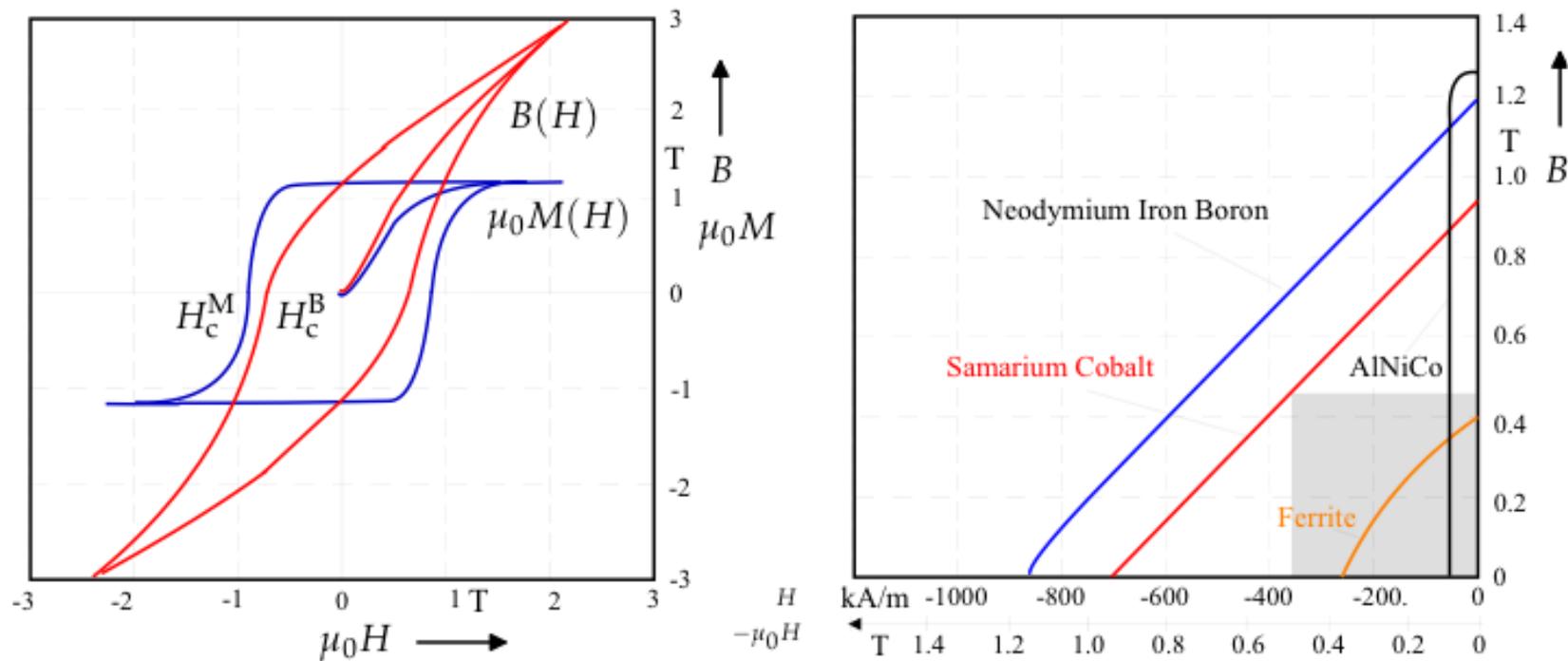
$$B_m = -\mu_0 \frac{s_m a_0}{s_0 a_m} H_m,$$

$$\frac{B_m}{\mu_0 H_m} = -\frac{s_m a_0}{s_0 a_m} = P$$

$$B_m a_m s_m = \mu_0 H_0 a_0 \frac{-H_0 s_0}{H_m}$$

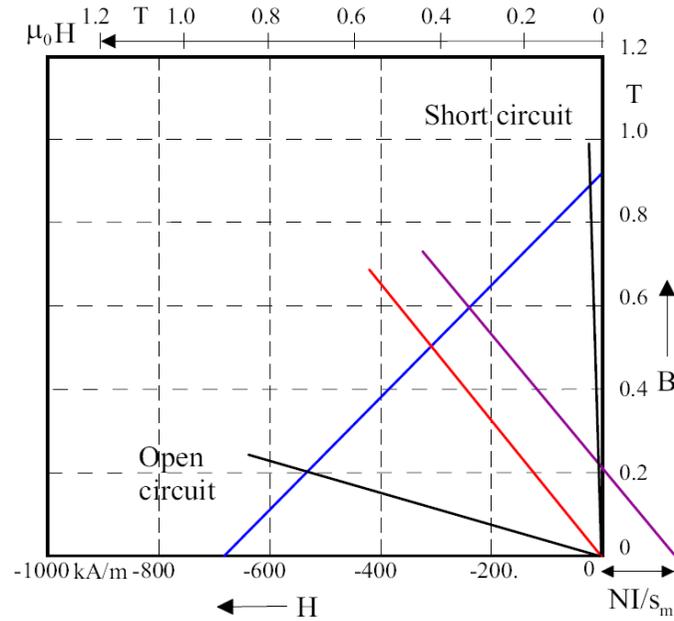
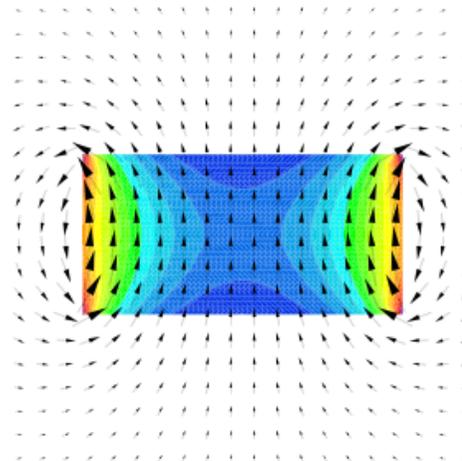
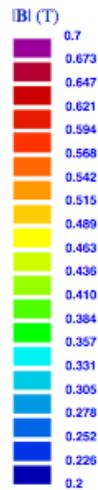
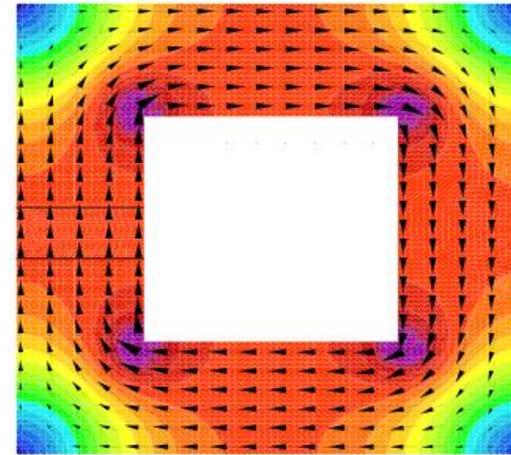
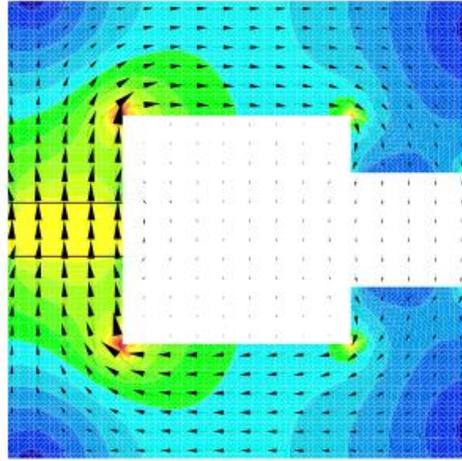
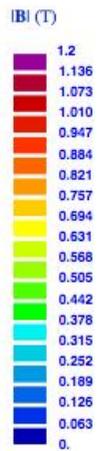
$$H_0 = \sqrt{\frac{(a_m s_m)(-B_m H_m)}{\mu_0 (a_0 s_0)}} = \sqrt{\frac{V_m (-B_m H_m)}{\mu_0 V_0}}$$

M(H) and B(H) for Permanent Magnets



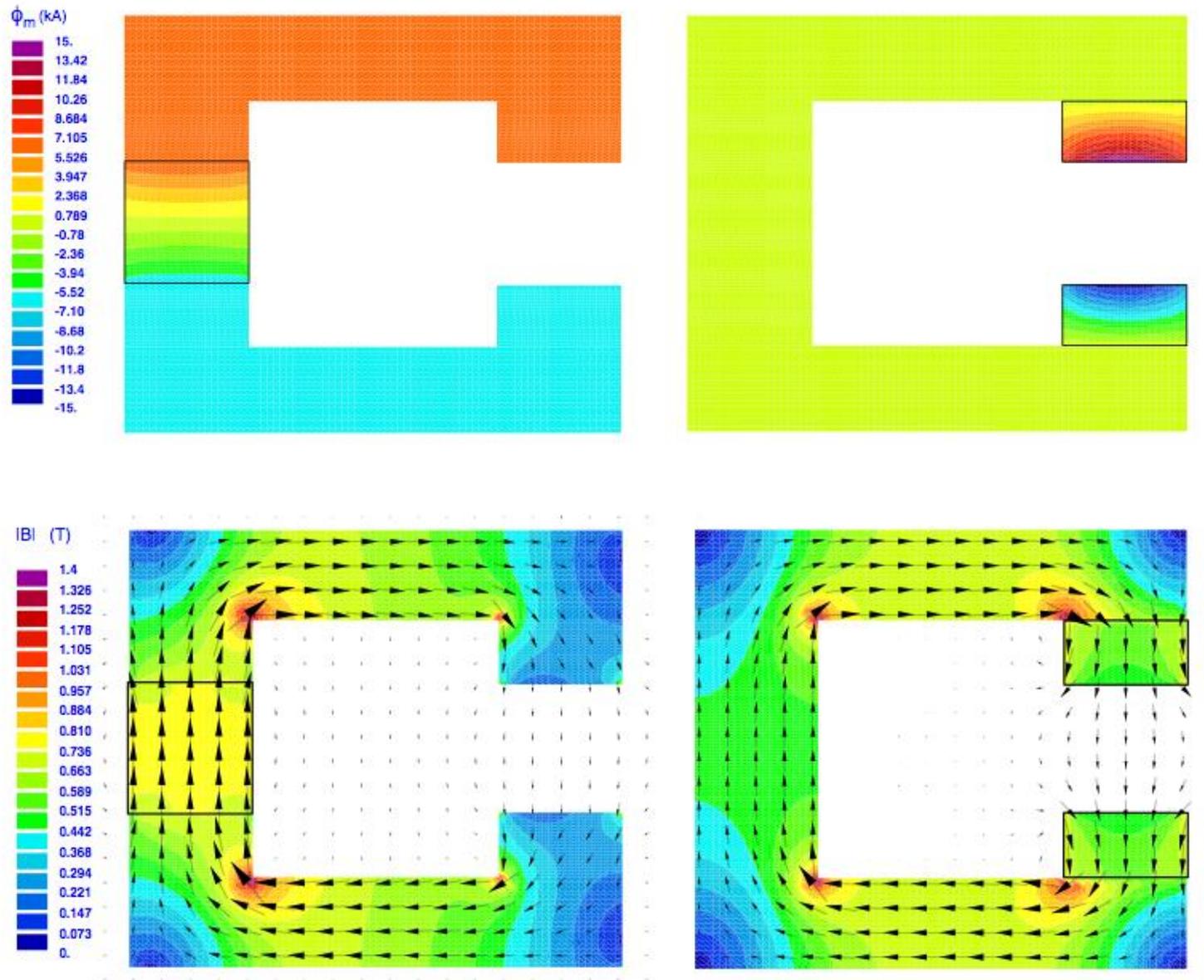
	B_r	$\mu_0 H_c^M$	H_c^B	$(BH)_{max}$	$(BH)_{max}^{id}$	T_c
	T	T	T	kJ m^{-3}	kJ m^{-3}	$^{\circ}\text{C}$
AlNiCo	1.3	0.06	0.06	50	336	857
Ferrite	0.4	0.4	0.37	30	32	447
SmCo ₅	0.9	2.5	0.87	160	161	727
Sm ₂ Co ₁₇	1.1	1.3	0.97	220	241	827
NdFeB	1.3	1.5	1.25	320	336	313

Permanent Magnet Circuits



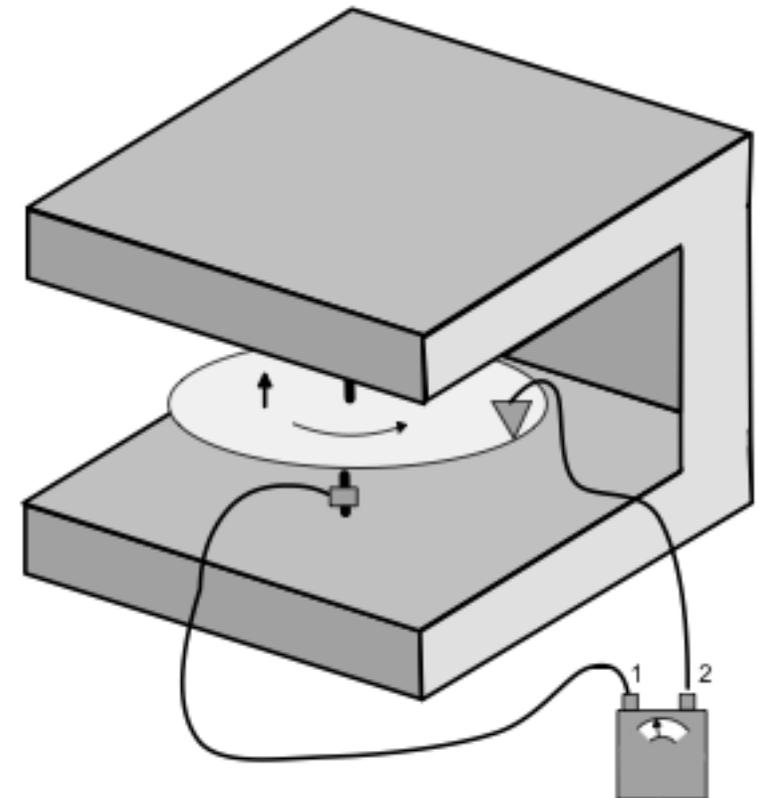
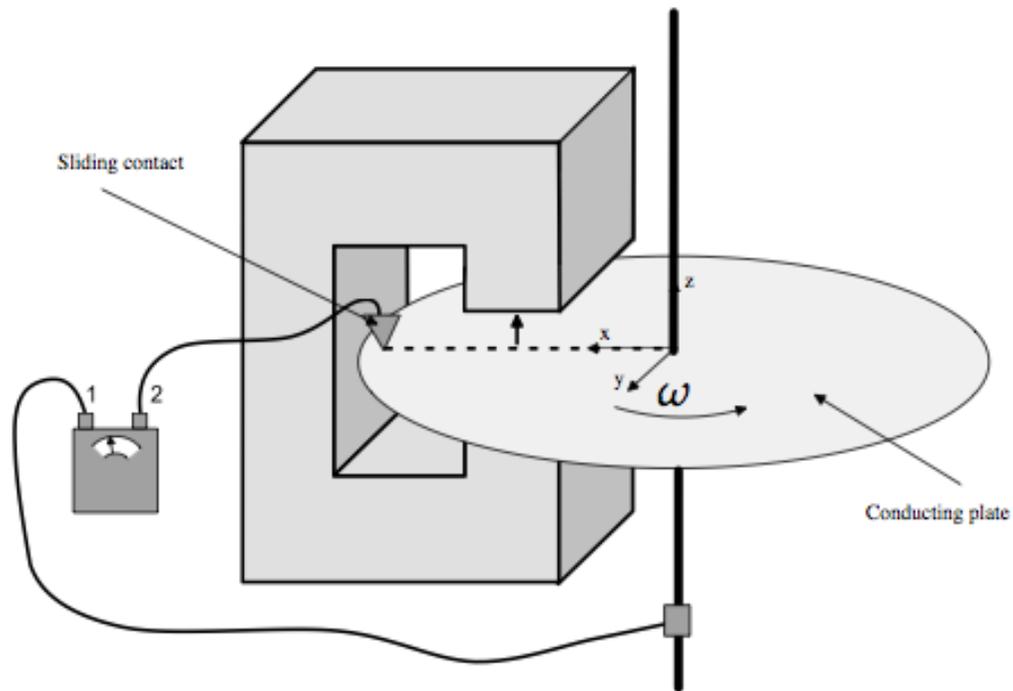
$$s = \frac{B_m}{H_m} = \mu_0 P = -\mu_0 \frac{s_m a_0}{s_0 a_m}$$

Optimal Position of Permanent Magnets



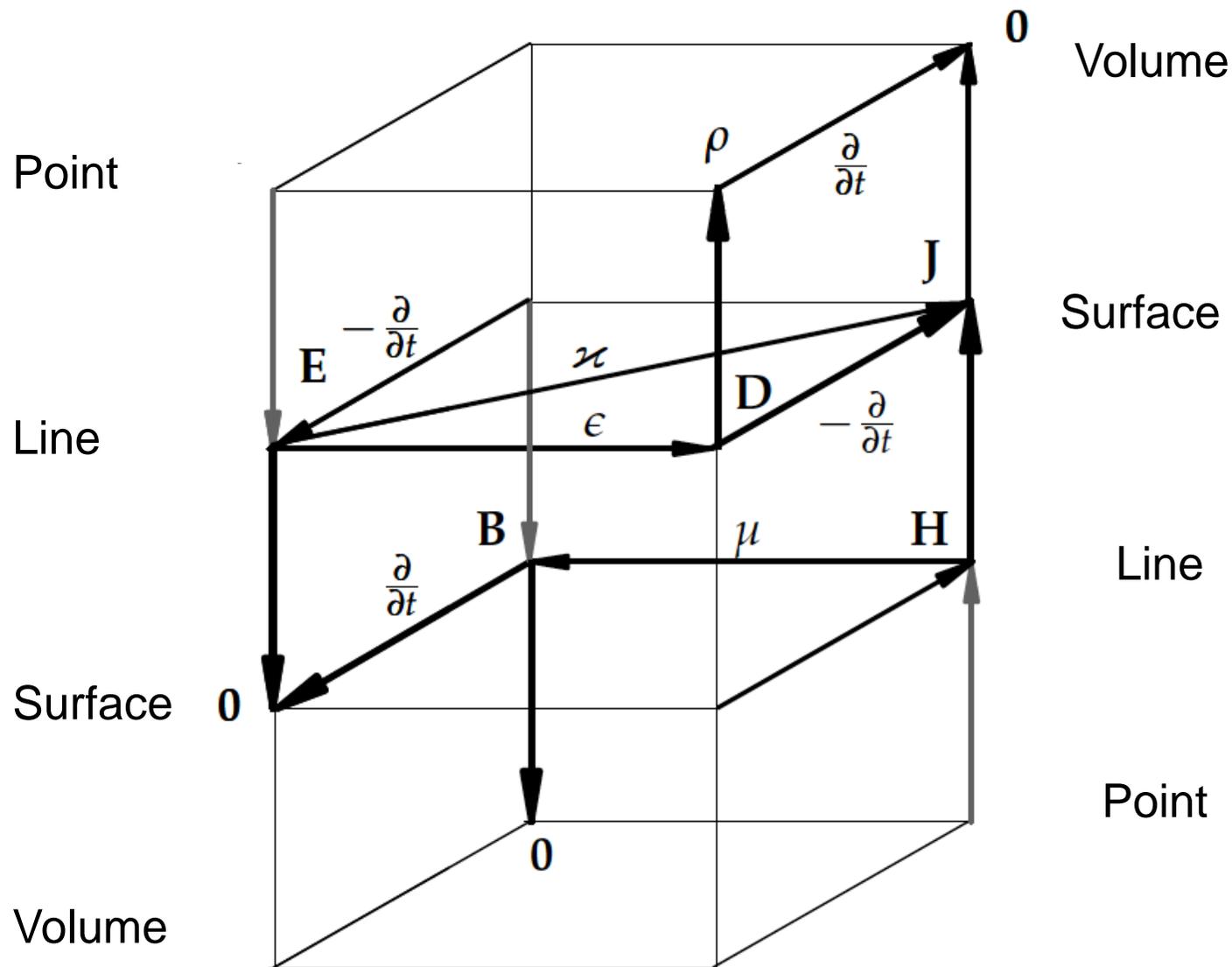
The Homopolar Generator

$$d\mathbf{F} = I \, d\mathbf{r} \times \mathbf{B}$$



Einstein: All physics is local

Maxwell's House Again



$$\text{grad } \phi := \frac{\partial \phi}{\partial x} \mathbf{e}_x + \frac{\partial \phi}{\partial y} \mathbf{e}_y + \frac{\partial \phi}{\partial z} \mathbf{e}_z$$

$$\text{curl } \mathbf{g} = \left(\frac{\partial g_z}{\partial y} - \frac{\partial g_y}{\partial z} \right) \mathbf{e}_x + \left(\frac{\partial g_x}{\partial z} - \frac{\partial g_z}{\partial x} \right) \mathbf{e}_y + \left(\frac{\partial g_y}{\partial x} - \frac{\partial g_x}{\partial y} \right) \mathbf{e}_z.$$

$$\text{div } \mathbf{g} = \frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \frac{\partial g_z}{\partial z}.$$

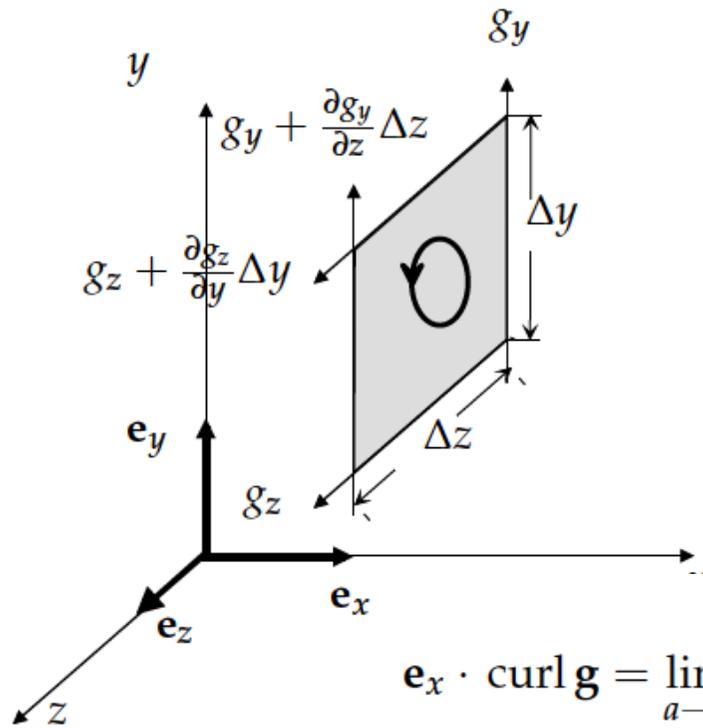
$$\int_{\mathcal{P}_1}^{\mathcal{P}_2} \mathbf{a} \cdot d\mathbf{r} = \int_{\mathcal{P}_1}^{\mathcal{P}_2} \text{grad } \phi \cdot d\mathbf{r} = \int_{\mathcal{P}_1}^{\mathcal{P}_2} d\phi = \phi(\mathcal{P}_2) - \phi(\mathcal{P}_1),$$

-

$$\mathbf{n} \cdot \text{curl } \mathbf{g} = \lim_{a \rightarrow 0} \frac{\int_{\partial \mathcal{A}} \mathbf{g} \cdot d\mathbf{r}}{a},$$

$$\text{div } \mathbf{g} = \lim_{V \rightarrow 0} \frac{\int_{\partial \mathcal{V}} \mathbf{g} \cdot d\mathbf{a}}{V},$$

Curl in Cartesian Coordinates



$$\mathbf{n} \cdot \text{curl } \mathbf{g} = \lim_{a \rightarrow 0} \frac{\int_{\partial \mathcal{A}} \mathbf{g} \cdot d\mathbf{r}}{a},$$

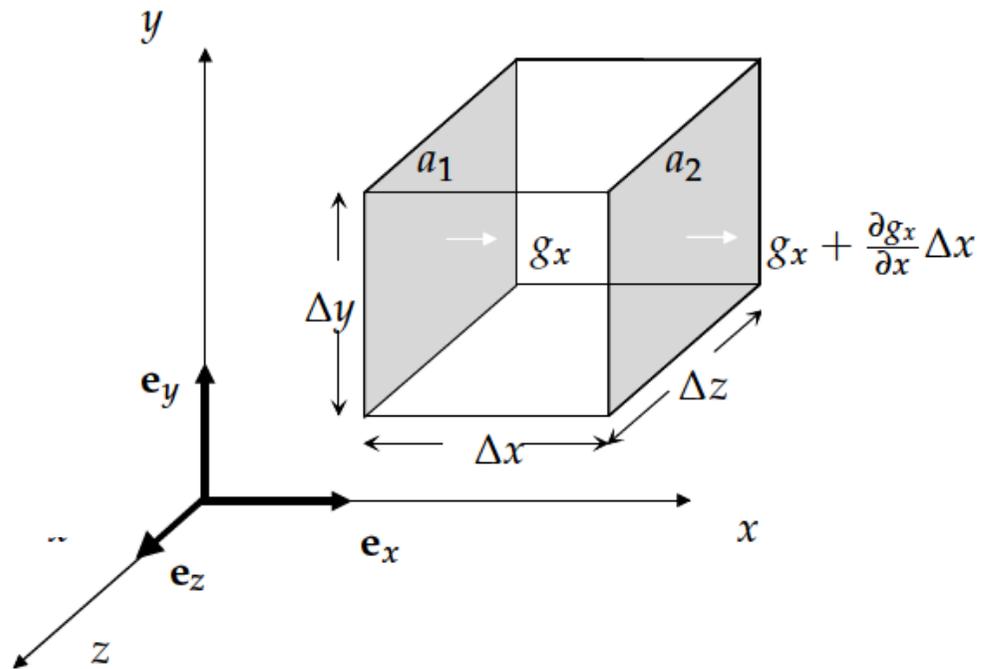
$$\mathbf{e}_x \cdot \text{curl } \mathbf{g} = \lim_{a \rightarrow 0} \frac{\int_{\partial \mathcal{A}} \mathbf{g} \cdot d\mathbf{r}}{a}$$

$$= \lim_{\Delta y, \Delta z \rightarrow 0} \frac{g_y \Delta y + \left(g_z + \frac{\partial g_z}{\partial y} \Delta y\right) \Delta z - \left(g_y + \frac{\partial g_y}{\partial z} \Delta z\right) \Delta y - g_z \Delta z}{\Delta y \Delta z}$$

$$= \frac{\partial g_z}{\partial y} - \frac{\partial g_y}{\partial z}.$$

$$\text{curl } \mathbf{g} = \left(\frac{\partial g_z}{\partial y} - \frac{\partial g_y}{\partial z}\right) \mathbf{e}_x + \left(\frac{\partial g_x}{\partial z} - \frac{\partial g_z}{\partial x}\right) \mathbf{e}_y + \left(\frac{\partial g_y}{\partial x} - \frac{\partial g_x}{\partial y}\right) \mathbf{e}_z.$$

Divergence in Cartesian Coordinates



$$\operatorname{div} \mathbf{g} = \lim_{V \rightarrow 0} \frac{\int_{\partial V} \mathbf{g} \cdot d\mathbf{a}}{V},$$

$$\lim_{V \rightarrow 0} \frac{\int_{a_1, a_2} g_x da}{V} = \lim_{\Delta x, \Delta y, \Delta z \rightarrow 0} \frac{-g_x \Delta y \Delta z + \left(g_x + \frac{\partial g_x}{\partial x} \Delta x\right) \Delta y \Delta z}{\Delta x \Delta y \Delta z} = \frac{\partial g_x}{\partial x}.$$

$$\operatorname{div} \mathbf{g} = \frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \frac{\partial g_z}{\partial z}.$$

$$\begin{aligned}\text{curl grad } \phi &= \text{curl} \left[\frac{1}{h_1} \frac{\partial \phi}{\partial u^1} \mathbf{e}_{u^1} + \frac{1}{h_2} \frac{\partial \phi}{\partial u^2} \mathbf{e}_{u^2} + \frac{1}{h_3} \frac{\partial \phi}{\partial u^3} \mathbf{e}_{u^3} \right] \\ &= \frac{1}{h_2 h_3} \left(\frac{\partial^2 \phi}{\partial u^2 \partial u^3} - \frac{\partial^2 \phi}{\partial u^3 \partial u^2} \right) \mathbf{e}_{u^1} \\ &\quad + \frac{1}{h_3 h_1} \left(\frac{\partial^2 \phi}{\partial u^3 \partial u^1} - \frac{\partial^2 \phi}{\partial u^1 \partial u^3} \right) \mathbf{e}_{u^2} \\ &\quad + \frac{1}{h_1 h_2} \left(\frac{\partial^2 \phi}{\partial u^1 \partial u^2} - \frac{\partial^2 \phi}{\partial u^2 \partial u^1} \right) \mathbf{e}_{u^3} = 0,\end{aligned}$$

Ugly and not even a universal proof

The Boundary Operator

The distance $d(\mathcal{P}, \Omega)$ of a point \mathcal{P} to a subset $\Omega \in A$ is defined by $\inf\{d(\mathcal{P}, \mathcal{Q})\}$ for all $\mathcal{Q} \in \Omega$. The interior of Ω is the set of all points $\mathcal{P} \in A$ such that $d(\mathcal{P}, A \setminus \Omega) > 0$. The *boundary* $\partial\Omega$ of Ω is the set of all points for which the distances $d(\mathcal{P}, \Omega) = 0$ and $d(\mathcal{P}, A \setminus \Omega) = 0$.

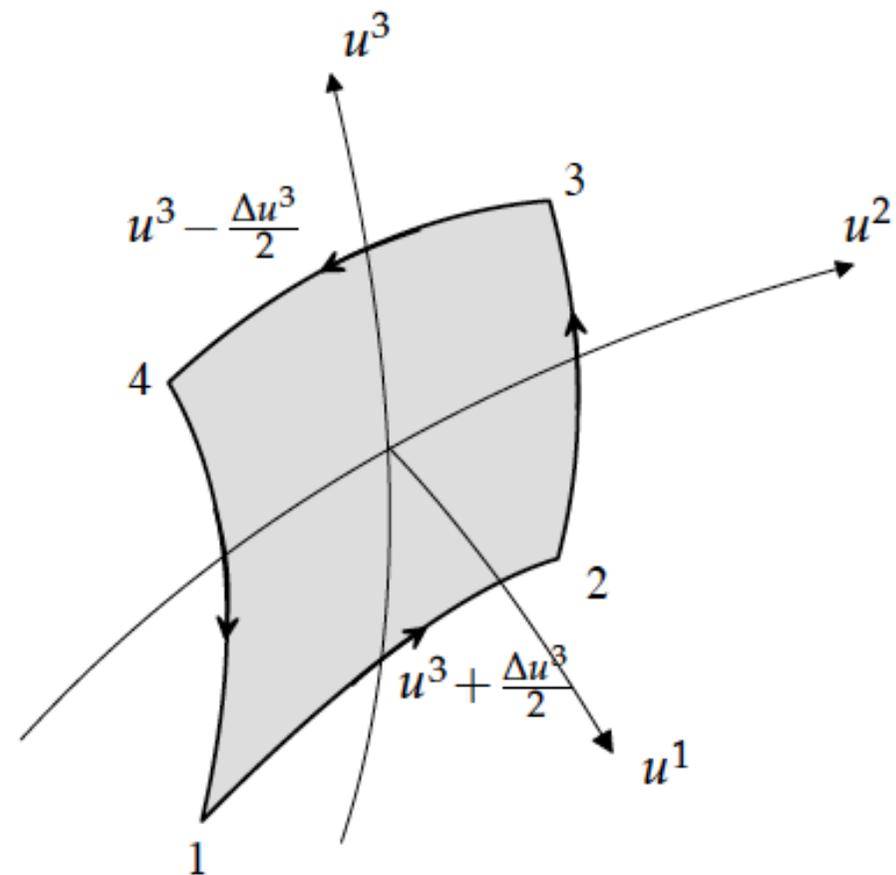
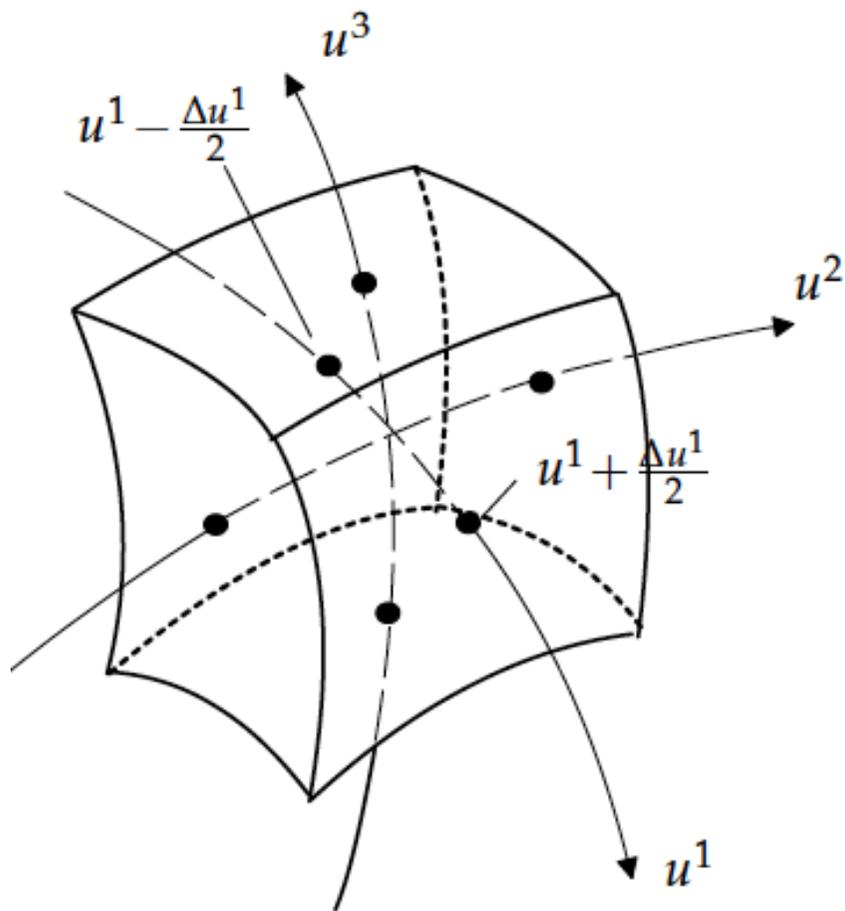
$$\partial(\partial\mathcal{V}) = \emptyset, \quad \partial(\partial\mathcal{A}) = \emptyset,$$

$$\int_{\mathcal{V}} \operatorname{div} \operatorname{curl} \mathbf{g} dV = \int_{\partial\mathcal{V}} \operatorname{curl} \mathbf{g} \cdot d\mathbf{a} = \int_{\partial(\partial\mathcal{V})} \mathbf{g} \cdot d\mathbf{r} = 0,$$

$$\int_{\mathcal{A}} \operatorname{curl} \operatorname{grad} \phi \cdot d\mathbf{a} = \int_{\partial\mathcal{A}} \operatorname{grad} \phi \cdot d\mathbf{r} = \phi|_{\partial(\partial\mathcal{A})} = 0,$$

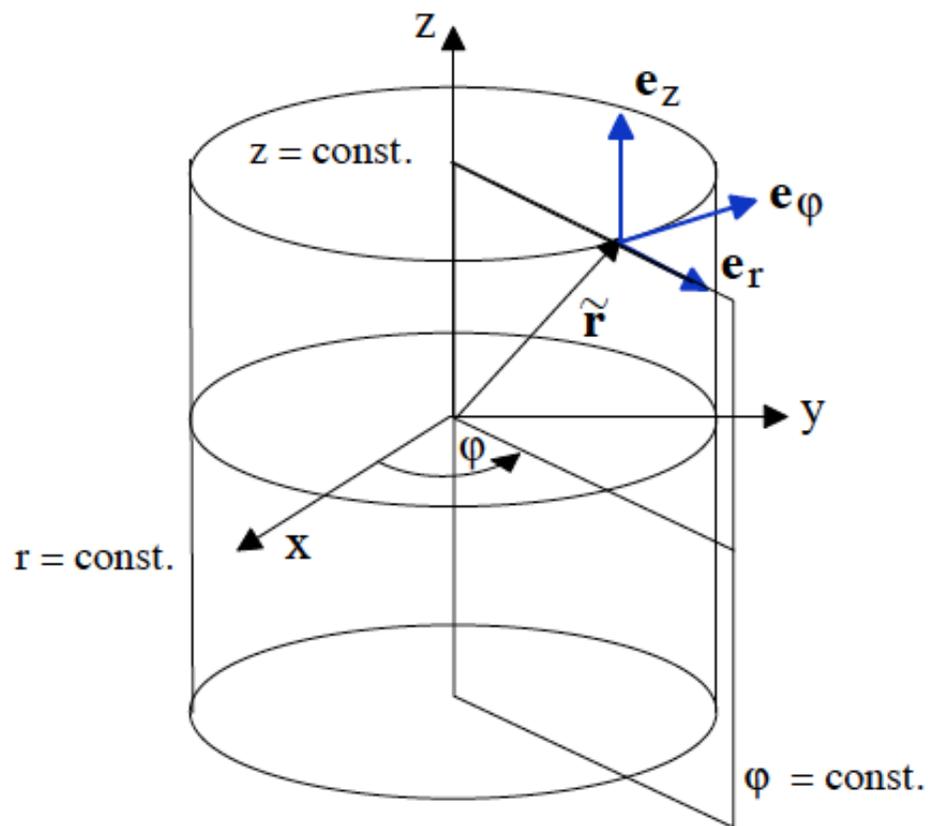
This gives a proof of the two important statements: Much nicer than writing in down in coordinates

Divergence and Curl in Curvilinear Coordinates



Curvilinear Coordinates (Cylindrical)

An easy example



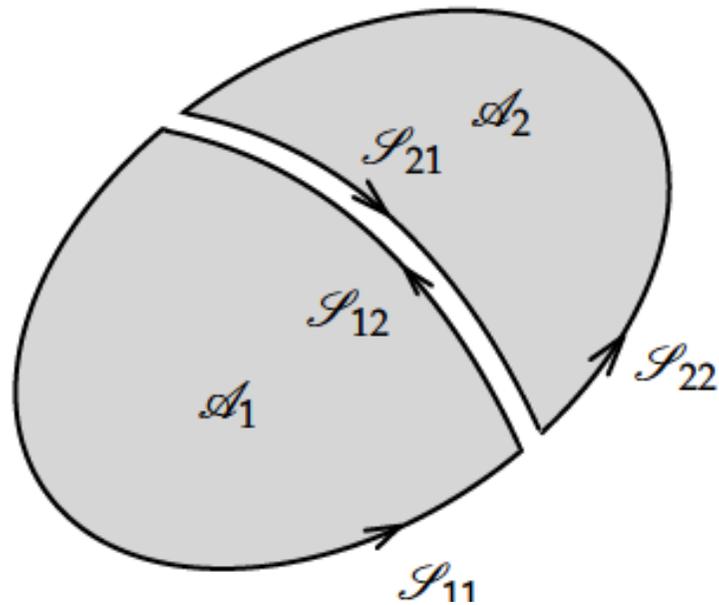
Cartesian	Cylindrical
x	$r \cos \varphi$
y	$r \sin \varphi$
z	z
$\sqrt{x^2 + y^2}$	r
$\arctan \frac{y}{x} + \alpha$	φ
z	z

Cartesian	Cylindrical
\mathbf{e}_x	$\cos \varphi \mathbf{e}_r - \sin \varphi \mathbf{e}_\varphi$
\mathbf{e}_y	$\sin \varphi \mathbf{e}_r + \cos \varphi \mathbf{e}_\varphi$
\mathbf{e}_z	\mathbf{e}_z
$\cos \varphi \mathbf{e}_x + \sin \varphi \mathbf{e}_y$	\mathbf{e}_r
$-\sin \varphi \mathbf{e}_x + \cos \varphi \mathbf{e}_y$	\mathbf{e}_φ
\mathbf{e}_z	\mathbf{e}_z

Differential Operators in Coordinates

	Cartesian	Cylindrical
u^1, u^2, u^3	x, y, z	r, φ, z
ds^2	$dx^2 + dy^2 + dz^2$	$dr^2 + r^2 d\varphi^2 + dz^2$
dV	$dx dy dz$	$r dr d\varphi dz$
$\text{grad } \phi$	$\frac{\partial \phi}{\partial x} \mathbf{e}_x + \frac{\partial \phi}{\partial y} \mathbf{e}_y + \frac{\partial \phi}{\partial z} \mathbf{e}_z$	$\frac{\partial \phi}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial \phi}{\partial \varphi} \mathbf{e}_\varphi + \frac{\partial \phi}{\partial z} \mathbf{e}_z$
$\text{div } \mathbf{g}$	$\frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \frac{\partial g_z}{\partial z}$	$\frac{1}{r} \frac{\partial}{\partial r} (r g_r) + \frac{1}{r} \frac{\partial g_\varphi}{\partial \varphi} + \frac{\partial g_z}{\partial z}$
$\text{curl } \mathbf{g}$	$\left(\frac{\partial g_z}{\partial y} - \frac{\partial g_y}{\partial z} \right) \mathbf{e}_x$ $+ \left(\frac{\partial g_x}{\partial z} - \frac{\partial g_z}{\partial x} \right) \mathbf{e}_y$ $+ \left(\frac{\partial g_y}{\partial x} - \frac{\partial g_x}{\partial y} \right) \mathbf{e}_z$	$\left(\frac{1}{r} \frac{\partial g_z}{\partial \varphi} - \frac{\partial g_\varphi}{\partial z} \right) \mathbf{e}_r$ $+ \left(\frac{\partial g_r}{\partial z} - \frac{\partial g_z}{\partial r} \right) \mathbf{e}_\varphi$ $+ \left(\frac{1}{r} \frac{\partial}{\partial r} (r g_\varphi) - \frac{1}{r} \frac{\partial g_r}{\partial \varphi} \right) \mathbf{e}_z$
$\nabla^2 \phi$	$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$	$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \varphi^2} + \frac{\partial^2 \phi}{\partial z^2}$

Kelvin-Stokes Theorem

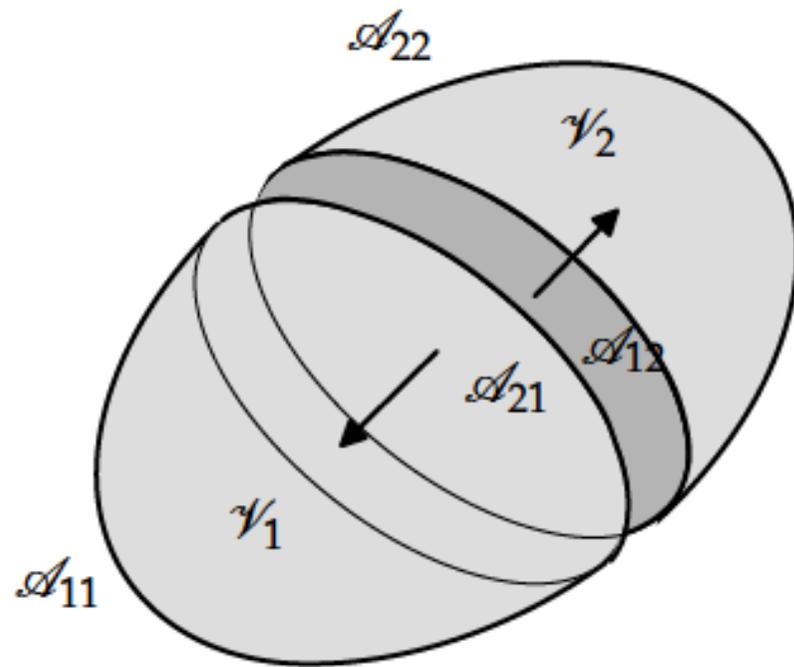


Smooth vector fields, smooth surfaces with simply connected, closed, piecewise-smooth and consistently oriented boundaries, and volumes with piecewise-smooth, closed and consistently oriented surfaces.

$$\int_{\partial \mathcal{A}} \mathbf{g} \cdot d\mathbf{r} = \int_{\mathcal{S}_1} \mathbf{g} \cdot d\mathbf{r} + \int_{\mathcal{S}_2} \mathbf{g} \cdot d\mathbf{r} = \int_{\mathcal{S}_{11}} \mathbf{g} \cdot d\mathbf{r}$$

$$\begin{aligned} \int_{\partial \mathcal{A}} \mathbf{g} \cdot d\mathbf{r} &= \lim_{I \rightarrow \infty} \sum_{i=1}^I \int_{\partial \mathcal{A}_i} \mathbf{g} \cdot d\mathbf{r} = \\ &= \lim_{I \rightarrow \infty} \sum_{i=1}^I (\text{curl } \mathbf{g})_i \cdot \mathbf{n} \Delta a_i = \int_{\mathcal{A}} \text{curl } \mathbf{g} \cdot d\mathbf{a}. \end{aligned}$$

Gauss' Theorem



Smooth vector fields, smooth surfaces with simply connected, closed, piecewise-smooth and consistently oriented boundaries, and volumes with piecewise-smooth, closed and consistently oriented surfaces.

$$\int_{\partial\mathcal{V}} \mathbf{g} \cdot d\mathbf{a} = \int_{\mathcal{A}_1} \mathbf{g} \cdot d\mathbf{a} + \int_{\mathcal{A}_2} \mathbf{g} \cdot d\mathbf{a} = \int_{\mathcal{A}_{11}} \mathbf{g} \cdot d\mathbf{a} + \int_{\mathcal{A}_{22}} \mathbf{g} \cdot d\mathbf{a}$$

$$\begin{aligned} \int_{\partial\mathcal{V}} \mathbf{g} \cdot d\mathbf{a} &= \lim_{I \rightarrow \infty} \sum_{i=1}^I \int_{\partial\mathcal{V}_i} \mathbf{g} \cdot d\mathbf{a} = \lim_{I \rightarrow \infty} \sum_{i=1}^I \Delta V_i \frac{1}{\Delta V_i} \int_{\partial\mathcal{V}_i} \mathbf{g} \cdot d\mathbf{a} \\ &= \lim_{I \rightarrow \infty} \sum_{i=1}^I (\operatorname{div} \mathbf{g})_i \Delta V_i = \int_{\mathcal{V}} \operatorname{div} \mathbf{g} dV. \end{aligned}$$

$$\mathbf{a}, \mathbf{b} \in \mathcal{V}(\Omega), \phi, \psi \in \mathcal{S}(\Omega),$$

$$\text{grad}(\phi + \psi) = \text{grad} \phi + \text{grad} \psi,$$

$$\text{grad}(\phi\psi) = \psi \text{grad} \phi + \phi \text{grad} \psi,$$

$$\text{grad}(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \text{grad})\mathbf{b} + (\mathbf{b} \cdot \text{grad})\mathbf{a} + \mathbf{a} \times \text{curl} \mathbf{b} + \mathbf{b} \times \text{curl} \mathbf{a},$$

$$\text{div}(\mathbf{a} + \mathbf{b}) = \text{div} \mathbf{a} + \text{div} \mathbf{b},$$

$$\text{div} \lambda \mathbf{a} = \lambda \text{div} \mathbf{a} + \mathbf{a} \cdot \text{grad} \lambda,$$

$$\text{div}(\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \text{curl} \mathbf{a} - \mathbf{a} \cdot \text{curl} \mathbf{b},$$

$$\text{curl} \lambda \mathbf{a} = \lambda \text{curl} \mathbf{a} - \mathbf{a} \times \text{grad} \lambda,$$

$$\text{curl}(\mathbf{a} + \mathbf{b}) = \text{curl} \mathbf{a} + \text{curl} \mathbf{b},$$

$$\text{curl}(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \text{div} \mathbf{b} - \mathbf{b} \text{div} \mathbf{a} + (\mathbf{b} \cdot \text{grad})\mathbf{a} - (\mathbf{a} \cdot \text{grad})\mathbf{b},$$

$$\text{curl} \text{curl} \mathbf{a} = \text{grad} \text{div} \mathbf{a} - \nabla^2 \mathbf{a},$$

$$\text{div} \text{grad} \phi = \nabla^2 \phi$$

$$\text{div} \text{curl} \mathbf{a} = 0,$$

$$\text{curl} \text{grad} \phi = 0.$$

Maxwell's Equations in Differential Form

$$\int_{\mathcal{A}} \text{curl } \mathbf{g} \cdot d\mathbf{a} = \int_{\partial\mathcal{A}} \mathbf{g} \cdot d\mathbf{r},$$
$$\int_{\mathcal{V}} \text{div } \mathbf{g} dV = \int_{\partial\mathcal{V}} \mathbf{g} \cdot d\mathbf{a},$$
$$\int_{\partial\mathcal{A}} \mathbf{H} \cdot d\mathbf{r} = \int_{\mathcal{A}} \mathbf{J} \cdot d\mathbf{a} + \frac{d}{dt} \int_{\mathcal{A}} \mathbf{D} \cdot d\mathbf{a},$$
$$\int_{\partial\mathcal{A}} \mathbf{E} \cdot d\mathbf{r} = -\frac{d}{dt} \int_{\mathcal{A}} \mathbf{B} \cdot d\mathbf{a},$$
$$\int_{\partial\mathcal{V}} \mathbf{B} \cdot d\mathbf{a} = 0,$$
$$\int_{\partial\mathcal{V}} \mathbf{D} \cdot d\mathbf{a} = \int_{\mathcal{V}} \rho dV.$$

$$\int_{\mathcal{A}} \text{curl } \mathbf{H} \cdot d\mathbf{a} = \int_{\mathcal{A}} \left(\mathbf{J} + \frac{\partial}{\partial t} \mathbf{D} \right) \cdot d\mathbf{a},$$

$$\int_{\mathcal{A}} \text{curl } \mathbf{E} \cdot d\mathbf{a} = -\int_{\mathcal{A}} \frac{\partial}{\partial t} \mathbf{B} \cdot d\mathbf{a},$$

$$\int_{\mathcal{V}} \text{div } \mathbf{B} dV = 0,$$

$$\int_{\mathcal{V}} \text{div } \mathbf{D} dV = \int_{\mathcal{V}} \rho dV.$$

$$\text{curl } \mathbf{H} = \mathbf{J} + \frac{\partial}{\partial t} \mathbf{D},$$

$$\text{curl } \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{B},$$

$$\text{div } \mathbf{B} = 0,$$

$$\text{div } \mathbf{D} = \rho.$$

$$\operatorname{div} \operatorname{curl} \mathbf{g} = 0.$$

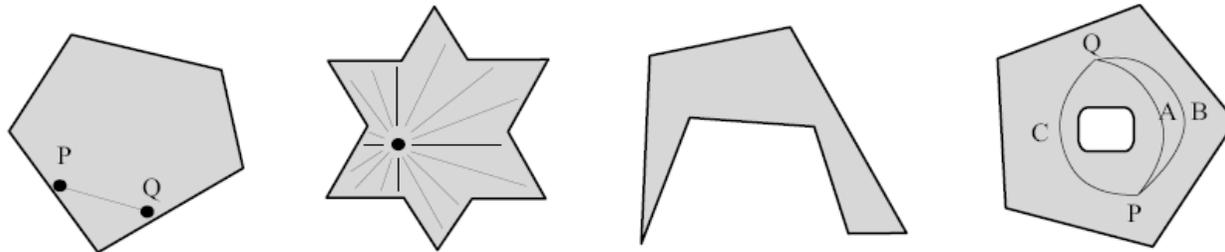
$$\operatorname{curl} \operatorname{grad} \phi = 0,$$

$$\partial(\partial\mathcal{V}) = \emptyset,$$

$$\partial(\partial\mathcal{A}) = \emptyset,$$

$$\int_{\mathcal{A}} \operatorname{curl} \operatorname{grad} \phi \cdot d\mathbf{a} = \int_{\partial\mathcal{A}} \operatorname{grad} \phi \cdot d\mathbf{r} = \phi|_{\partial(\partial\mathcal{A})} = 0,$$

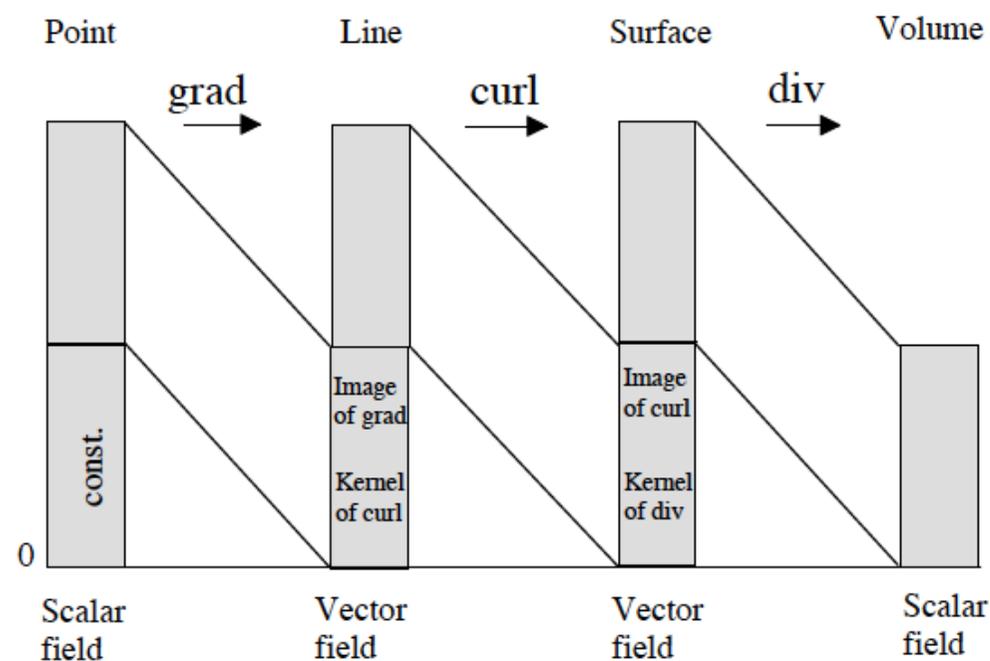
$$\int_{\mathcal{V}} \operatorname{div} \operatorname{curl} \mathbf{g} dV = \int_{\partial\mathcal{V}} \operatorname{curl} \mathbf{g} \cdot d\mathbf{a} = \int_{\partial(\partial\mathcal{V})} \mathbf{g} \cdot d\mathbf{r} = 0,$$



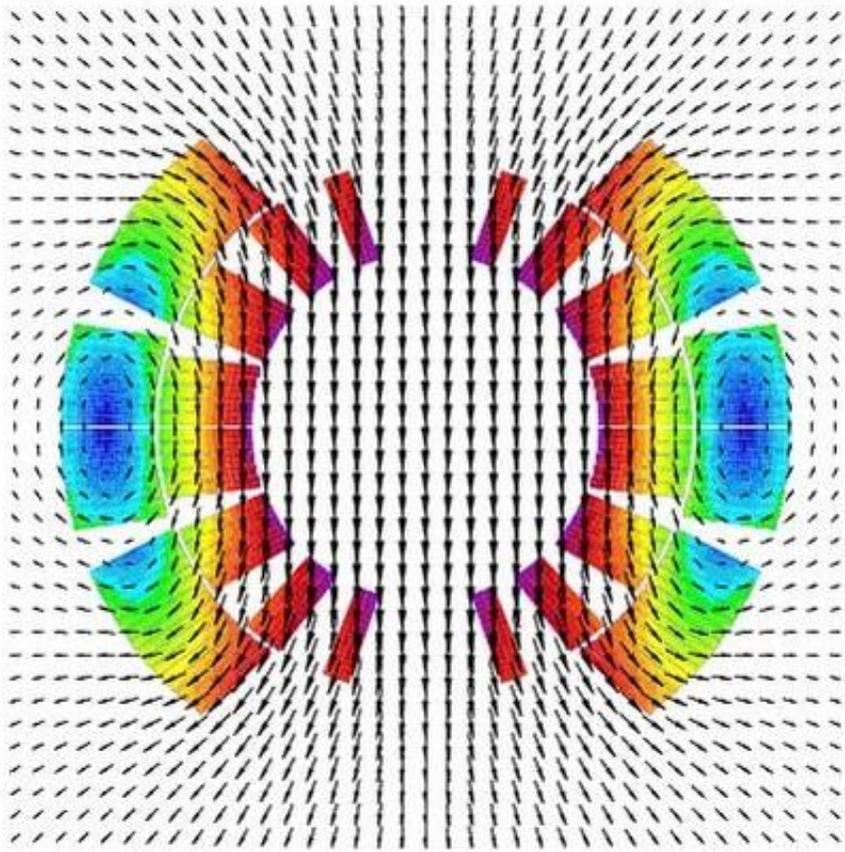
In 3D: also connected boundaries

$$\operatorname{div} \mathbf{b} = 0 \quad \rightarrow \quad \mathbf{b} = \operatorname{curl} \mathbf{a}.$$

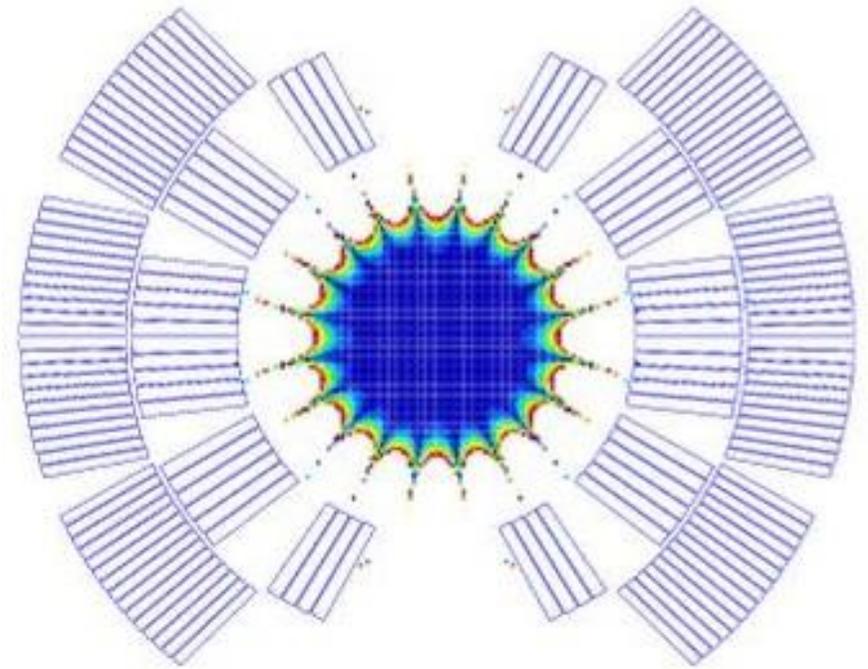
$$\operatorname{curl} \mathbf{h} = 0 \quad \rightarrow \quad \mathbf{h} = \operatorname{grad} \phi.$$



Interrupt: Can we make physical sense out of these potentials?



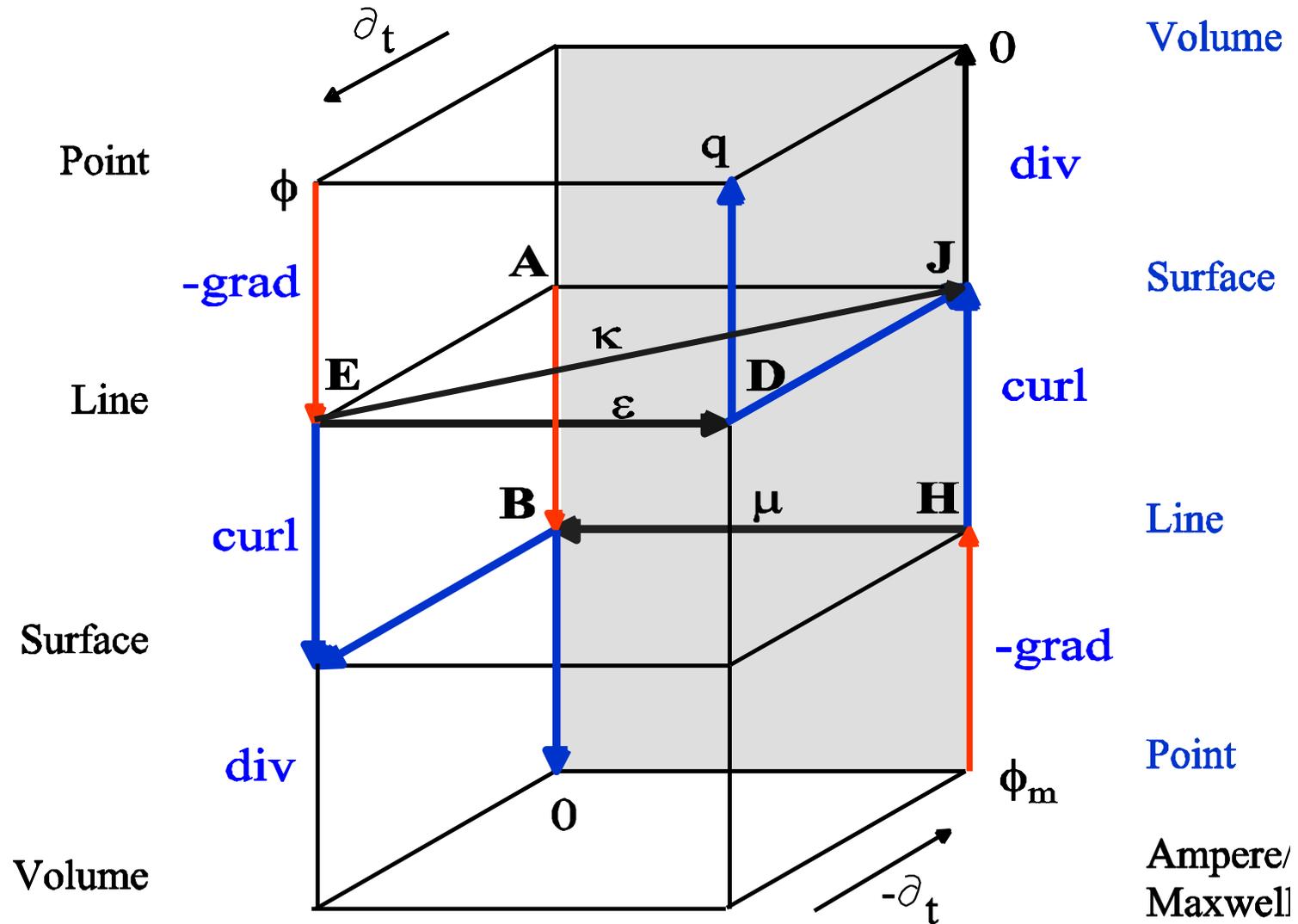
Field map



Good field region

Maxwell's House

Faraday



Cash-Back 2

The Laplace Equation and the Field Quality in Accelerator Magnets

Maxwell's Facade

$$\text{curl} \frac{1}{\mu} \text{curl} \mathbf{A} = \mathbf{J}$$

$$\frac{1}{\mu_0} \text{curl} \text{curl} \mathbf{A} = \mathbf{J}$$

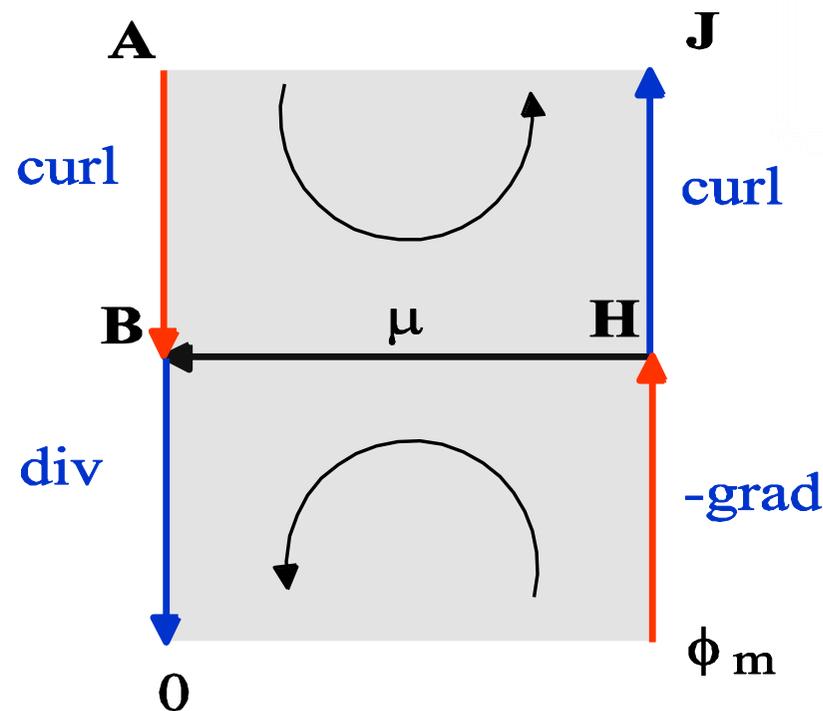
$$\nabla^2 \mathbf{A} - \text{grad} \text{div} \mathbf{A} = 0$$

$$\nabla^2 A_z = 0$$

$$\text{div} \mu \text{grad} \phi_m = 0$$

$$\mu_0 \text{div} \text{grad} \phi_m = 0$$

$$\nabla^2 \phi_m = 0$$



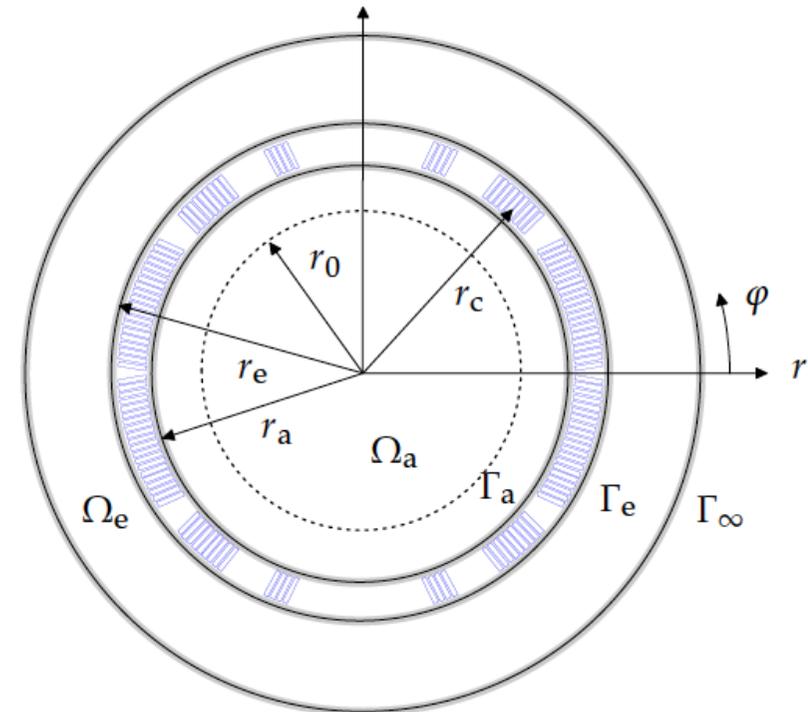
Solving of Boundary Value Problems

1. Governing equation in the air domain

$$\nabla^2 A_z = 0,$$

2. Chose a suitable coordinate system

$$r^2 \frac{\partial^2 A_z}{\partial r^2} + r \frac{\partial A_z}{\partial r} + \frac{\partial^2 A_z}{\partial \varphi^2} = 0,$$



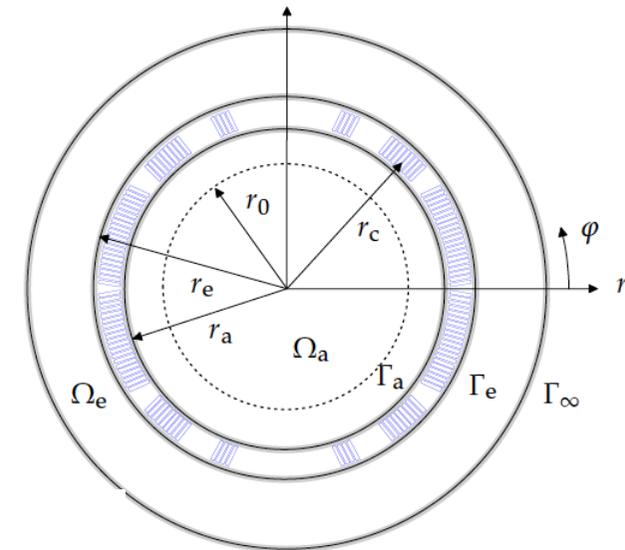
3. Make a guess, look it up in a book, use the method of separation:
That is: find **eigenfunctions**. **Coefficients are not know yet**

$$A_z(r, \varphi) = \sum_{n=1}^{\infty} (\mathcal{E}_n r^n + \mathcal{F}_n r^{-n}) (\mathcal{G}_n \sin n\varphi + \mathcal{H}_n \cos n\varphi).$$

Solving of Boundary Value Problems

4. Incorporate a bit of knowledge and rename

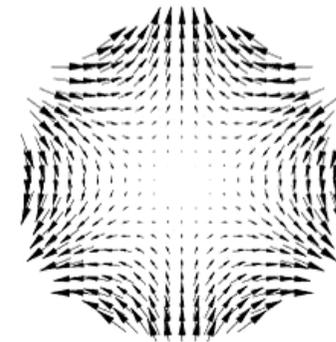
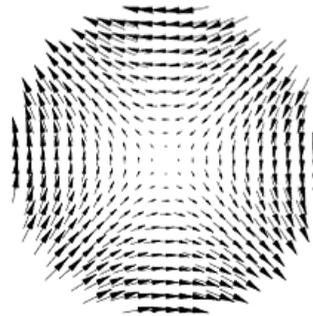
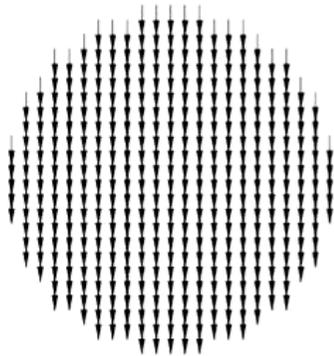
$$A_z(r, \varphi) = \sum_{n=1}^{\infty} r^n (\mathcal{A}_n \sin n\varphi + \mathcal{B}_n \cos n\varphi).$$



2. Calculate a field component

$$B_r(r, \varphi) = \frac{1}{r} \frac{\partial A_z}{\partial \varphi} = \sum_{n=1}^{\infty} nr^{n-1} (\mathcal{A}_n \cos n\varphi - \mathcal{B}_n \sin n\varphi),$$

$$B_\varphi(r, \varphi) = -\frac{\partial A_z}{\partial r} = -\sum_{n=1}^{\infty} nr^{n-1} (\mathcal{A}_n \sin n\varphi + \mathcal{B}_n \cos n\varphi),$$

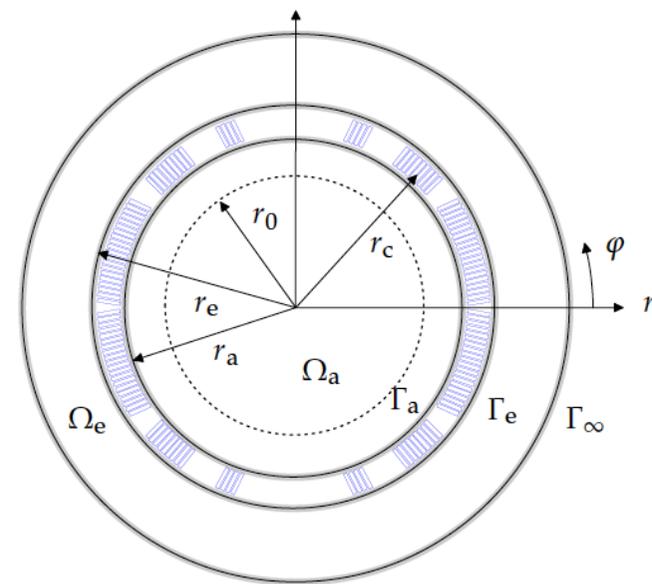


Solving of Boundary Value Problems

$$B_r(r, \varphi) = \frac{1}{r} \frac{\partial A_z}{\partial \varphi} = \sum_{n=1}^{\infty} n r^{n-1} (\mathcal{A}_n \cos n\varphi - \mathcal{B}_n \sin n\varphi),$$

3. Measure or calculate the field on a reference radius and perform Fourier analysis (develop into the eigenfunctions). **Coefficients known here.**

$$B_r(r_0, \varphi) = \sum_{n=1}^{\infty} (B_n(r_0) \sin n\varphi + A_n(r_0) \cos n\varphi),$$



4: Compare the known and unknown coefficients

$$B_r(r, \varphi) = \frac{1}{r} \frac{\partial A_z}{\partial \varphi} = \sum_{n=1}^{\infty} n r^{n-1} (\mathcal{A}_n \cos n\varphi - \mathcal{B}_n \sin n\varphi),$$

$$B_r(r_0, \varphi) = \sum_{n=1}^{\infty} (B_n(r_0) \sin n\varphi + A_n(r_0) \cos n\varphi),$$

$$\mathcal{A}_n = \frac{1}{n r_0^{n-1}} A_n(r_0), \quad \mathcal{B}_n = \frac{-1}{n r_0^{n-1}} B_n(r_0).$$

5. Put this into the original solution for the entire air domain

$$A_z(r, \varphi) = - \sum_{n=1}^{\infty} \frac{r_0}{n} \left(\frac{r}{r_0} \right)^n (B_n(r_0) \cos n\varphi - A_n(r_0) \sin n\varphi).$$

6: Calculate fields and potential in the entire air domain

$$A_z(r, \varphi) = - \sum_{n=1}^{\infty} \frac{r_0}{n} \left(\frac{r}{r_0} \right)^n (B_n(r_0) \cos n\varphi - A_n(r_0) \sin n\varphi).$$

$$B_r(r, \varphi) = \sum_{n=1}^{\infty} \left(\frac{r}{r_0} \right)^{n-1} (B_n(r_0) \sin n\varphi + A_n(r_0) \cos n\varphi)$$

$$B_\varphi(r, \varphi) = \sum_{n=1}^{\infty} \left(\frac{r}{r_0} \right)^{n-1} (B_n(r_0) \cos n\varphi - A_n(r_0) \sin n\varphi)$$

$$B_x(r, \varphi) = \sum_{n=1}^{\infty} \left(\frac{r}{r_0} \right)^{n-1} (B_n(r_0) \sin(n-1)\varphi + A_n(r_0) \cos(n-1)\varphi)$$

$$B_y(r, \varphi) = \sum_{n=1}^{\infty} \left(\frac{r}{r_0} \right)^{n-1} (B_n(r_0) \cos(n-1)\varphi - A_n(r_0) \sin(n-1)\varphi)$$

Fourier Series (an Infinite Dimensional Vector Space)

$$B_r(r_0, \varphi) = \sum_{n=1}^{\infty} (B_n(r_0) \sin n\varphi + A_n(r_0) \cos n\varphi),$$

$$A_n(r_0) = \frac{1}{\pi} \int_0^{2\pi} B_r(r_0, \varphi) \cos n\varphi \, d\varphi, \quad n = 1, 2, 3, \dots,$$

$$B_n(r_0) = \frac{1}{\pi} \int_0^{2\pi} B_r(r_0, \varphi) \sin n\varphi \, d\varphi, \quad n = 1, 2, 3, \dots$$

And on the computer: Discrete setting (don't bother with the FFT)

$$\varphi_k = \frac{2\pi k}{N}, \quad k = 0, 1, 2, \dots, N-1.$$

$$A_n(r_0) \approx \frac{2}{N} \sum_{k=0}^{N-1} B_r(r_0, \varphi_k) \cos n\varphi_k,$$

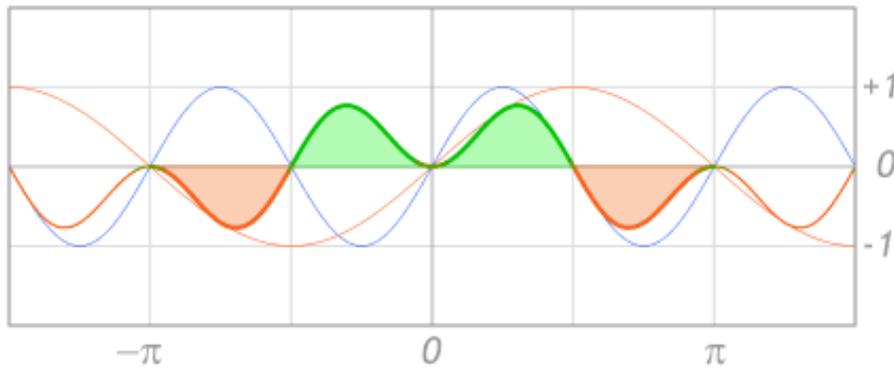
$$B_n(r_0) \approx \frac{2}{N} \sum_{k=0}^{N-1} B_r(r_0, \varphi_k) \sin n\varphi_k.$$

Interrupt: Expansions; orthogonality, completeness, convergence

The Road Map to Convergence of Fourier Series

- The trigonometric functions are orthogonal
- The Fourier polynomial P_n of grade n is the best approximation of f in V_n
- The projections onto the trigonometric functions (scalar product) induces a norm (the RMS error)
- Riemann Lebesgue Lemma: Within this norm, the coefficients converge to zero.
- 3 Convergence theorems
 - For a C^1 function P_n converges uniformly to $f(x)$
 - For “clean jumps” P_n converges pointwise to $0.5 (f_+(x) + f_-(x))$
 - The P_n converges for every square integrable function in the RMS sense

Trigonometric Functions as Orthogonal Function Set (from Wikipedia)



$$\int_{-\pi}^{+\pi} \sin(2x) \sin(1x) dx = 0$$

$$\int_{-\pi}^{\pi} \cos(mx) \cos(nx) dx = \pi \delta_{mn},$$

$$\int_{-\pi}^{\pi} \sin(mx) \sin(nx) dx = \pi \delta_{mn},$$

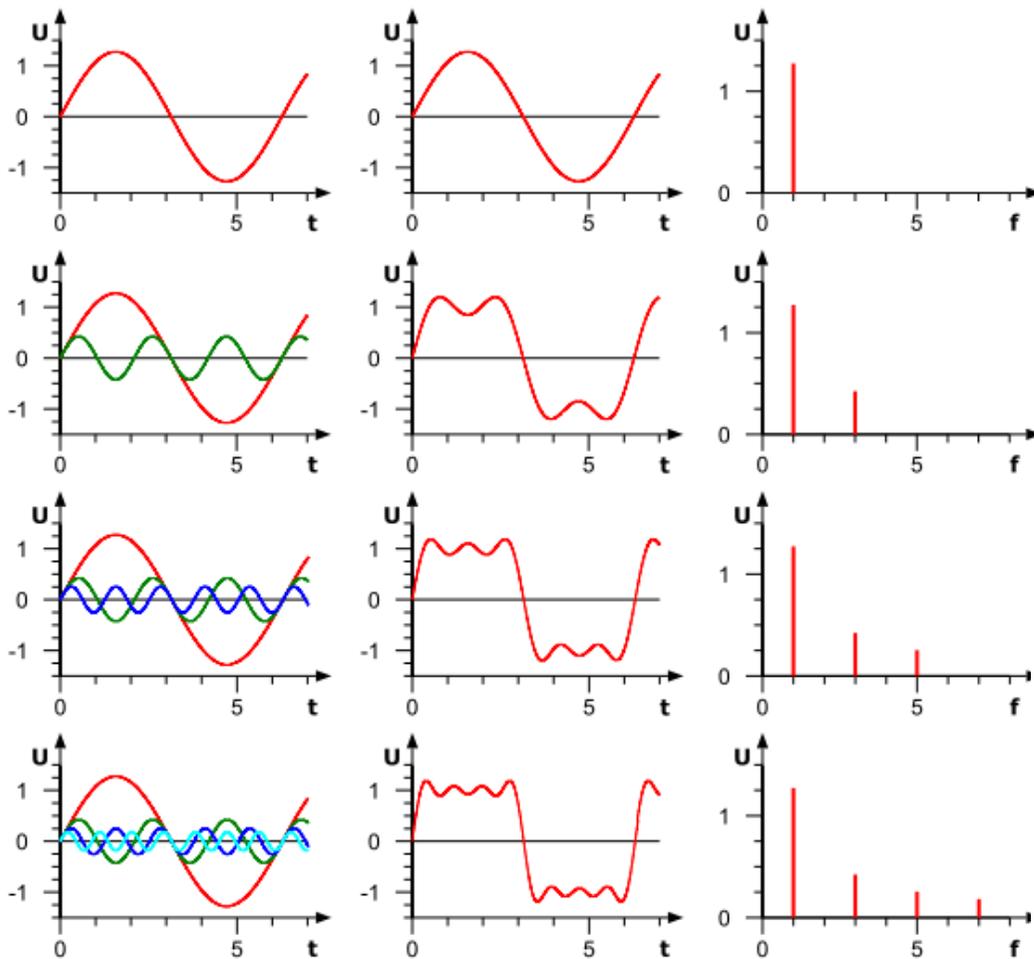
(where δ_{mn} is the **Kronecker delta**), and

$$\int_{-\pi}^{\pi} \cos(mx) \sin(nx) dx = 0;$$

$$\cos \alpha \sin \beta = \frac{1}{2} (\sin(\alpha + \beta) - \sin(\alpha - \beta))$$

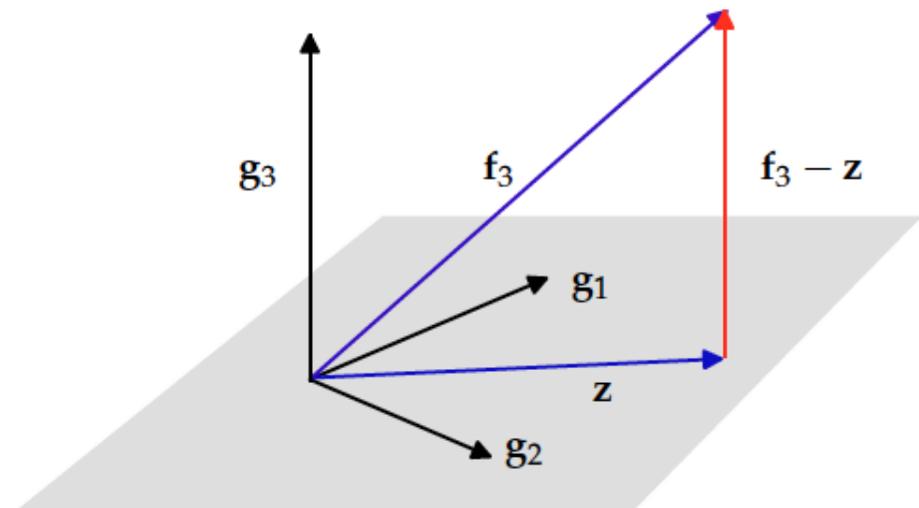
$$\begin{aligned} \int_{-L}^L \cos \frac{m\pi x}{L} \sin \frac{n\pi x}{L} dx &= \frac{1}{2} \int_{-L}^L \sin \frac{(m+n)\pi x}{L} dx - \frac{1}{2} \int_{-L}^L \sin \frac{(m-n)\pi x}{L} dx \\ &= \frac{1}{2} \left(\frac{-\cos \frac{(m+n)\pi x}{L}}{\frac{(m+n)\pi}{L}} \right) \Big|_{-L}^L - \frac{1}{2} \left(\frac{-\cos \frac{(m-n)\pi x}{L}}{\frac{(m-n)\pi}{L}} \right) \Big|_{-L}^L = 0 \end{aligned}$$

The Fourier Polynomial P_n is the best Approximation in V_n



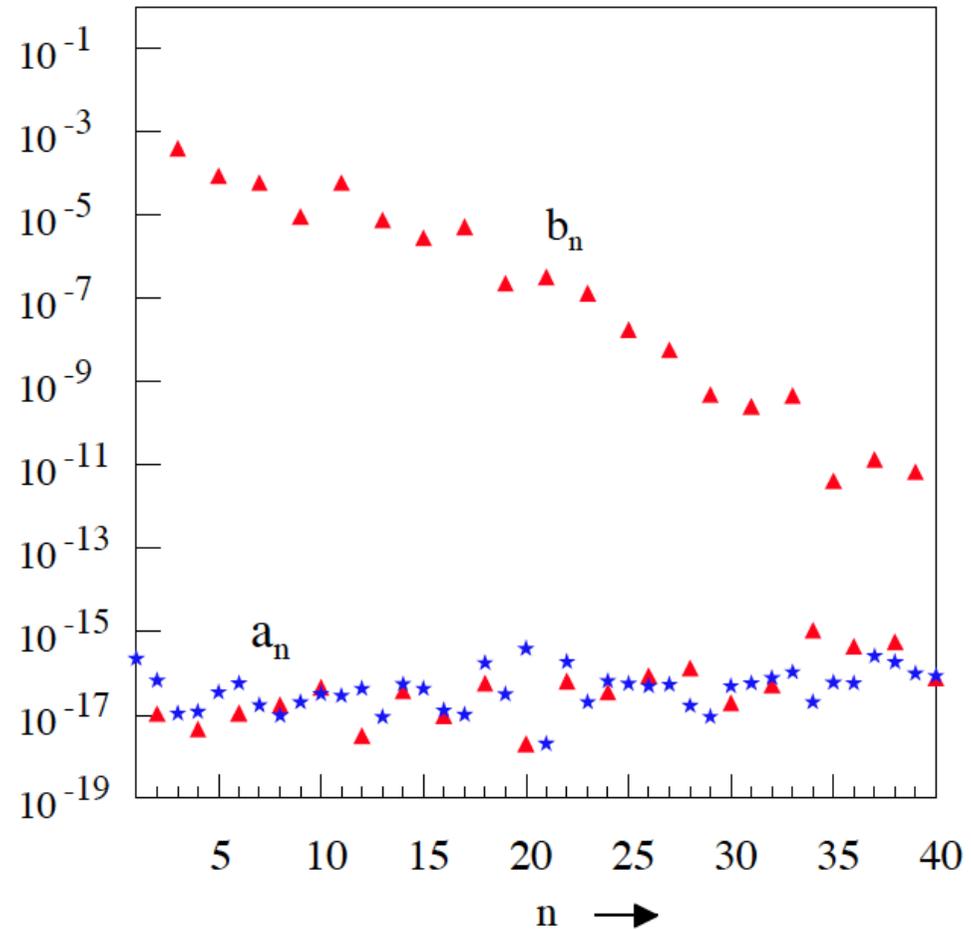
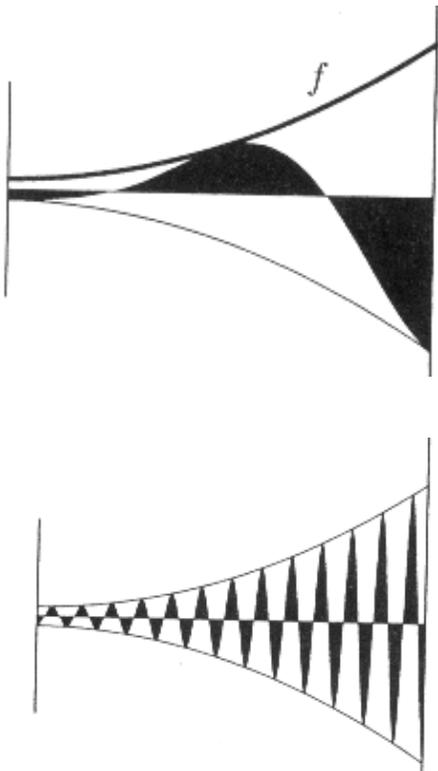
Projection of the square wave onto the “shape” of the trigonometric functions

$f_3 - z$ is the shortest distance to the projective plane

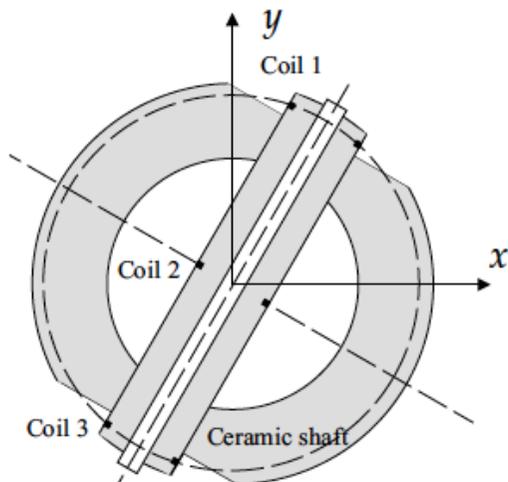
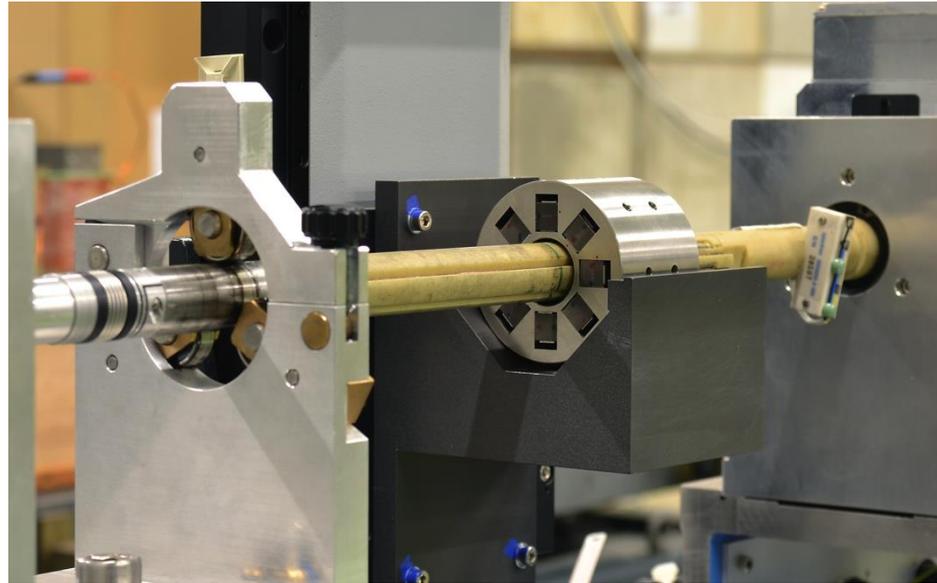


The Riemann Lebesgue Lemma

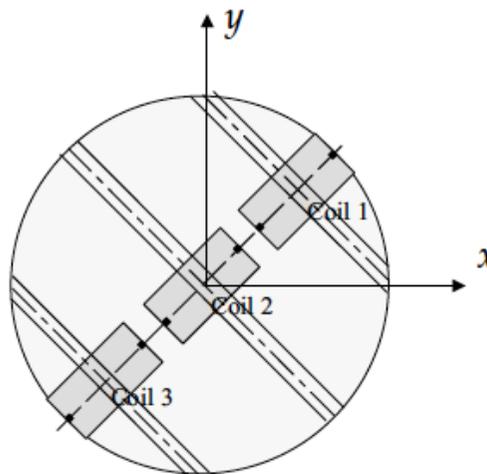
The Fourier coefficients
tend to zero as n goes to
infinity



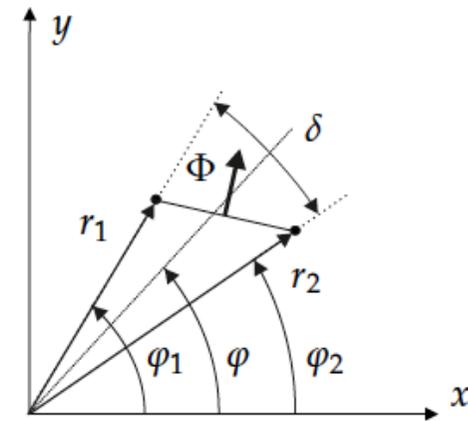
Rotating Coil Measurements



Tangential coil
Radial flux



Radial coil
Tangential flux



Series Measurements of the LHC Magnets



Stephan Russenschuck, CERN TE-MS-C-MM, 1211 Geneva 23
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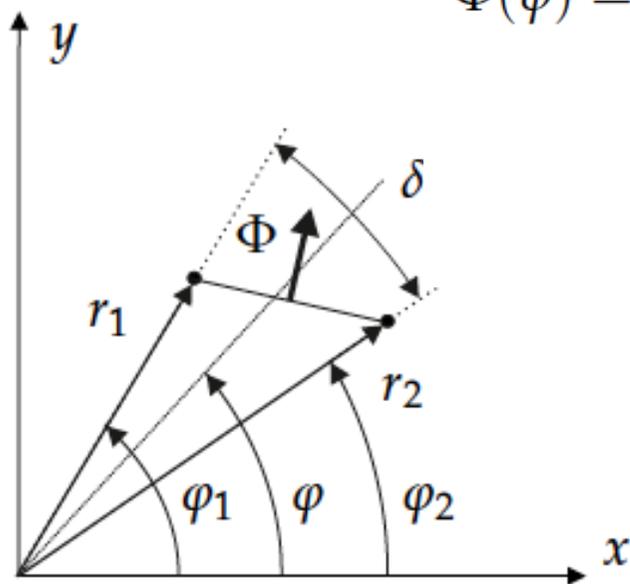
Rotating Coil Measurements

$$\varphi = \omega t + \Theta, \quad \Phi(\varphi) = N \int_{\mathcal{A}} \mathbf{B} \cdot d\mathbf{a} = N \int_{\mathcal{A}} \text{curl } \mathbf{A} \cdot d\mathbf{a} = N \int_{\partial \mathcal{A}} \mathbf{A} \cdot d\mathbf{r}$$

$$= N\ell [A_z(\mathcal{P}_1) - A_z(\mathcal{P}_2)],$$

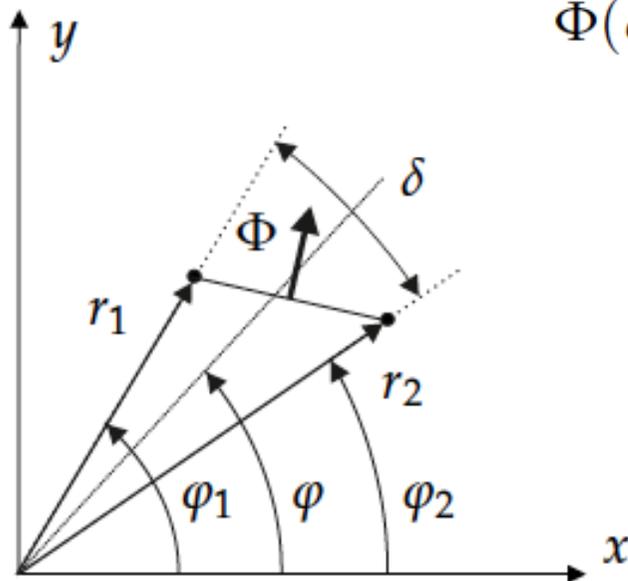
2 Dim version of Stoke's Theorem

$$\Phi(\varphi) = N\ell \left[\sum_{n=1}^{\infty} \frac{r_0}{n} \left(\frac{r_2}{r_0} \right)^n (B_n(r_0) \cos n\varphi_2 - A_n(r_0) \sin n\varphi_2) \right. \\ \left. - \sum_{n=1}^{\infty} \frac{r_0}{n} \left(\frac{r_1}{r_0} \right)^n (B_n(r_0) \cos n\varphi_1 - A_n(r_0) \sin n\varphi_1) \right],$$



Rotating Coil Measurements

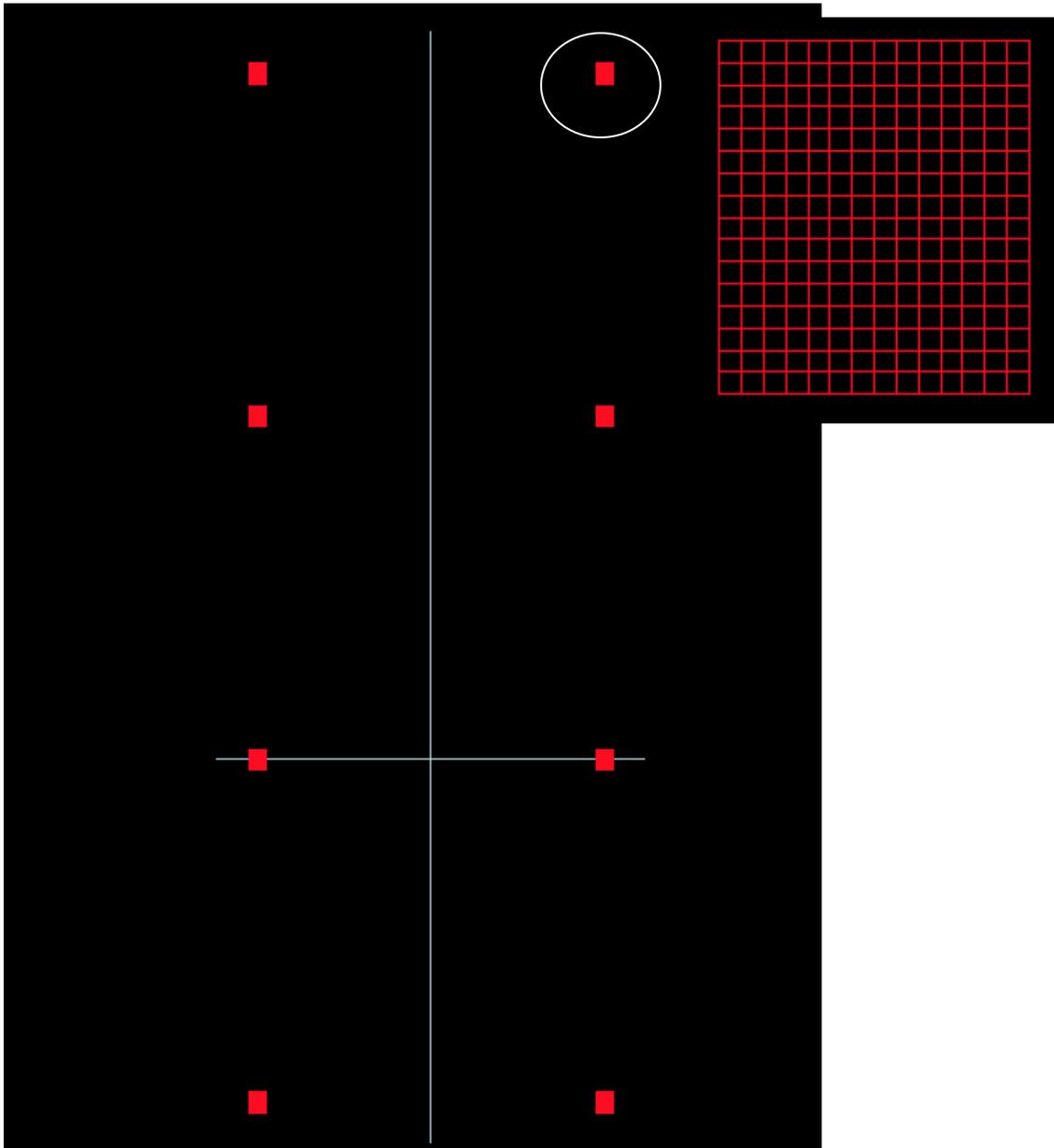
$$\Phi(\varphi) = N\ell \left[\sum_{n=1}^{\infty} \frac{r_0}{n} \left(\frac{r_2}{r_0} \right)^n (B_n(r_0) \cos n\varphi_2 - A_n(r_0) \sin n\varphi_2) - \sum_{n=1}^{\infty} \frac{r_0}{n} \left(\frac{r_1}{r_0} \right)^n (B_n(r_0) \cos n\varphi_1 - A_n(r_0) \sin n\varphi_1) \right],$$



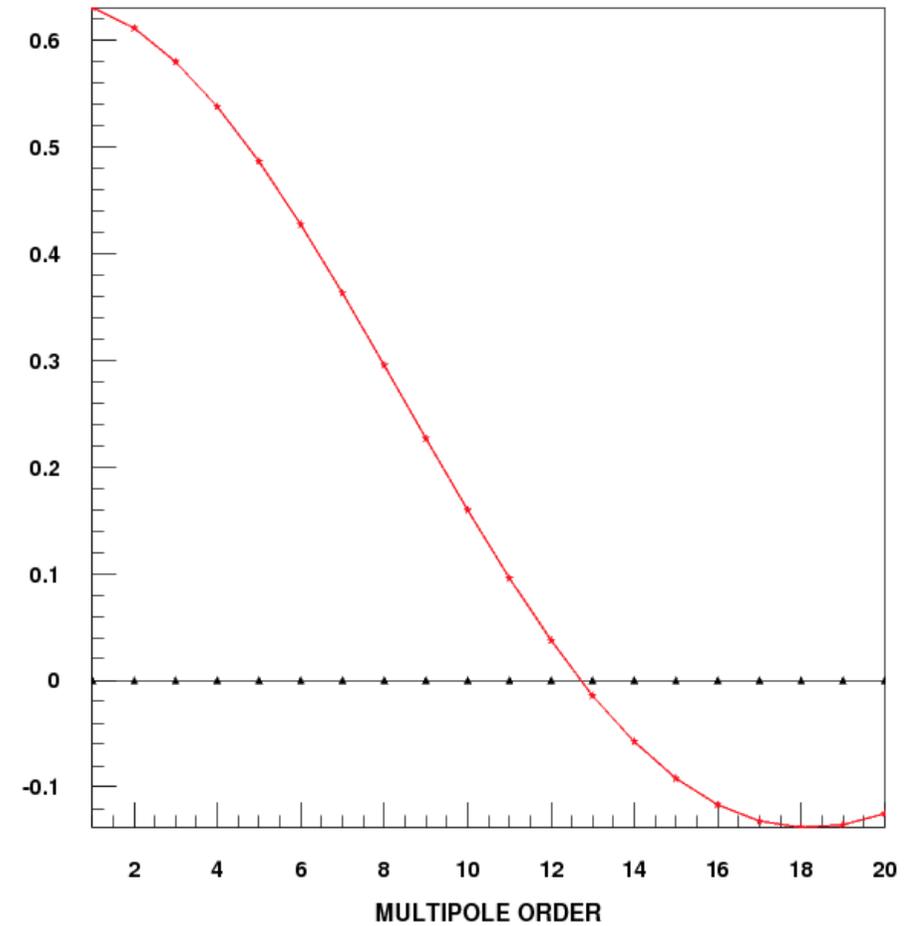
$$\Phi(\varphi) = \sum_{n=1}^{\infty} \frac{\ell}{r_0^{n-1}} \left[K_n^{\text{rad}} (B_n(r_0) \cos n\varphi - A_n(r_0) \sin n\varphi) + K_n^{\text{tan}} (B_n(r_0) \sin n\varphi + A_n(r_0) \cos n\varphi) \right],$$

$$K_n^{\text{tan}} = \frac{2N}{n} r_c^n \sin \left(\frac{n\delta}{2} \right),$$

Rotating Coil Measurements



$$S_n := \frac{\ell}{r_0^{n-1}} K_n.$$



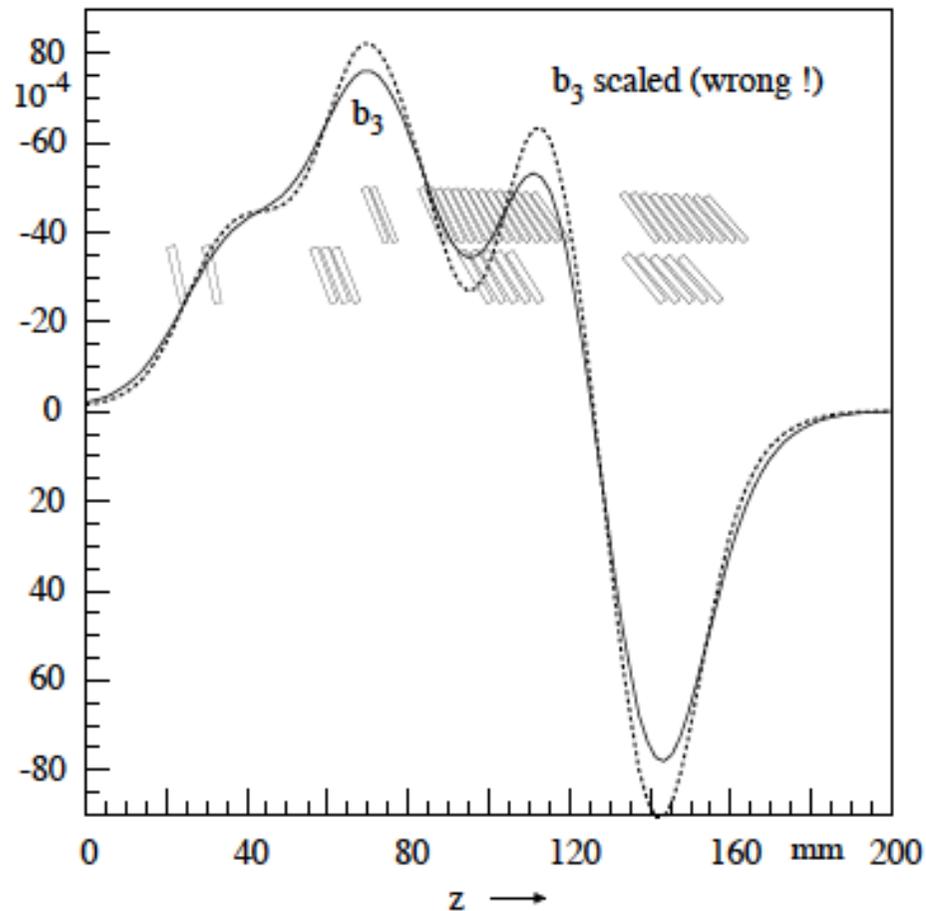
Multipoles and Scaling Laws

$$B_r(r, \varphi) = \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} (B_n(r_0) \sin n\varphi + A_n(r_0) \cos n\varphi)$$

$$B_r(r, \varphi) = B_N \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-N} (b_n(r_0) \sin n\varphi + a_n(r_0) \cos n\varphi).$$

$$A_n(r_1) = \left(\frac{r_1}{r_0}\right)^{n-1} A_n(r_0), \quad B_n(r_1) = \left(\frac{r_1}{r_0}\right)^{n-1} B_n(r_0),$$

$$b_n(r_1) = \frac{B_n(r_1)}{B_N(r_1)} = \frac{\left(\frac{r_1}{r_0}\right)^{n-1} B_n(r_0)}{\left(\frac{r_1}{r_0}\right)^{N-1} B_N(r_0)} = \left(\frac{r_1}{r_0}\right)^{n-N} b_n(r_0),$$



Local transverse harmonics calculated at different reference radii and scaled with the 2D laws

$$b_n(r_1) = \left(\frac{r_1}{r_0}\right)^{n-N} b_n(r_0),$$

wrong

$$\nabla^2 \phi_m(x, y, z) = \frac{\partial^2 \phi_m(x, y, z)}{\partial x^2} + \frac{\partial^2 \phi_m(x, y, z)}{\partial y^2} + \frac{\partial^2 \phi_m(x, y, z)}{\partial z^2} = 0.$$

$$\bar{\phi}_m(x, y) := \int_{-z_0}^{z_0} \phi_m(x, y, z) dz.$$

$$\begin{aligned} \frac{\partial^2 \bar{\phi}_m(x, y)}{\partial x^2} + \frac{\partial^2 \bar{\phi}_m(x, y)}{\partial y^2} &= \int_{-z_0}^{z_0} \left(\frac{\partial^2 \phi_m}{\partial x^2} + \frac{\partial^2 \phi_m}{\partial y^2} \right) dz \\ &= \int_{-z_0}^{z_0} \left(-\frac{\partial^2 \phi_m}{\partial z^2} \right) dz = - \left. \frac{\partial \phi_m}{\partial z} \right|_{-z_0}^{z_0} \\ &= H_z(-z_0) - H_z(z_0) \stackrel{!}{=} 0. \end{aligned}$$

The 2D scaling laws hold for the **integrated** harmonics

The imaginary number is a fine and wonderful resource of the human spirit, almost an amphibian between being and not being.

Gottfried Wilhelm Leibnitz (1646-1716)

Theorem 9.2 *Real and imaginary parts of a holomorphic function are harmonic functions.*

Proof. If $f(z) = f(x, y) = u(x, y) + iv(x, y)$ is holomorphic, the Cauchy-Riemann equations yield

$$\nabla^2 u = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) = \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial y} \right) + \frac{\partial}{\partial y} \left(-\frac{\partial v}{\partial x} \right) = 0, \quad (9.55)$$

$$\nabla^2 v = \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial v}{\partial y} \right) = \frac{\partial}{\partial x} \left(-\frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} \right) = 0.$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

$$f^{(1)}(z_0) = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} i$$

$$\mathbf{H} = -\operatorname{grad} \phi = -\frac{\partial \phi}{\partial x} \mathbf{e}_x - \frac{\partial \phi}{\partial y} \mathbf{e}_y,$$

$$\mathbf{B} = \operatorname{curl}(\mathbf{e}_z A_z) = \frac{\partial A_z}{\partial y} \mathbf{e}_x - \frac{\partial A_z}{\partial x} \mathbf{e}_y.$$

This implies

$$\frac{\partial A_z}{\partial y} = -\mu_0 \frac{\partial \phi}{\partial x} \quad \text{and} \quad \frac{\partial A_z}{\partial x} = \mu_0 \frac{\partial \phi}{\partial y},$$

Wich are the Cauchy Riemann equations of

$$w(z) := u(x, y) + iv(x, y) = A_z(x, y) + i\mu_0 \phi(x, y).$$

$$-\frac{dw}{dz} = -\frac{\partial A_z}{\partial x} - i\mu_0 \frac{\partial \phi}{\partial x} = i\frac{\partial A_z}{\partial y} - \mu_0 \frac{\partial \phi}{\partial y} = B_y(x, y) + iB_x(x, y) =: B(z).$$

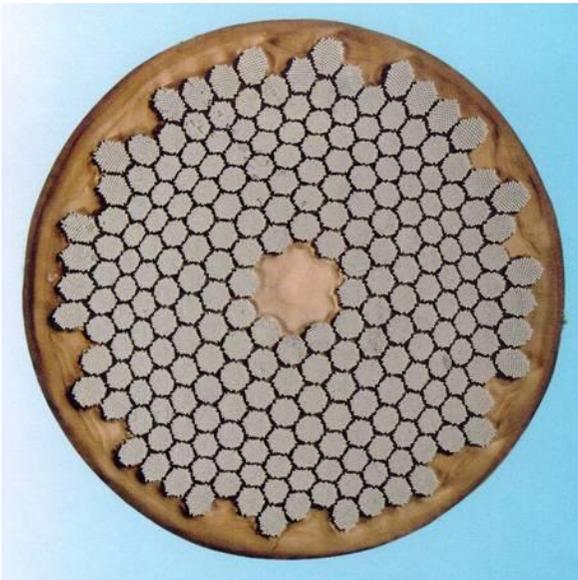
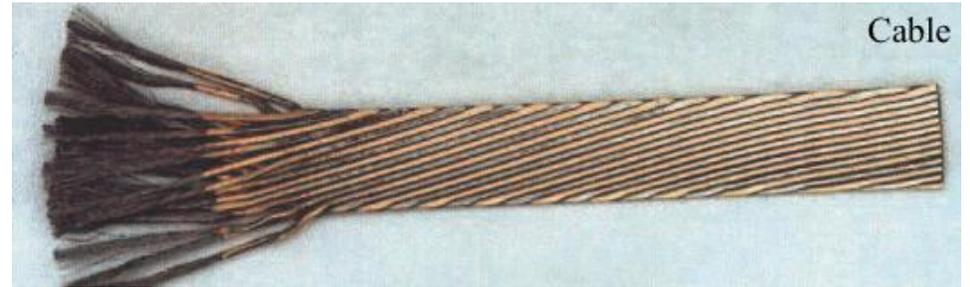
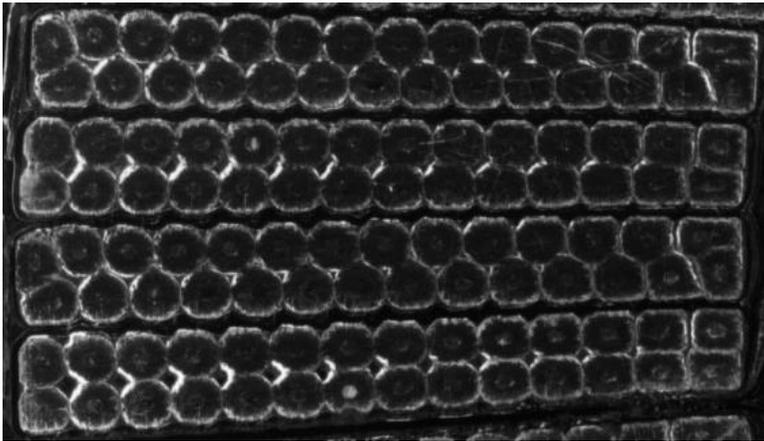
$$B_x = B_r \cos \varphi - B_\varphi \sin \varphi, \quad B_y = B_r \sin \varphi + B_\varphi \cos \varphi,$$

$$B_y + iB_x = (B_\varphi + iB_r)e^{-i\varphi}.$$

$$\begin{aligned} B_y + iB_x &= \sum_{n=1}^{\infty} (B_n(r_0) + iA_n(r_0)) \left(\frac{r}{r_0}\right)^{n-1} e^{i(n-1)\varphi} \\ &= \sum_{n=1}^{\infty} (B_n(r_0) + iA_n(r_0)) \left(\frac{z}{r_0}\right)^{n-1} \\ &= B_N \sum_{n=1}^{\infty} (b_n(r_0) + ia_n(r_0)) \left(\frac{z}{r_0}\right)^{n-1}, \end{aligned}$$

- We have studied the mathematical foundations of magnetic fields
 - Vectorfields
 - One dimensional techniques for normal conducting magnets
 - The Laplace equation
 - Field harmonics in accelerator magnets (including solenoids and wigglers)
- So far we assumed that the fields on the domain boundary where known (from measurements or calculations)
- Now we need to address how to calculate these fields
 - The field of line currents and the Biot Savart law
 - Development of these field solutions into the Eigenfunctions

Rutherford (Roebel) Kabel, Strand, Nb-Ti Filament



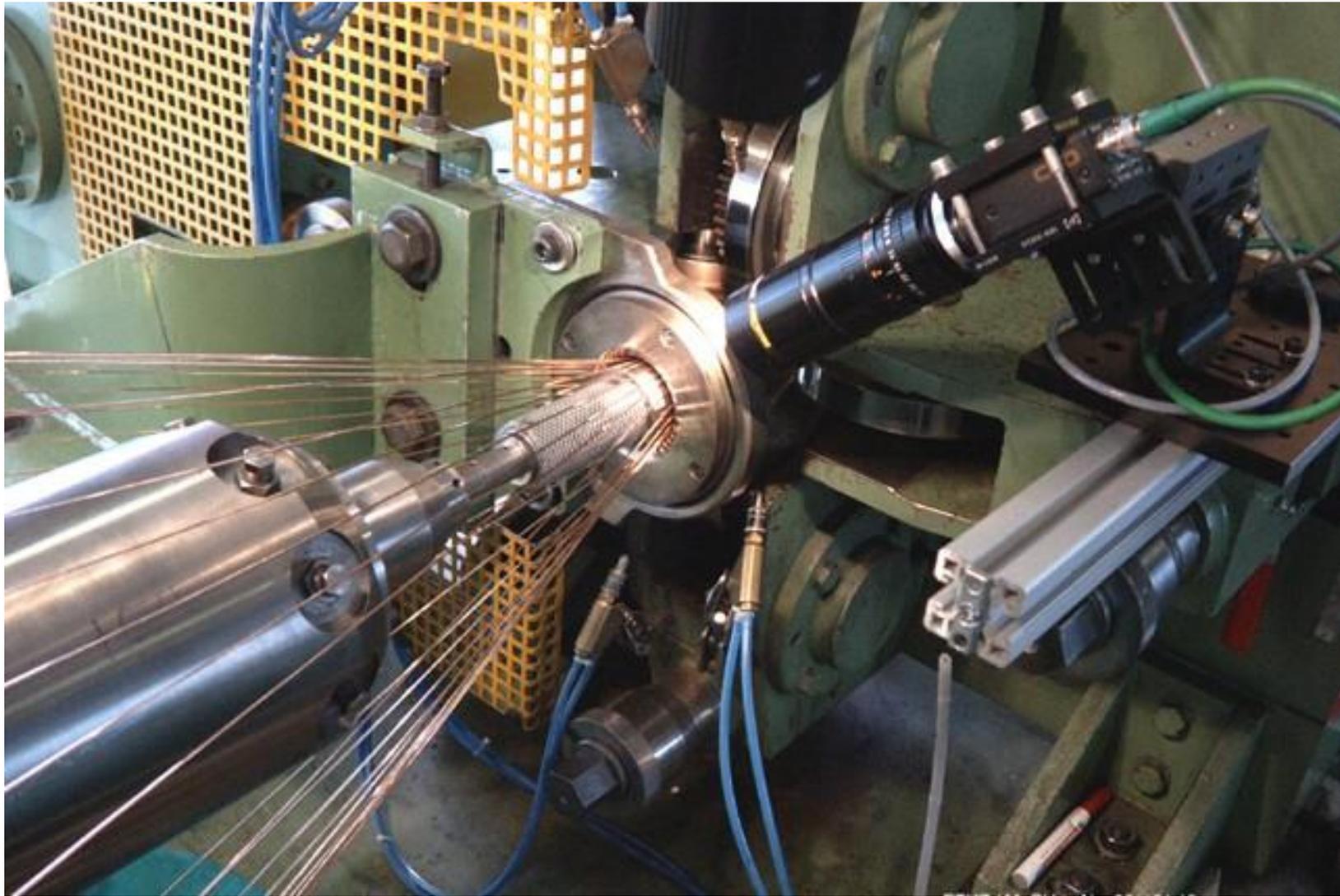
200 nm 

Cabling Machine for Rutherford Cables



Stephan Russenschuck, CERN TE-MS-C-MM, 1211 Geneva 23
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Turk's Head



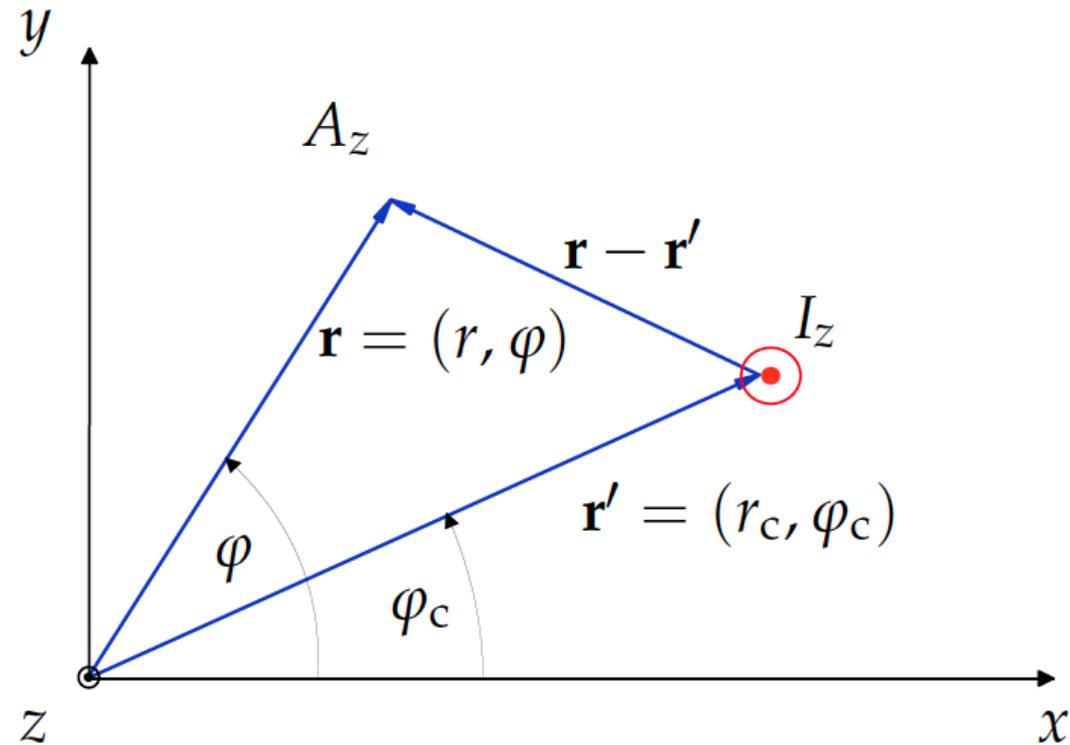
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The Field of Line Currents

$$\mathbf{r} \mapsto \phi(|\mathbf{r} - \mathbf{r}'|)$$
$$\mathbf{r}' \mapsto \phi(|\mathbf{r} - \mathbf{r}'|)$$

$$\text{grad } \phi(|\mathbf{r} - \mathbf{r}'|) = -\text{grad}_{\mathbf{r}'} \phi(|\mathbf{r} - \mathbf{r}'|),$$
$$\text{div } \mathbf{a}(|\mathbf{r} - \mathbf{r}'|) = -\text{div}_{\mathbf{r}'} \mathbf{a}(|\mathbf{r} - \mathbf{r}'|),$$
$$\text{curl } \mathbf{a}(|\mathbf{r} - \mathbf{r}'|) = -\text{curl}_{\mathbf{r}'} \mathbf{a}(|\mathbf{r} - \mathbf{r}'|),$$
$$\nabla^2 \phi(|\mathbf{r} - \mathbf{r}'|) = \nabla_{\mathbf{r}'}^2 \phi(|\mathbf{r} - \mathbf{r}'|).$$

Why bother?
Reciprocity; except for
sign it does not matter if
we exchange the
source and field points



$$\mathcal{L}_{\mathbf{r}'}\phi(\mathbf{r}') = -f(\mathbf{r}')$$

$$\mathcal{L}_{\mathbf{r}'}G(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}'),$$



$$\int_{\mathcal{V}} \mathcal{L}_{\mathbf{r}'}G(\mathbf{r}, \mathbf{r}') f(\mathbf{r})dV = - \int_{\mathcal{V}} \delta(\mathbf{r} - \mathbf{r}')f(\mathbf{r})dV = -f(\mathbf{r}').$$

$$\mathcal{L}_{\mathbf{r}'}\phi(\mathbf{r}') = \int_{\mathcal{V}} \mathcal{L}_{\mathbf{r}'}G(\mathbf{r}, \mathbf{r}')f(\mathbf{r})dV = \mathcal{L}_{\mathbf{r}'} \int_{\mathcal{V}} G(\mathbf{r}, \mathbf{r}')f(\mathbf{r})dV,$$

$$\phi(\mathbf{r}') = \int_{\mathcal{V}} G(\mathbf{r}, \mathbf{r}')f(\mathbf{r})dV.$$

$$G_2(\mathbf{r}, \mathbf{r}') = \frac{1}{2\pi} \ln \left(\frac{|\mathbf{r} - \mathbf{r}'|}{r_{\text{ref}}} \right),$$

$$G_3(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|}$$

$$\phi(\mathbf{r}') = \int_{\mathcal{V}} G(\mathbf{r}, \mathbf{r}') f(\mathbf{r}) dV.$$

$$\phi(\mathbf{r}) = \int_{\mathcal{V}} G(\mathbf{r}, \mathbf{r}') f(\mathbf{r}') dV'.$$

But what if boundaries are present?

$$\begin{aligned} \phi(\mathbf{r}) = & \int_{\mathcal{V}} G(\mathbf{r}, \mathbf{r}') f(\mathbf{r}') dV' \\ & + \int_{\partial\mathcal{V}} \left(-\phi(\mathbf{r}') \partial_{\mathbf{n}'} G(\mathbf{r}, \mathbf{r}') + G(\mathbf{r}, \mathbf{r}') \partial_{\mathbf{n}'} \phi(\mathbf{r}') \right) da'. \end{aligned}$$

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}, \quad G_3(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi |\mathbf{r} - \mathbf{r}'|}$$

$$A_i(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} \frac{J_i(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV',$$

$$\mathbf{A}(\mathbf{r}) = A_x \mathbf{e}_x + A_y \mathbf{e}_y + A_z \mathbf{e}_z = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'.$$

This works only in Cartesian Coordinates

$$\begin{aligned} \mathbf{B}(\mathbf{r}) &= \text{curl } \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} \text{curl} \left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) dV' \\ &= \frac{\mu_0}{4\pi} \int_{\mathcal{V}} \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \text{curl } \mathbf{J}(\mathbf{r}') - \mathbf{J}(\mathbf{r}') \times \text{grad} \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \right) dV' \\ &= \frac{\mu_0}{4\pi} \int_{\mathcal{V}} \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dV'. \end{aligned}$$

But wait a minute: Are we finished? Are we sure that the divergence of the vector potential is zero as it was required for the Laplace equation?

$$\begin{aligned}\operatorname{div} \mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \int_{\mathcal{V}'} \operatorname{div} \left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) dV' \\ &= \frac{\mu_0}{4\pi} \int_{\mathcal{V}'} \left(\mathbf{J}(\mathbf{r}') \cdot \operatorname{grad} \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) + \frac{1}{|\mathbf{r} - \mathbf{r}'|} \operatorname{div} \mathbf{J}(\mathbf{r}') \right) dV' \\ &= \frac{\mu_0}{4\pi} \int_{\mathcal{V}'} \mathbf{J}(\mathbf{r}') \cdot \operatorname{grad} \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) dV' \\ &= -\frac{\mu_0}{4\pi} \int_{\mathcal{V}'} \mathbf{J}(\mathbf{r}') \cdot \operatorname{grad}_{\mathbf{r}'} \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) dV' \\ &= -\frac{\mu_0}{4\pi} \int_{\mathcal{V}'} \left(\operatorname{div}_{\mathbf{r}'} \left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) - \frac{1}{|\mathbf{r} - \mathbf{r}'|} \operatorname{div}_{\mathbf{r}'} \mathbf{J}(\mathbf{r}') \right) dV' \\ &= -\frac{\mu_0}{4\pi} \int_{\mathcal{V}'} \operatorname{div}_{\mathbf{r}'} \left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) dV' = -\frac{\mu_0}{4\pi} \int_{\partial \mathcal{V}'} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \cdot d\mathbf{a}' .\end{aligned}$$

Current loops must always be closed and must not leave the problem domain

$$\mathbf{A}(\mathbf{r}) = A_x \mathbf{e}_x + A_y \mathbf{e}_y + A_z \mathbf{e}_z = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'.$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int_{\mathcal{C}} \frac{d\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$$

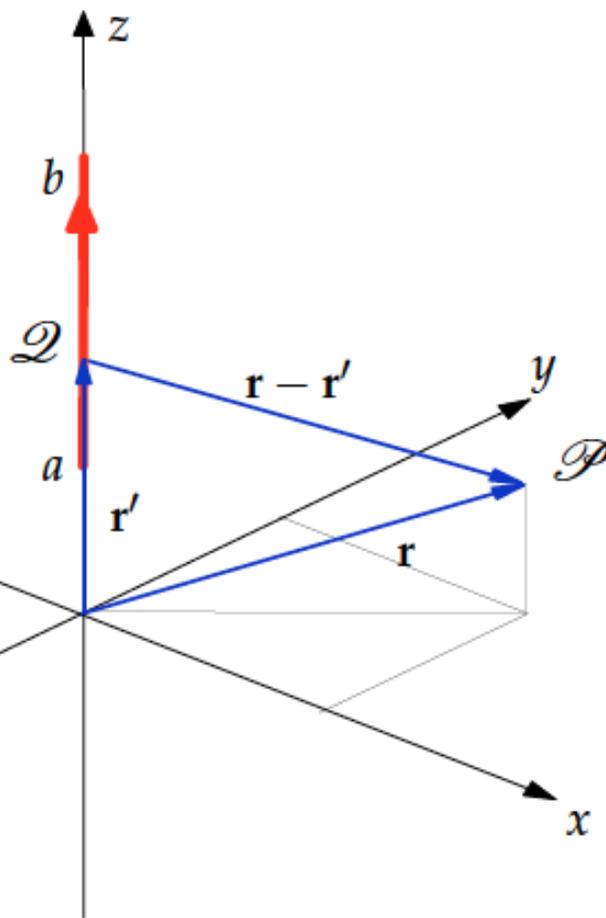
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int_{\mathcal{C}} \frac{d\mathbf{r}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3},$$

Vector Potential of a Line Current

$$A_z(x, y, z) = \frac{\mu_0 I}{4\pi} \int_a^b \frac{dz_c}{|\mathbf{r} - \mathbf{r}'|} = \frac{\mu_0 I}{4\pi} \int_a^b \frac{dz_c}{\sqrt{x^2 + y^2 + (z - z_c)^2}}$$

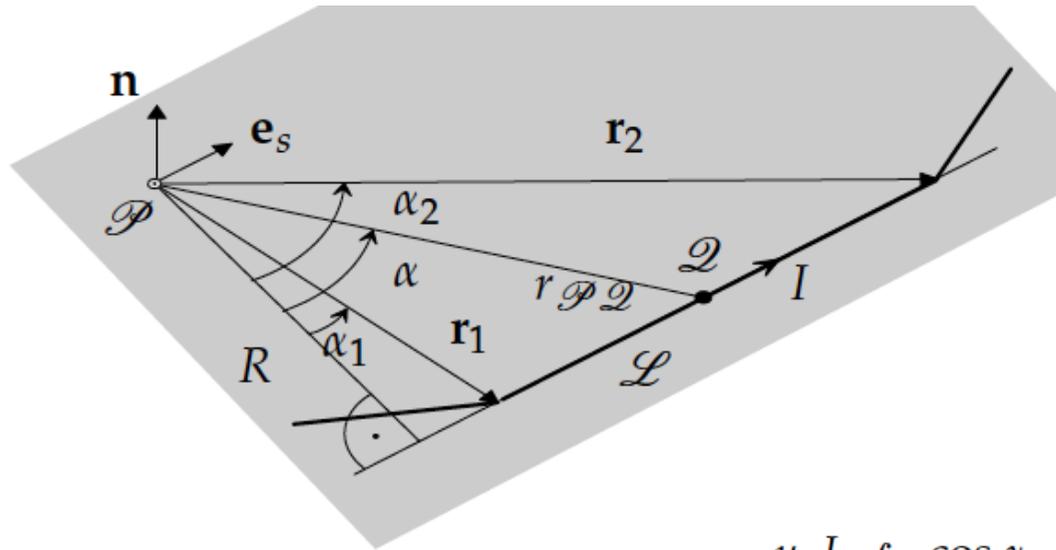
$$= \frac{-\mu_0 I}{4\pi} \ln \left((z - z_c) + \sqrt{x^2 + y^2 + (z - z_c)^2} \right) \Big|_a^b$$

$$= \frac{\mu_0 I}{4\pi} \ln \frac{z - a + \sqrt{x^2 + y^2 + (z - a)^2}}{z - b + \sqrt{x^2 + y^2 + (z - b)^2}}.$$



Caution: Infinitely long line currents have infinite energy

Field of a Line Current Segment



$$\begin{aligned}
 \mathbf{B}(\mathcal{P}) &= \frac{\mu_0 I}{4\pi} \int_{\mathcal{L}} \frac{\cos \alpha}{r_{\mathcal{P}\mathcal{Q}}^2} d\mathbf{r}' = \frac{\mu_0 I}{4\pi R} \mathbf{n} \int_{\alpha_1}^{\alpha_2} \cos \alpha d\alpha = \frac{\mu_0 I}{4\pi R} (\sin \alpha_2 - \sin \alpha_1) \mathbf{n} \\
 &= \frac{\mu_0 I}{4\pi} \frac{\cos \alpha_2 + \cos \alpha_1}{R} \frac{\sin \alpha_2 - \sin \alpha_1}{\cos \alpha_2 + \cos \alpha_1} \mathbf{n} \\
 &= \frac{\mu_0 I}{4\pi} \left(\frac{1}{|\mathbf{r}_1|} + \frac{1}{|\mathbf{r}_2|} \right) \frac{\sin(\alpha_2 - \alpha_1)}{1 + \cos(\alpha_2 - \alpha_1)} \mathbf{n} \\
 &= \frac{\mu_0 I}{4\pi} \left(\frac{1}{|\mathbf{r}_1|} + \frac{1}{|\mathbf{r}_2|} \right) \frac{\sin(\alpha_2 - \alpha_1)}{1 + \frac{\mathbf{r}_1 \cdot \mathbf{r}_2}{|\mathbf{r}_1| |\mathbf{r}_2|}} \frac{\mathbf{r}_1 \times \mathbf{r}_2}{|\mathbf{r}_1| |\mathbf{r}_2| \sin(\alpha_2 - \alpha_1)} \\
 &= \frac{\mu_0 I}{4\pi} \frac{|\mathbf{r}_1| + |\mathbf{r}_2|}{|\mathbf{r}_1| |\mathbf{r}_2| + \mathbf{r}_1 \cdot \mathbf{r}_2} \frac{\mathbf{r}_1 \times \mathbf{r}_2}{|\mathbf{r}_1| |\mathbf{r}_2|} ,
 \end{aligned}$$

Field of a Line Current

$$\begin{aligned} \lim_{a,b \rightarrow \pm\infty} \ln \frac{z - a + \sqrt{x^2 + y^2 + (z - a)^2}}{z - b + \sqrt{x^2 + y^2 + (z - b)^2}} &= \lim_{a,b \rightarrow \pm\infty} \ln \frac{-a + |a| \sqrt{1 + \frac{x^2 + y^2}{a^2}}}{-b + |b| \sqrt{1 + \frac{x^2 + y^2}{b^2}}} \\ &= \lim_{a,b \rightarrow \pm\infty} \ln \frac{-a - a(1 + \frac{x^2 + y^2}{2a^2} + \dots)}{-b + b(1 + \frac{x^2 + y^2}{2b^2} + \dots)} = \lim_{a,b \rightarrow \pm\infty} \ln \frac{-2a}{-b + b + \frac{x^2 + y^2}{2b}} \\ &= \lim_{a,b \rightarrow \pm\infty} \ln \frac{-4ab}{x^2 + y^2}. \end{aligned}$$

$$A_z(x, y) = \lim_{a,b \rightarrow \pm\infty} \frac{\mu_0 I}{4\pi} \ln \left(\frac{-4ab}{x_0^2 + y_0^2} \right) - \frac{\mu_0 I}{4\pi} \ln \left(\frac{x^2 + y^2}{x_0^2 + y_0^2} \right).$$

$$\mathbf{A}(x, y) = -\frac{\mu_0 I}{4\pi} \ln \left(\frac{x^2 + y^2}{x_0^2 + y_0^2} \right) \mathbf{e}_z = -\frac{\mu_0 I}{2\pi} \ln \left(\frac{r}{r_{\text{ref}}} \right) \mathbf{e}_z,$$

**Problem solved, but reference radius has physical significance:
Return path for sum-currents**

Field of a Line Current

$$A_{\varphi}(r, z) = \frac{\mu_0 I r_c}{\pi \sqrt{(r + r_c)^2 + z^2}} \int_0^{\pi/2} \frac{2 \sin^2 \psi - 1}{\sqrt{1 - k^2 \sin^2 \psi}} d\psi.$$

Appearance of elliptic integrals:
To be solved numerically.

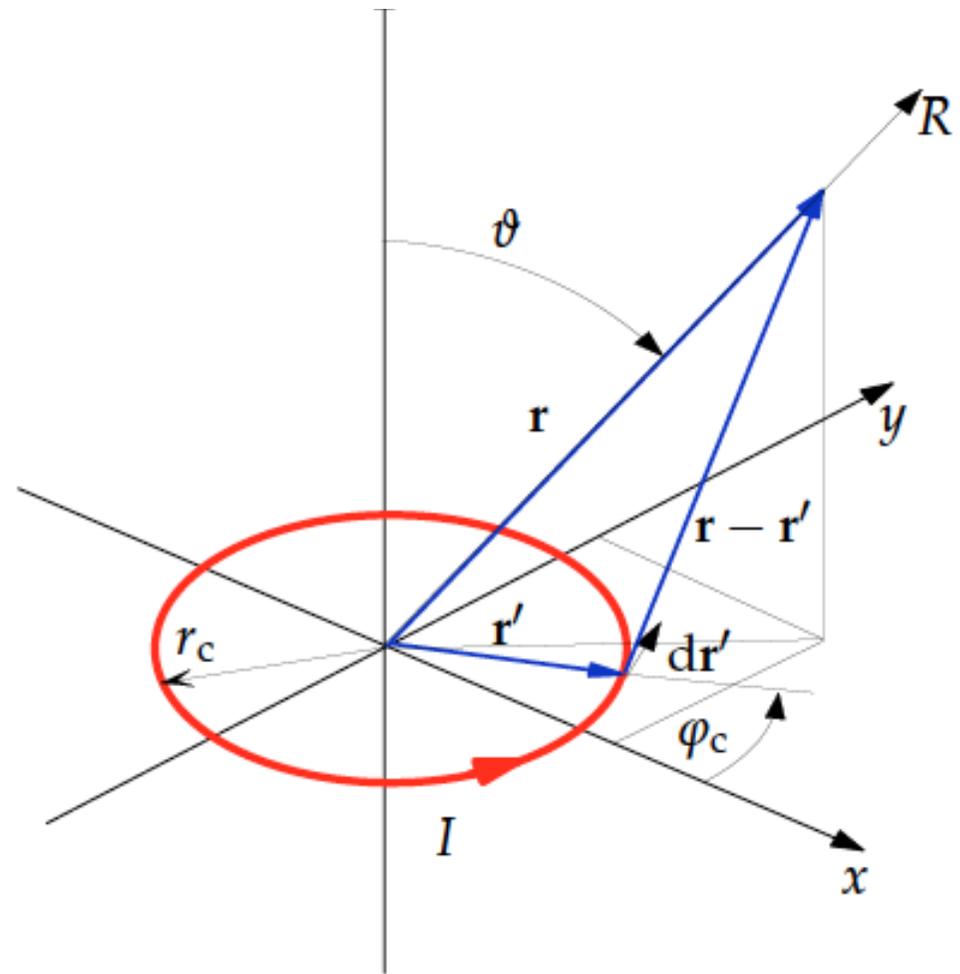
On axis:

$$A_{\varphi}(r, z) = \frac{\mu_0 I r_c^2}{4} \frac{r}{(r_c^2 + z^2)^{\frac{3}{2}}},$$

$$B_z(z) = \frac{\mu_0 I}{2} \frac{r_c^2}{(r_c^2 + z^2)^{\frac{3}{2}}}.$$

In the center:

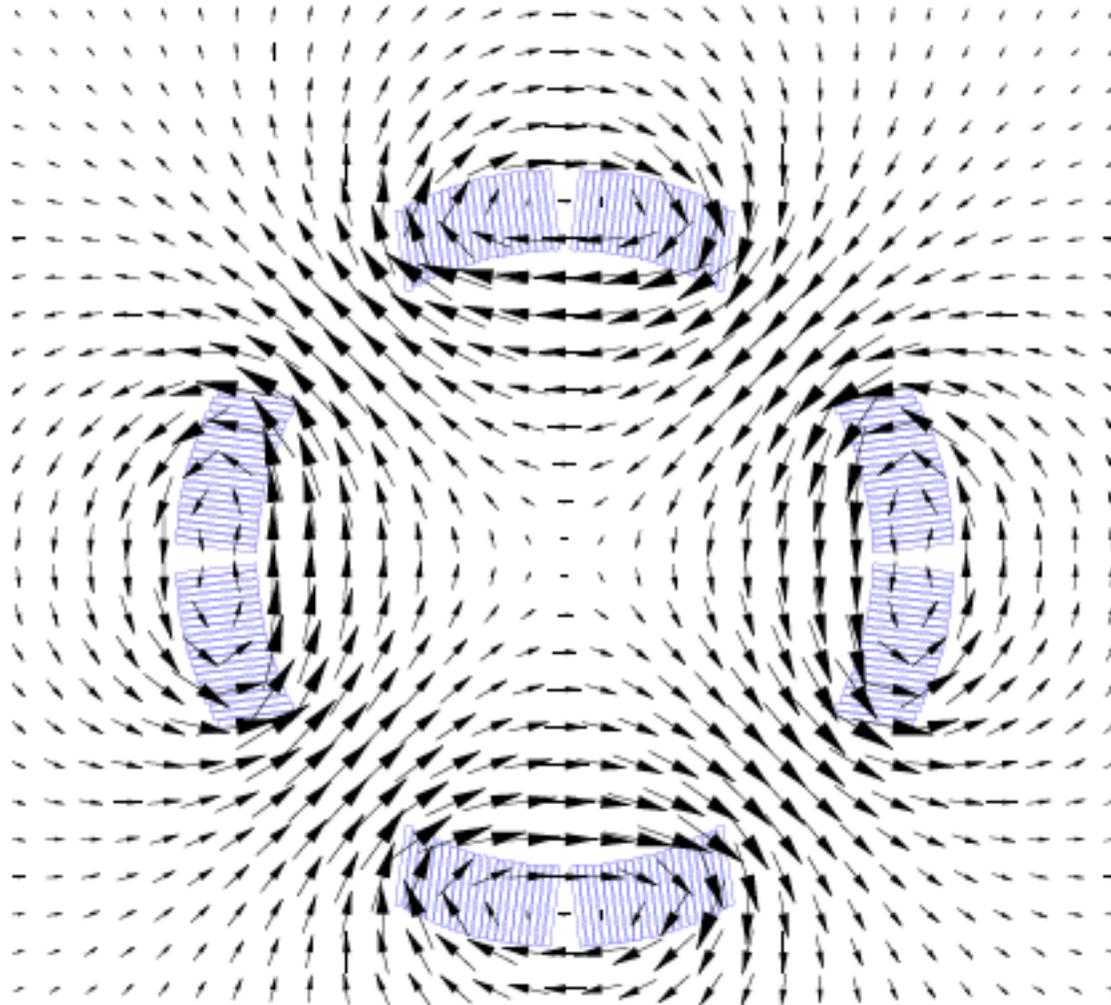
$$B_z(z = 0) = \frac{\mu_0 I}{2r_c}.$$



Cash-Back 3

The Coil Field of Superconducting Accelerator Magnets

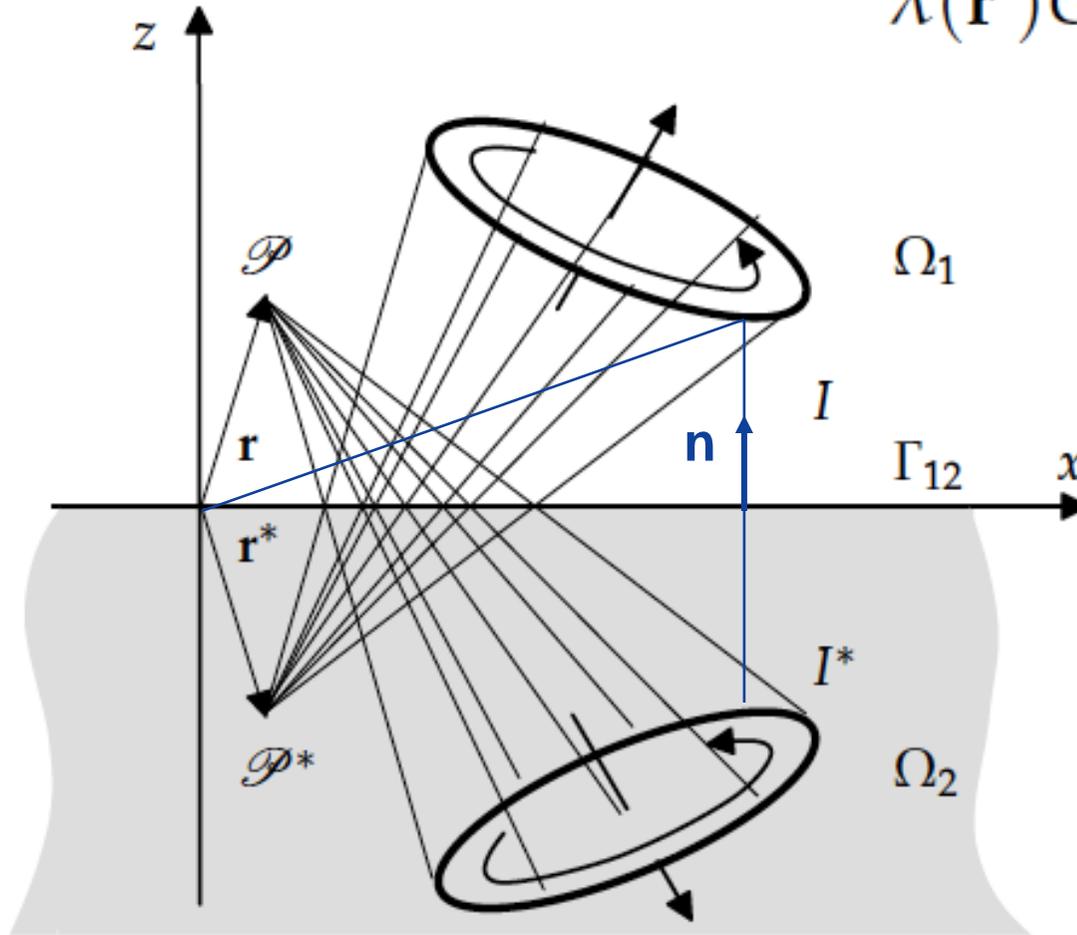
The Imaging Current Method



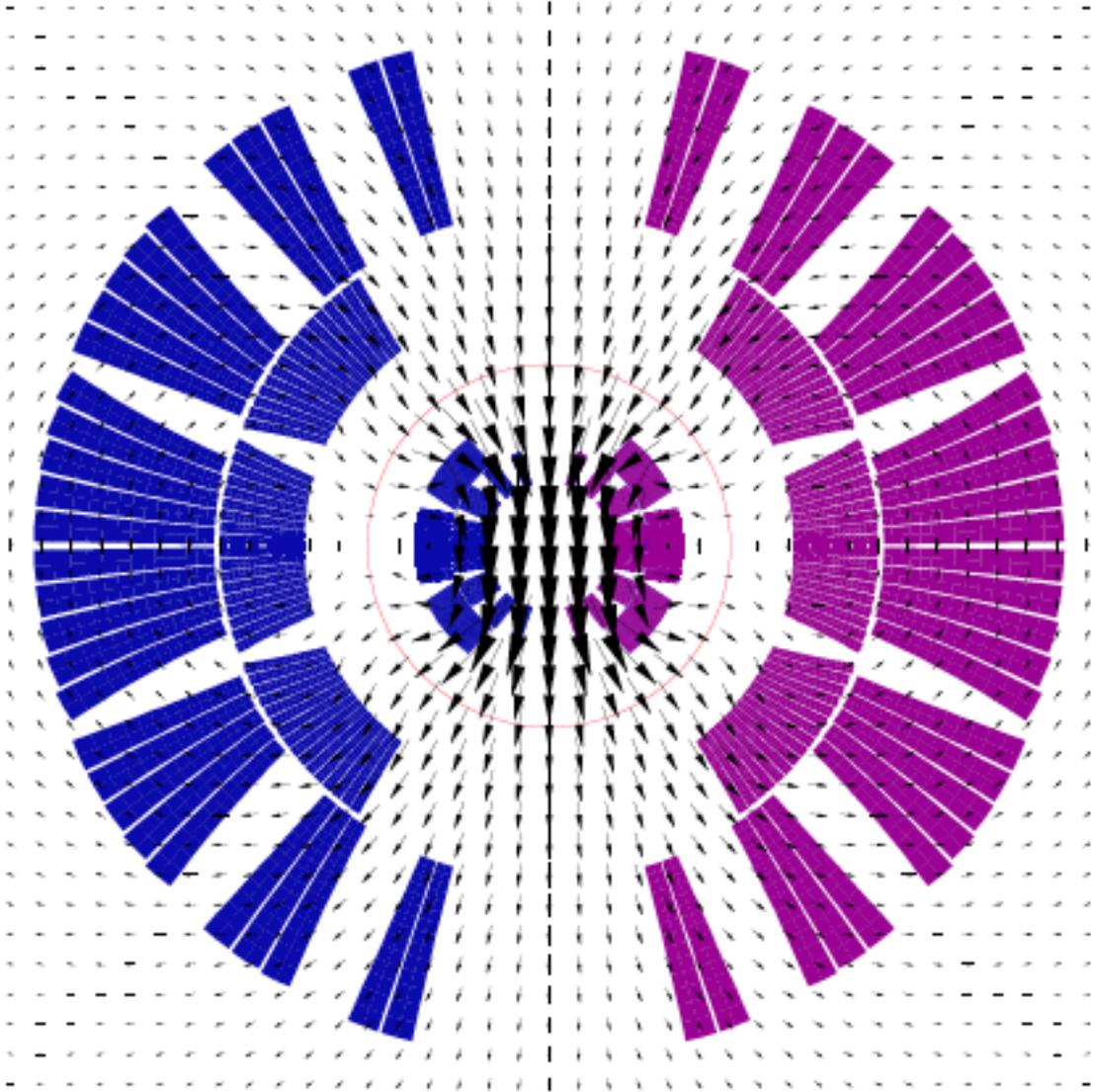
- Domain 1: Domain with current sources
- Domain 2: Highly permeable material
 - All imaging currents must be in domain 2
 - The sources and the images must create a field that satisfies the continuity conditions at the interface between domains 1 and 2
 - The image of the image must be the original source
 - The field generated in domain 1 is identical to the source field plus the field from the (iron) magnetization.
 - The field generated in domain 2 has no physical significance

The Imaging Current Method

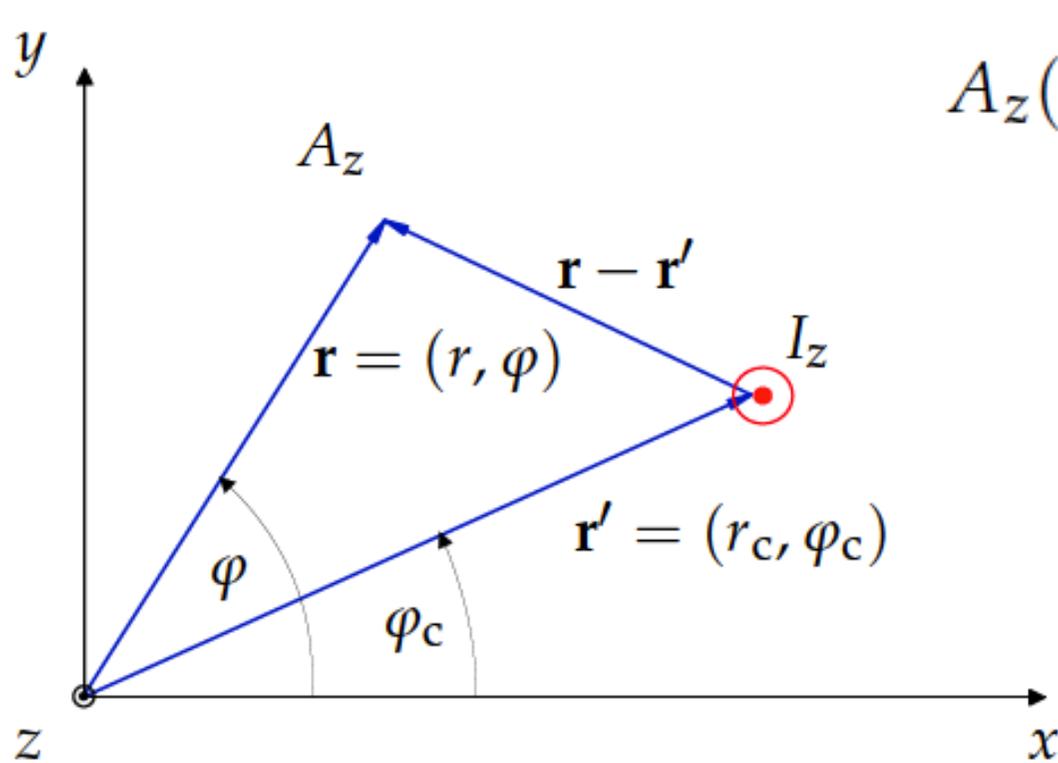
$$\lambda(\mathbf{r}')G(\mathbf{r}, T\mathbf{r}') = \lambda(\mathbf{r})G(T\mathbf{r}, \mathbf{r}')$$



The Imaging Current Method



The Field of a Line Current (2D)

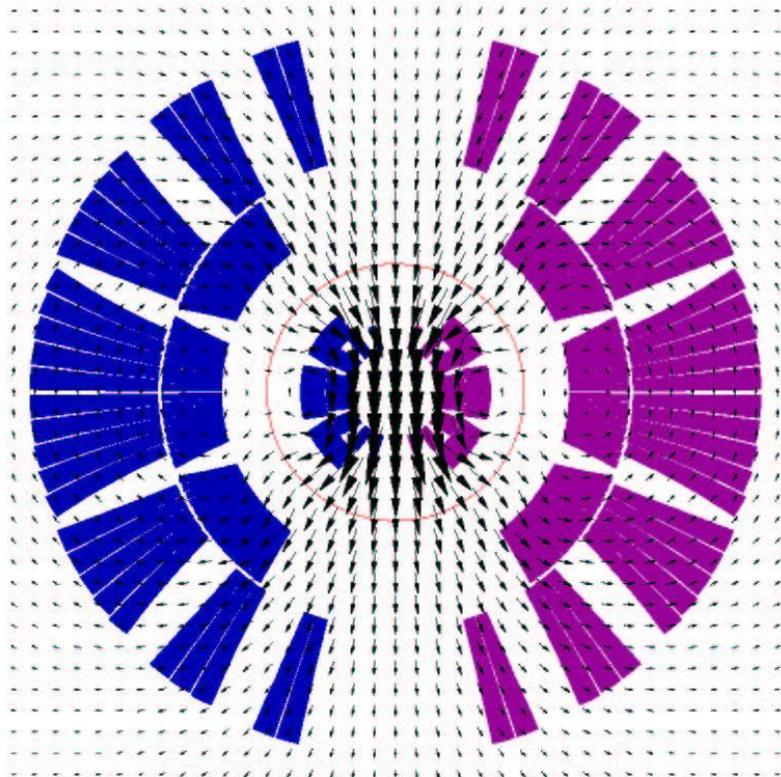


$$A_z(\mathbf{r}) = -\frac{\mu_0 I}{2\pi} \ln \left(\frac{|\mathbf{r} - \mathbf{r}'|}{r_{\text{ref}}} \right)$$

$$A_z(r, \varphi) = -\frac{\mu_0 I}{2\pi} \ln \left(\frac{r_c}{r_{\text{ref}}} \right) + \frac{\mu_0 I}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{r}{r_c} \right)^n \cos n(\varphi - \varphi_c)$$

$$B_n(r_0) = -\frac{\mu_0 I}{2\pi r_c} \left(\frac{r_0}{r_c} \right)^{n-1} \cos n\varphi_c, \quad A_n(r_0) = \frac{\mu_0 I}{2\pi r_c} \left(\frac{r_0}{r_c} \right)^{n-1} \sin n\varphi_c.$$

$$B_n(r_0) = - \sum_{k=1}^K \frac{\mu_0 I_k}{2\pi} \frac{r_0^{n-1}}{r_{c,k}^n} \left(1 + \lambda_\mu \left(\frac{r_{c,k}}{r_y} \right)^{2n} \right) \cos n\varphi_{c,k},$$



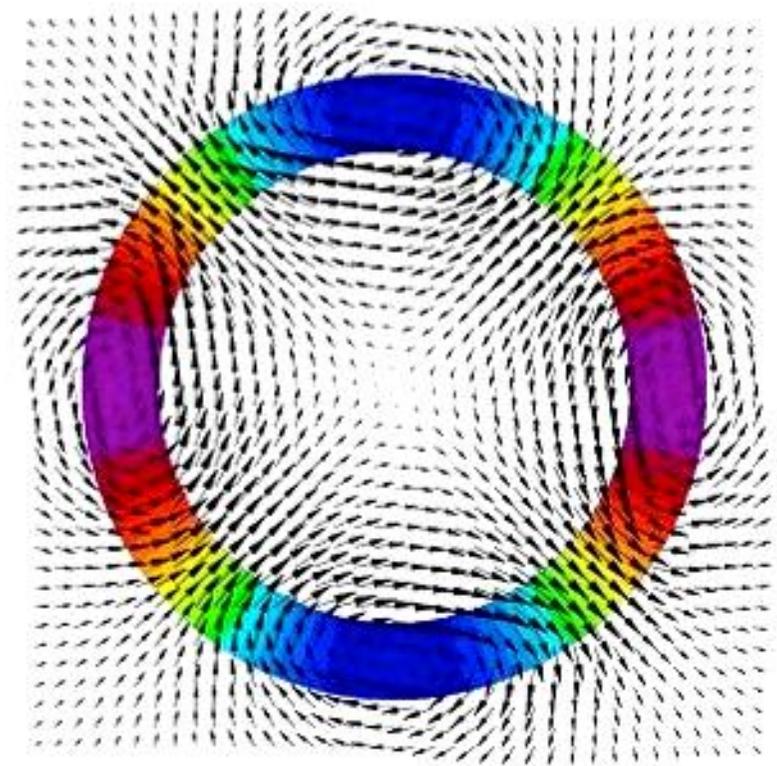
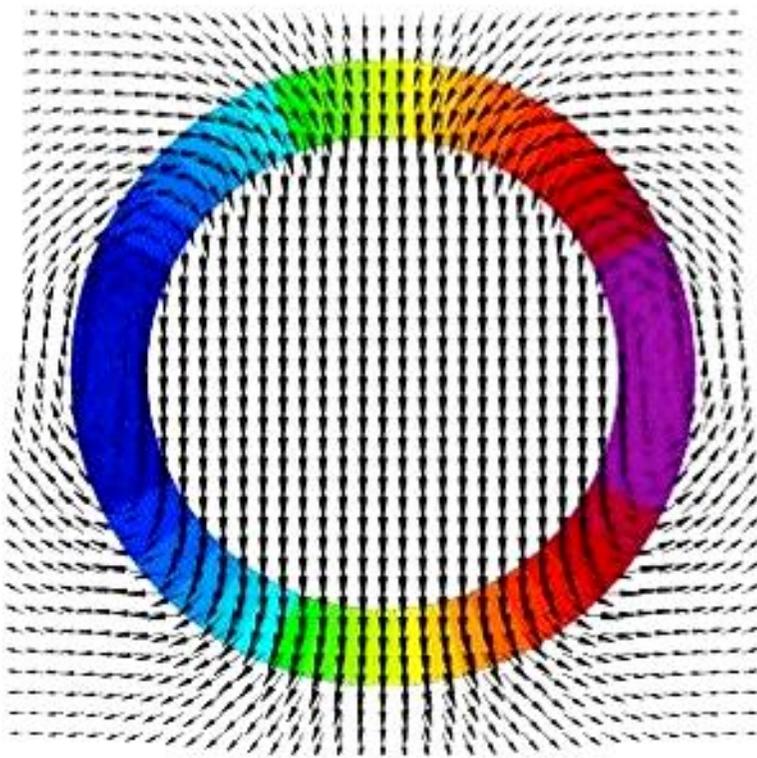
$$\lambda_\mu I := \frac{\mu_r - 1}{\mu_r + 1} I.$$

$$\frac{B_N^{\text{imag}}}{B_N + B_N^{\text{imag}}} \approx \left(1 + \left(\frac{r_y}{r} \right)^{2N} \right)^{-1}.$$

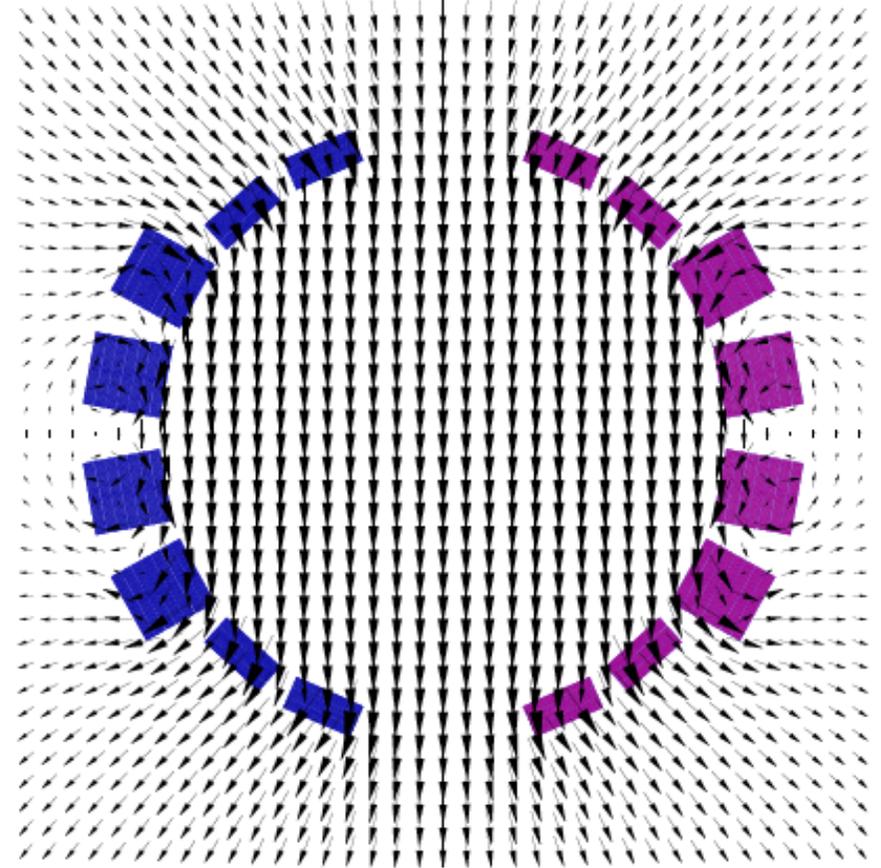
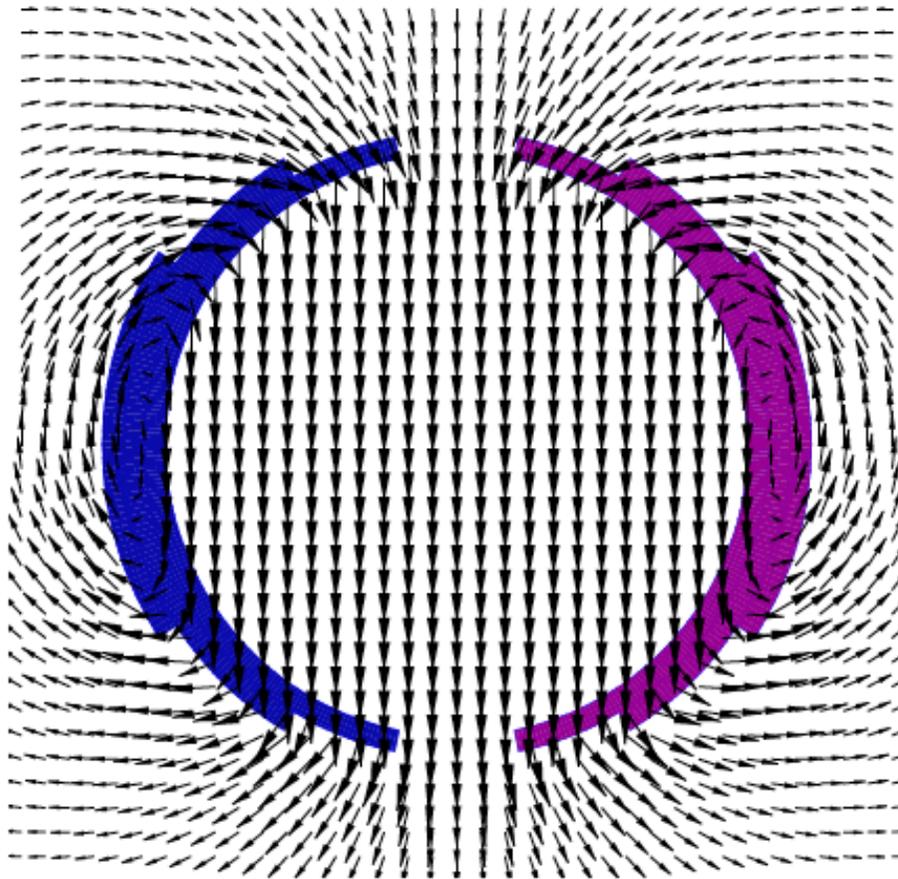
Ideal Current Distributions

$$B_n(r_0) = \int_{r_a}^{r_e} \int_0^{2\pi} -\frac{\mu_0 J_E}{2\tau} J_c(B) = d(B_{c2} - B) \cos n\varphi_c r_c d\varphi_c dr_c$$

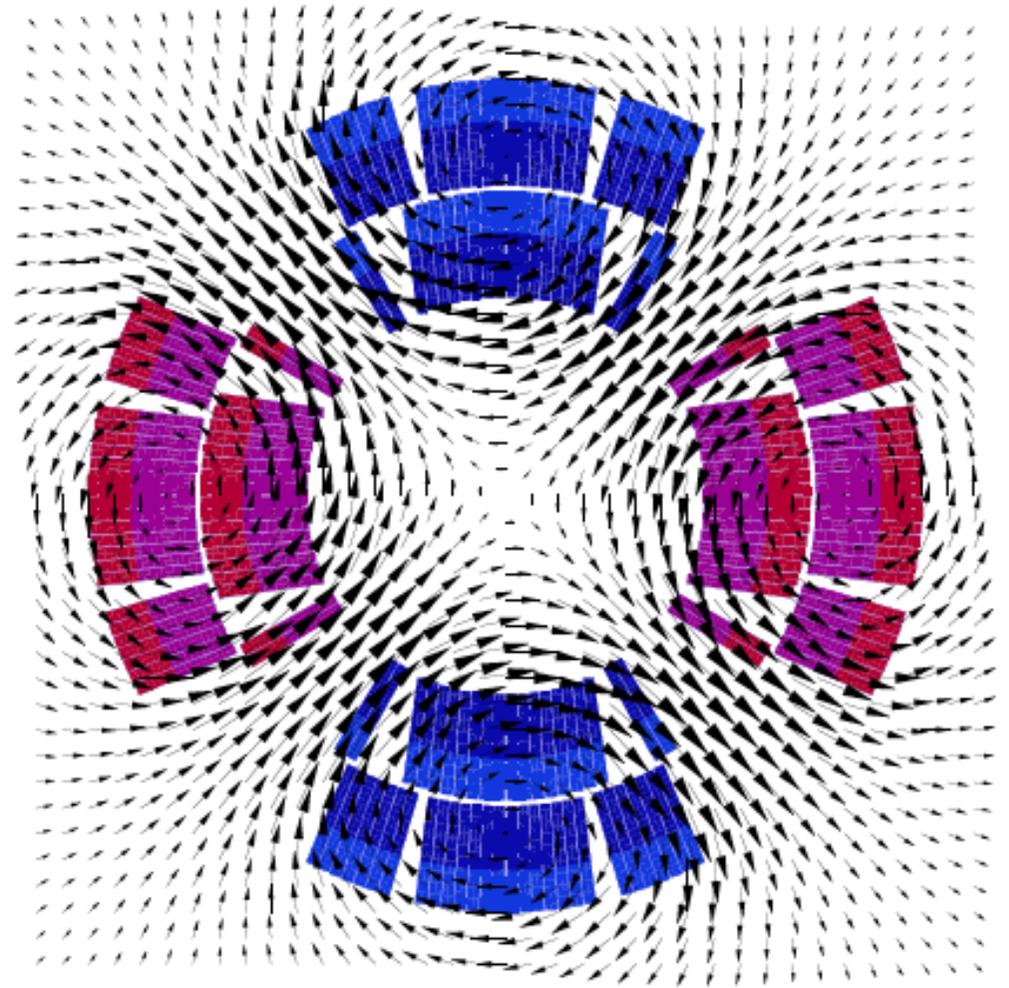
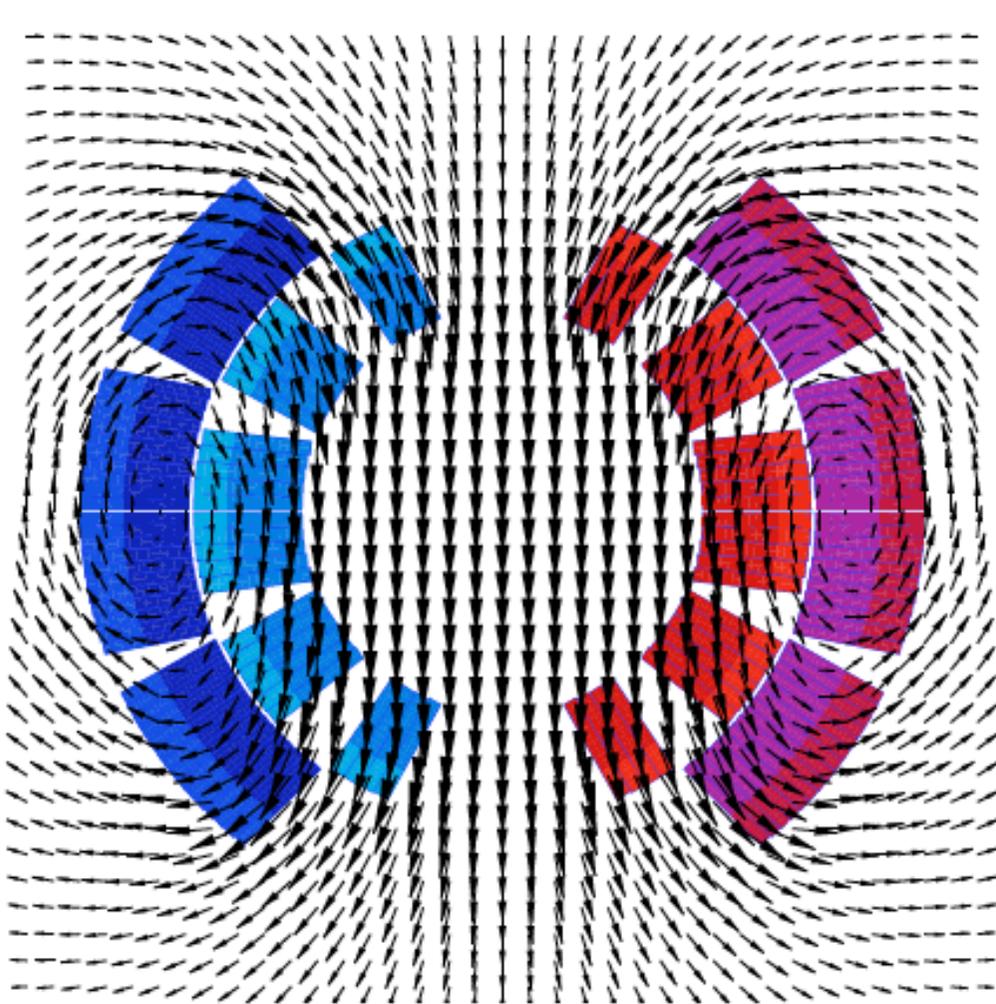
$$B = \frac{\mu_0}{2} \lambda_{\text{tot}} J_c (r_e - r_a) = \frac{\mu_0}{2} \lambda_{\text{tot}} d (B_{c2} - B) (r_e - r_a),$$



Coil-Block Approximations



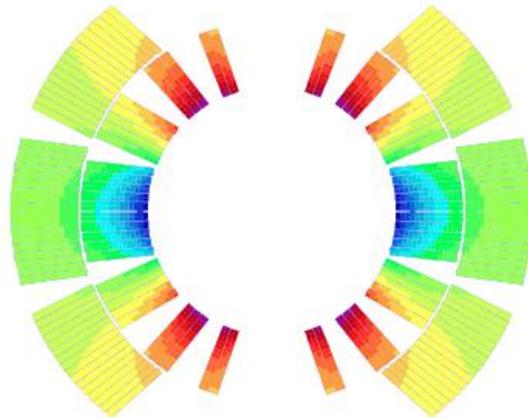
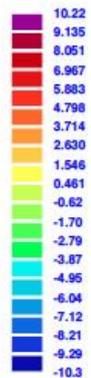
Coil-Block Approximations



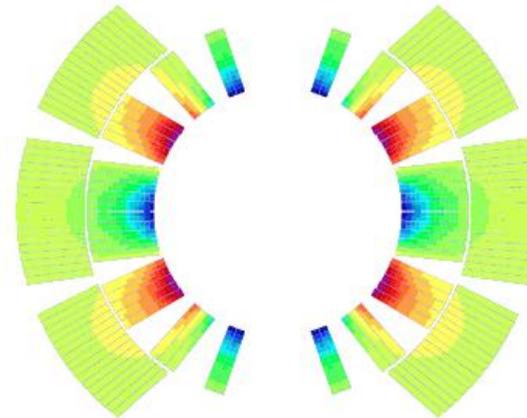
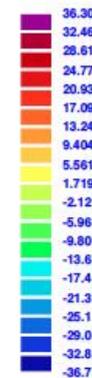
Generation of Multipole Field Errors

$$B_n(r_0) = - \sum_{k=1}^K \frac{\mu_0 I_k}{2\pi} \frac{r_0^{n-1}}{r_{c,k}^n} \left(1 + \lambda_\mu \left(\frac{r_{c,k}}{r_y} \right)^{2n} \right) \cos n\varphi_{c,k},$$

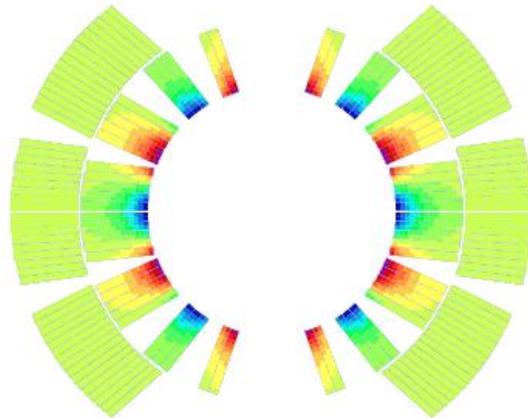
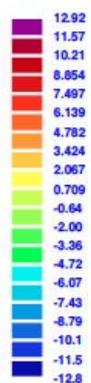
B3 (10E-4 T)



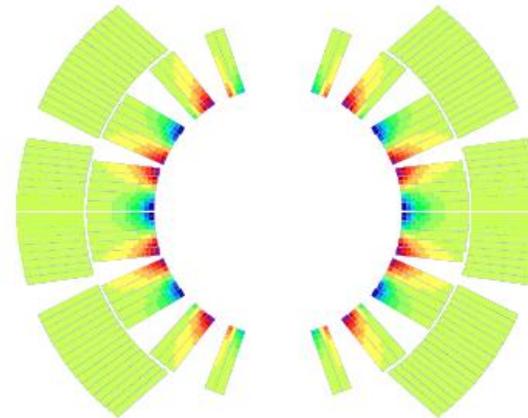
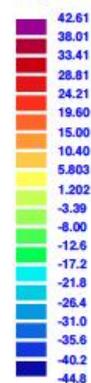
B5 (10E-5 T)



B7 (10E-5 T)



B9 (10E-6 T)



Sensitivity to Manufacturing Errors

$$B_n(r_0) = - \sum_{k=1}^K \frac{\mu_0 I_k}{2\pi} \frac{r_0^{n-1}}{r_{c,k}^n} \left(1 + \lambda_\mu \left(\frac{r_{c,k}}{r_y} \right)^{2n} \right) \cos n\varphi_{c,k},$$

$$\frac{\partial B_n(r_0)}{\partial \varphi_c} = - \frac{\mu_0 I_k}{2\pi} \frac{nr_0^{n-1}}{r_c^n} \left(1 + \left(\frac{r_c}{r_y} \right)^{2n} \right) \sin n\varphi_c,$$

$$\frac{\partial B_n(r_0)}{\partial r_c} = \frac{\mu_0 I_k}{2\pi} \frac{nr_0^{n-1}}{r_c^{n+1}} \left(1 - \left(\frac{r_c}{r_y} \right)^{2n} \right) \cos n\varphi_c.$$

Increase of the azimuthal coil size by 0.1 mm produces (in units of 10^{-4}):

$$b_1 = -14. \quad b_3 = 1.2 \quad b_5 = 0.03$$

Specified tolerances on coils: ± 0.025 mm

Coil Winding and Curing



$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}'} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'.$$

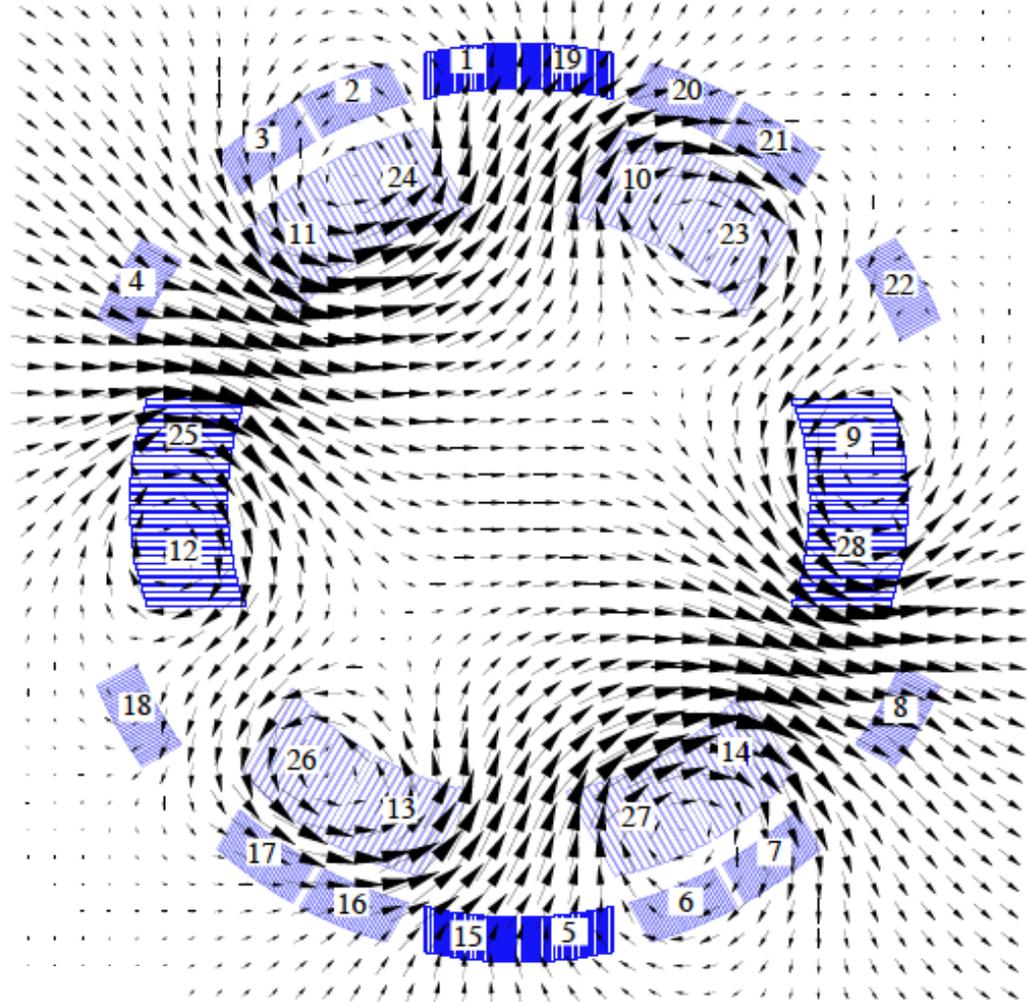
$$W = \frac{\mu_0}{8\pi} \int_{\mathcal{V}} \int_{\mathcal{V}'} \frac{\mathbf{J}(\mathbf{r}) \cdot \mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV' dV.$$

$$\begin{aligned} W &= \sum_{i=1}^n \sum_{j=1}^n W_{ij} = \frac{\mu_0}{8\pi} \sum_{i=1}^n \sum_{j=1}^n \int_{\mathcal{V}} \int_{\mathcal{V}'} \frac{\mathbf{J}_i(\mathbf{r}) \cdot \mathbf{J}_j(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV' dV \\ &= \frac{\mu_0}{8\pi} \sum_{i=1}^n \sum_{j=1}^n I_i I_j \int_{\mathcal{V}} \int_{\mathcal{V}'} \frac{\mathbf{J}_i(\mathbf{r}) \cdot \mathbf{J}_j(\mathbf{r}')}{I_i I_j |\mathbf{r} - \mathbf{r}'|} dV' dV. \end{aligned}$$

$$W = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n L_{ij} I_i I_j. \quad L_{ij} := \frac{\mu_0}{4\pi I_i I_j} \int_{\mathcal{V}} \int_{\mathcal{V}'} \frac{\mathbf{J}_i(\mathbf{r}) \cdot \mathbf{J}_j(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV' dV$$

Inductance

On the computer:



$$L_{ii} = \frac{2W_{ii}}{I^2} .$$

$$L_{ij} = \frac{1}{2} \left(\frac{2W_{ij}}{I^2} - L_{ii} - L_{jj} \right) .$$

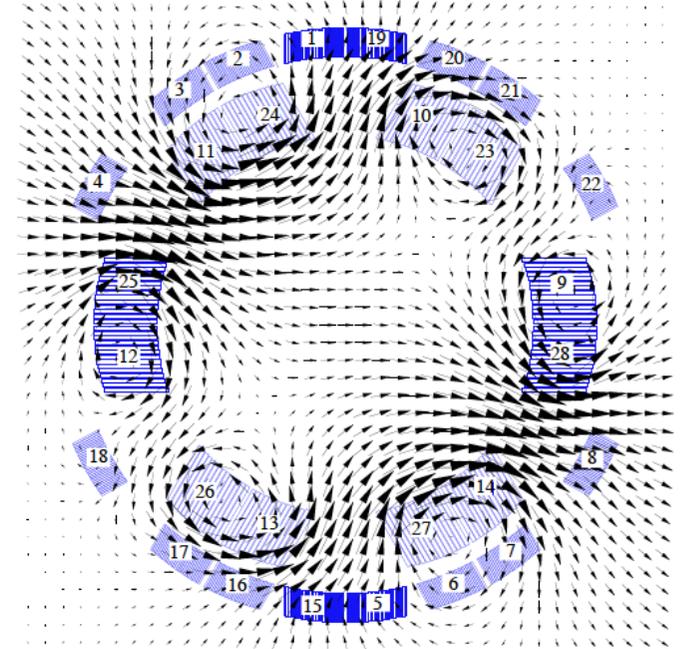
Inductance

$$\Phi_i = \sum_{j=1}^n L_{ij} I_j.$$

$$U_i = \frac{d\Phi_i}{dt} = \sum_{j=1}^n L_{ij} \frac{dI_j}{dt},$$

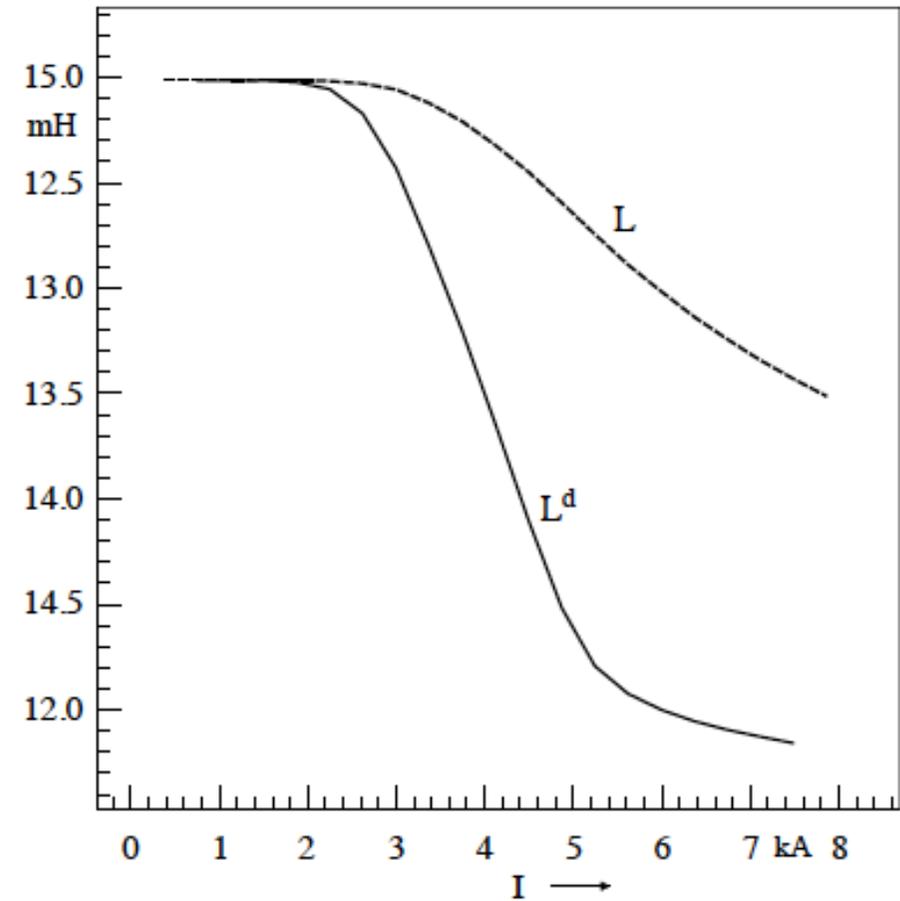
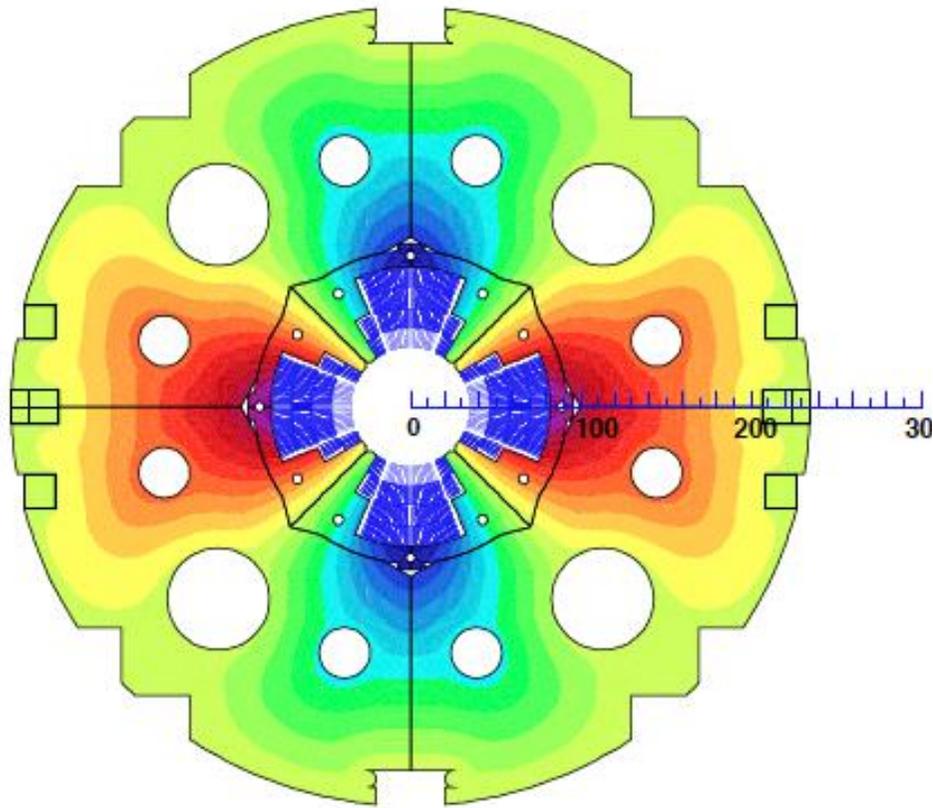
Careful: only
in the linear
case

$$U_{\text{Dipole}} = U_1 + U_2 = \sum_{j=3}^8 L_{1j} \frac{dI_j}{dt} + \sum_{j=3}^8 L_{2j} \frac{dI_j}{dt} = 0.$$



Coil	1	2	3	4	5	6	7	8
1	12.601	6.517	-0.245	0.252	0.478	-0.478	-0.252	0.245
2	6.517	12.601	-0.478	-0.252	0.245	-0.245	0.252	0.478
3	-0.245	-0.478	0.136	0.027	-0.010	0.009	-0.010	0.027
4	0.252	-0.252	0.027	0.136	0.027	-0.010	0.009	-0.010
5	0.478	0.245	-0.010	0.027	0.136	0.027	-0.010	0.009
6	-0.478	-0.245	0.009	-0.010	0.027	0.136	0.027	-0.010
7	-0.252	0.252	-0.010	0.009	-0.010	0.027	0.136	0.027
8	0.245	0.478	0.027	-0.010	0.009	-0.010	0.027	0.136

Inductance in nonlinear Circuits



$$U = L^d \frac{dI}{dt}$$

Inductance in nonlinear Circuits

$$U(t) = \frac{d\Phi}{dt} = \frac{d(L I)}{dt} = L \frac{dI}{dt} + I \frac{dL}{dt}.$$

Always true. Now calculate the differential of L

$$dL = \frac{\partial L}{\partial I} dI + \frac{\partial L}{\partial t} dt.$$

If the coil is moving

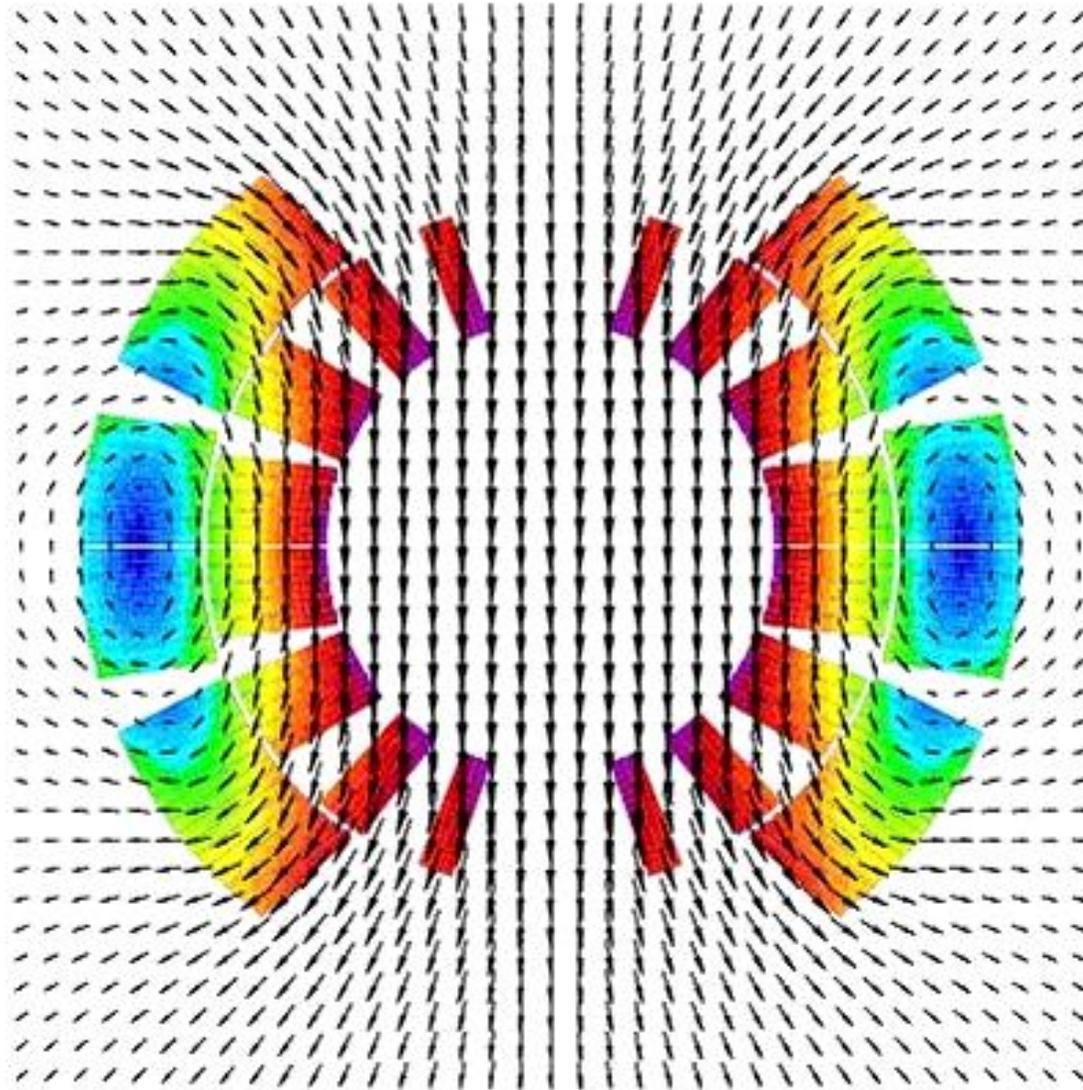
$$U(t) = \left(\frac{\partial L}{\partial I} I + L \right) \frac{dI}{dt} + I \frac{\partial L}{\partial t}.$$

$$L^d = L + I \frac{\partial L}{\partial I} = \frac{d\Phi}{dI}.$$

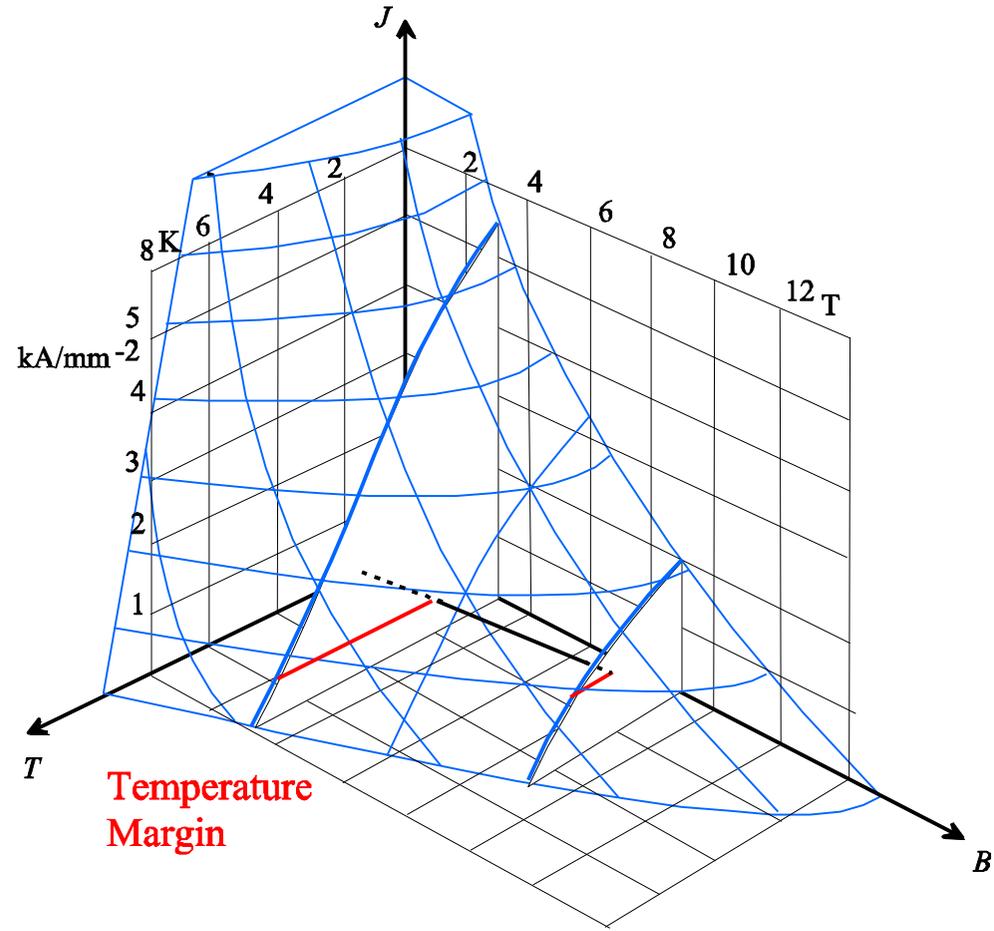
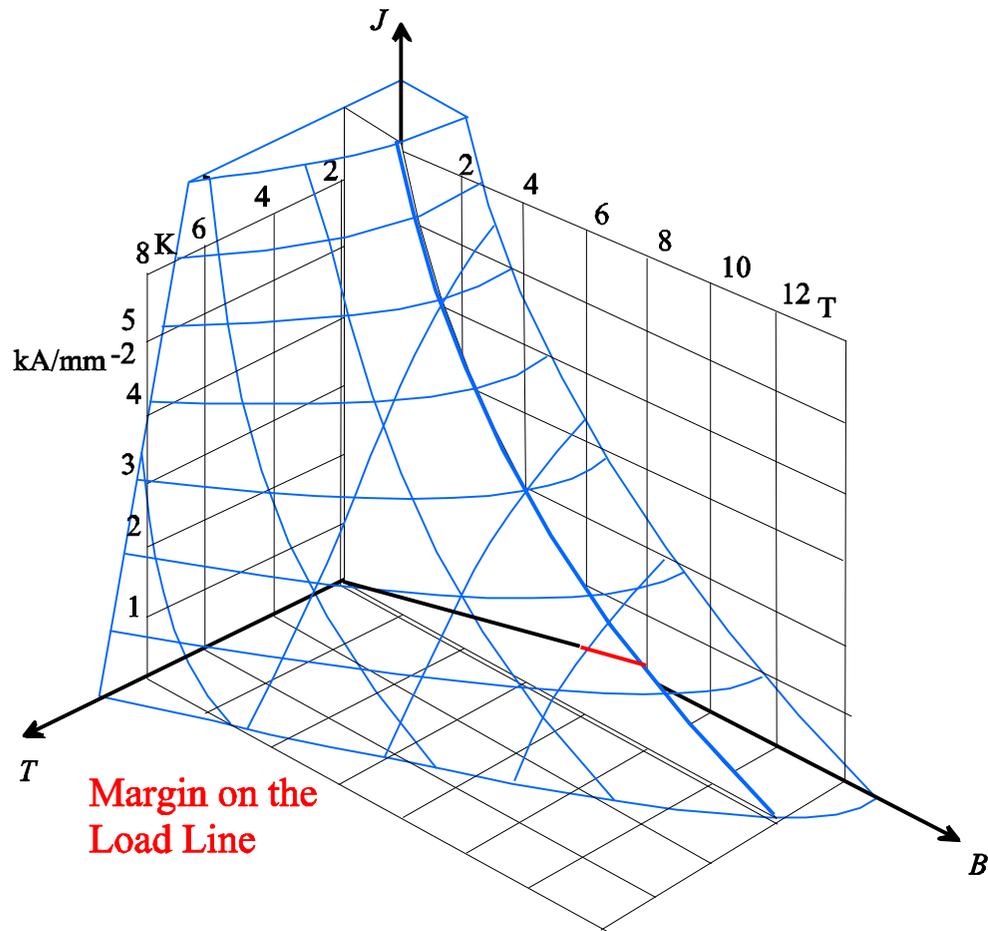
$$U = L^d \frac{dI}{dt}.$$

- The real world is often too difficult for back-of-the-envelope calculations
- We must use the computer to add the effect of 1000 strands
- For a nonlinear continuum we must use numerical approximation techniques

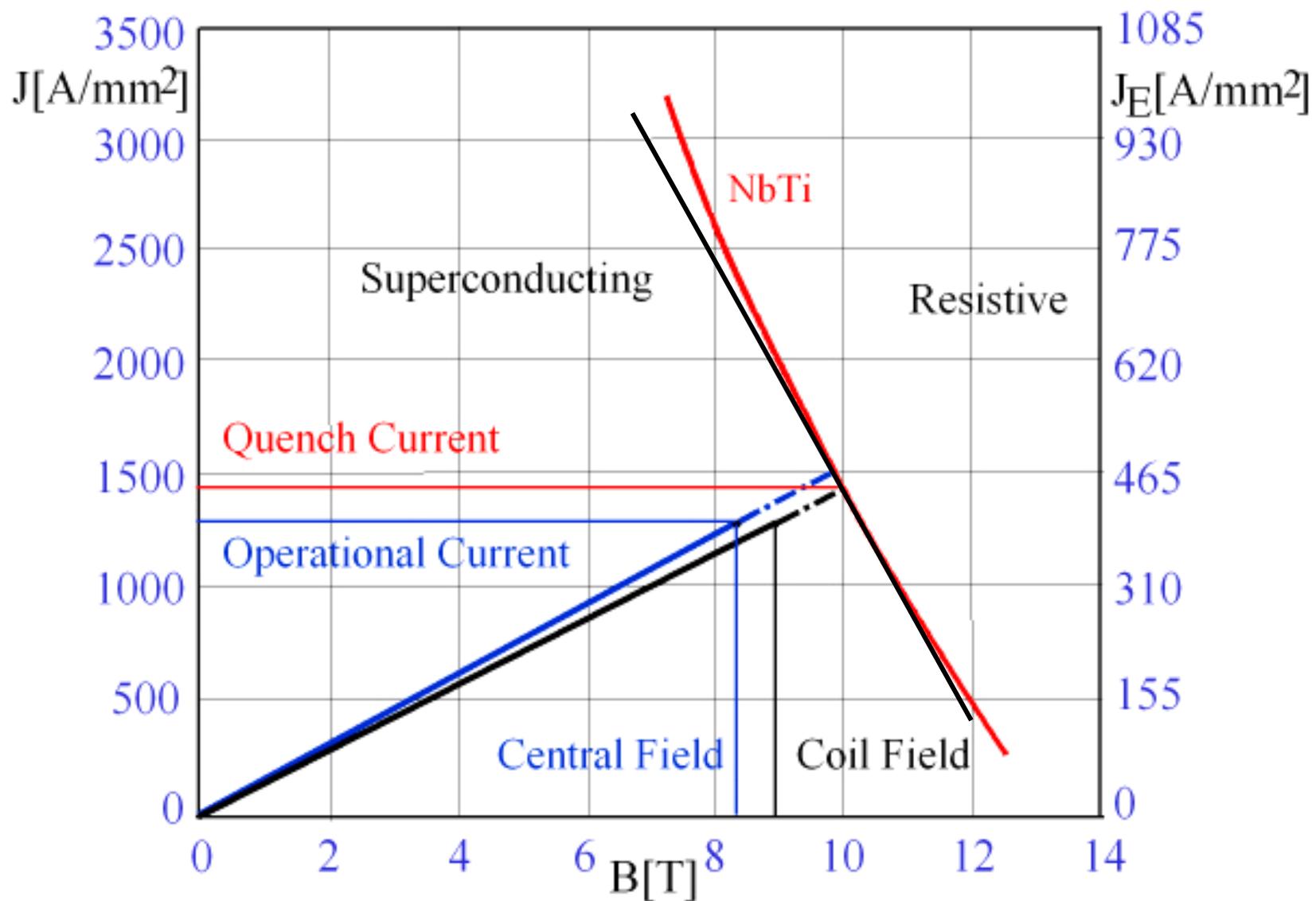
Field Distribution in the LHC Main Dipoles



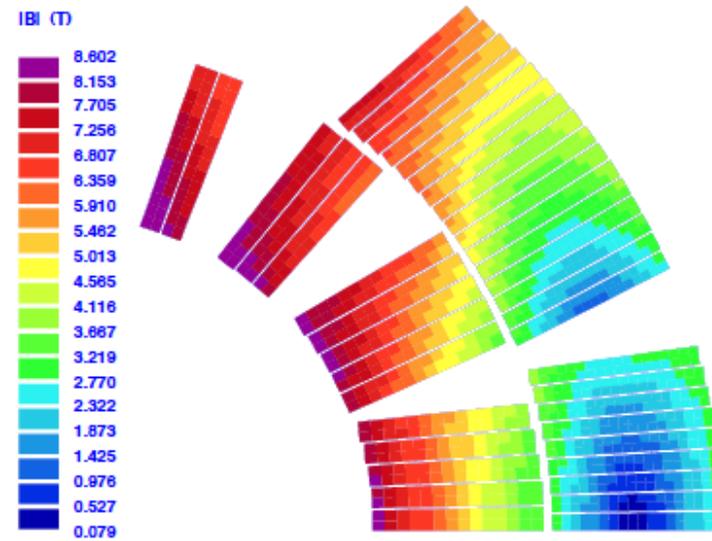
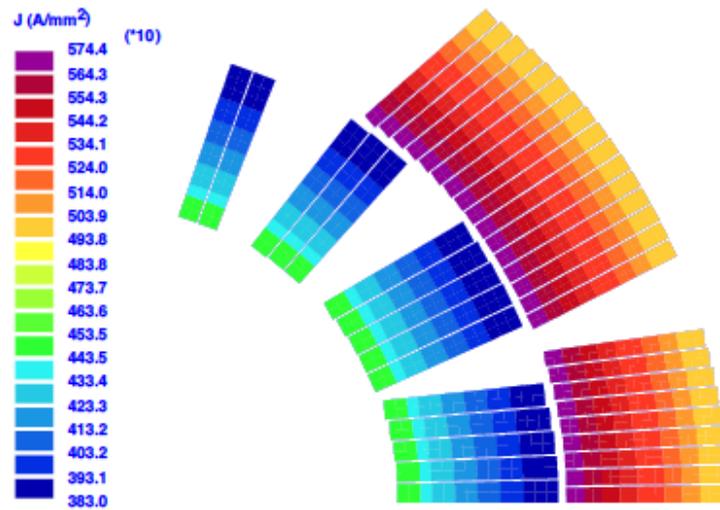
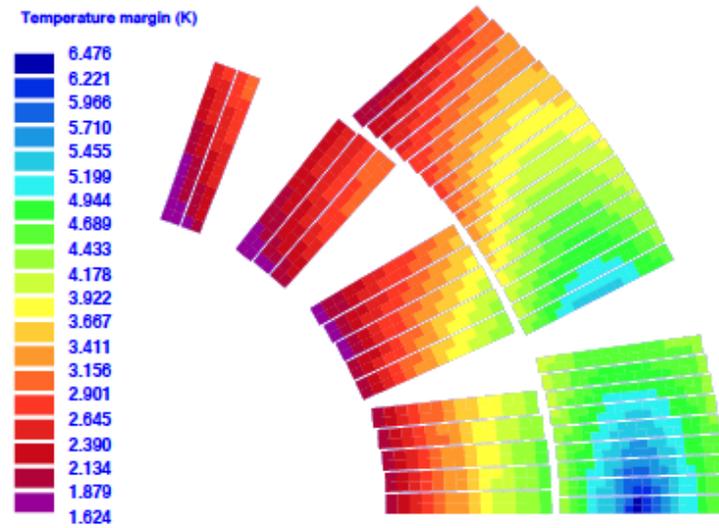
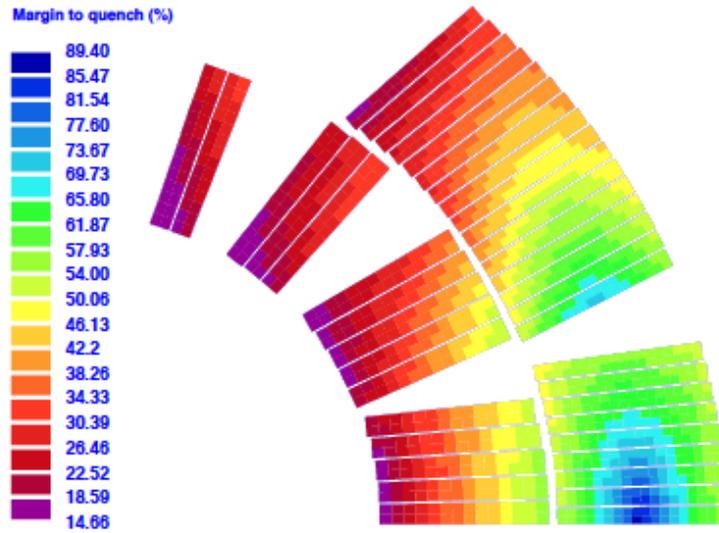
Critical Surface of NbTi



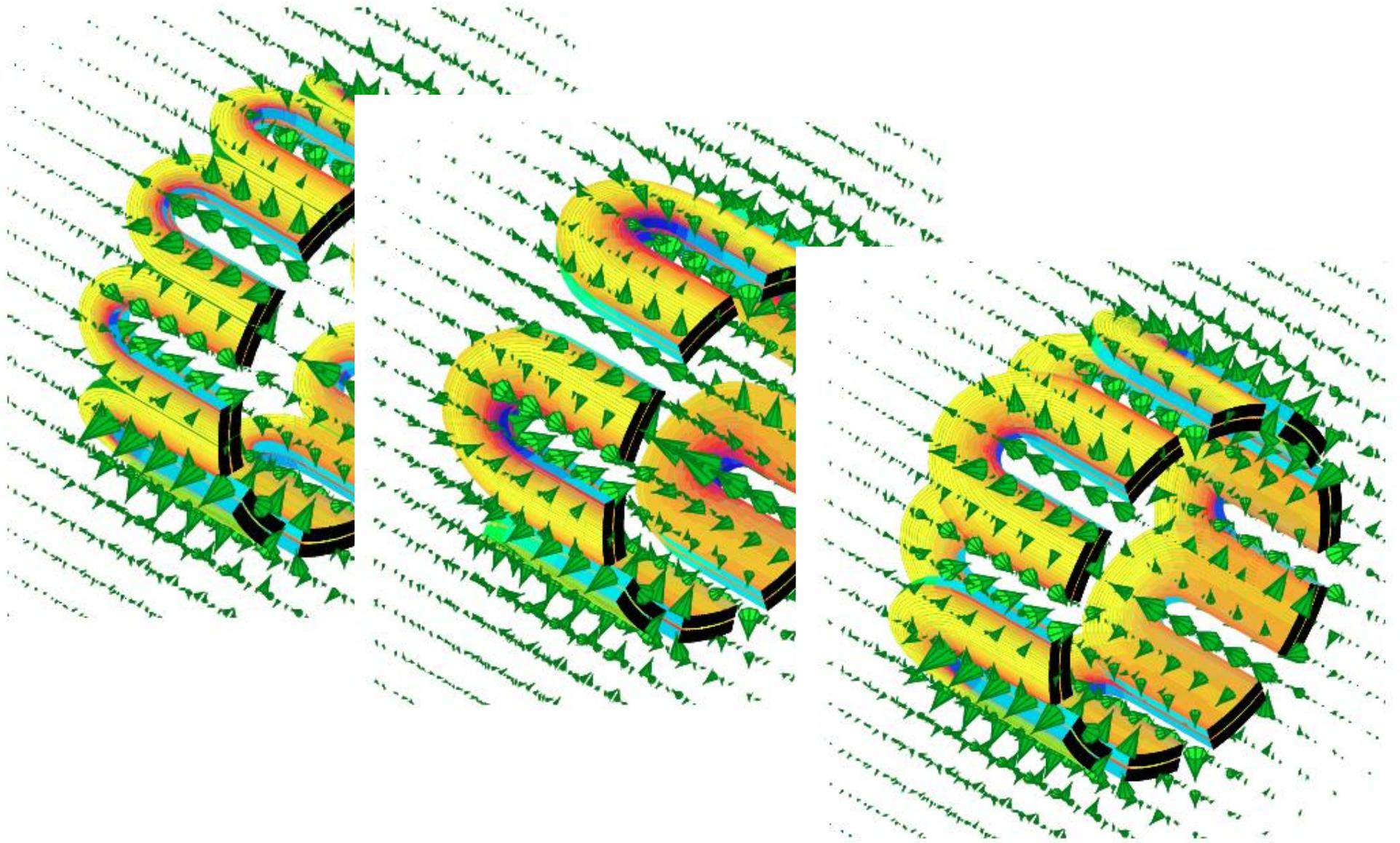
Working Point of LHC Dipole Magnets



Margins



Margins

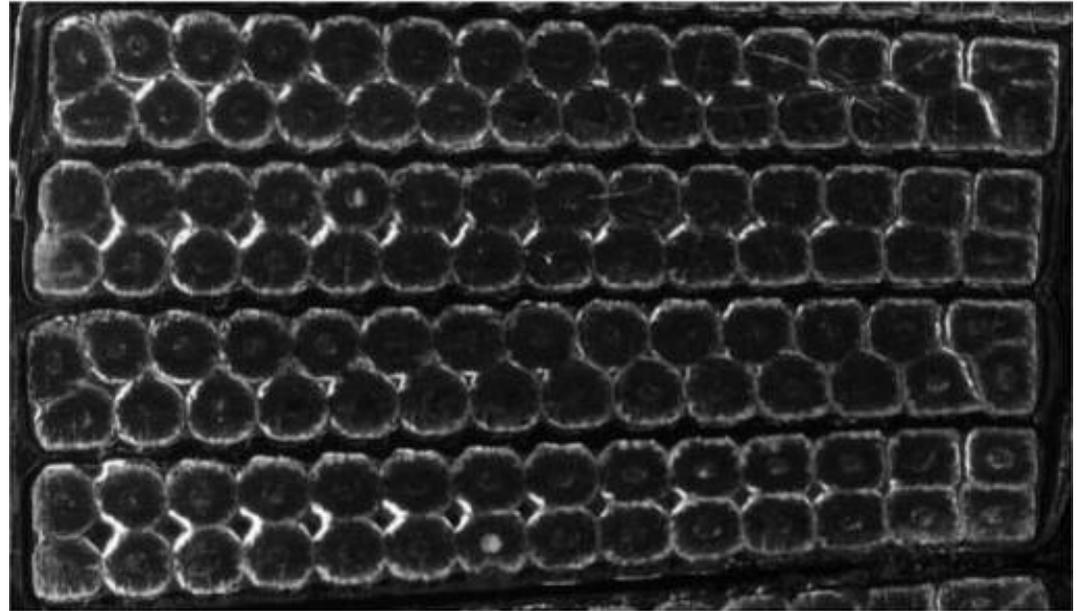


Coil Block, Pinning Centers in Filaments

Magnetization



200 nm 



Grading of current density

Superconductor Properties

→ Hard Superconductors (Type 2)

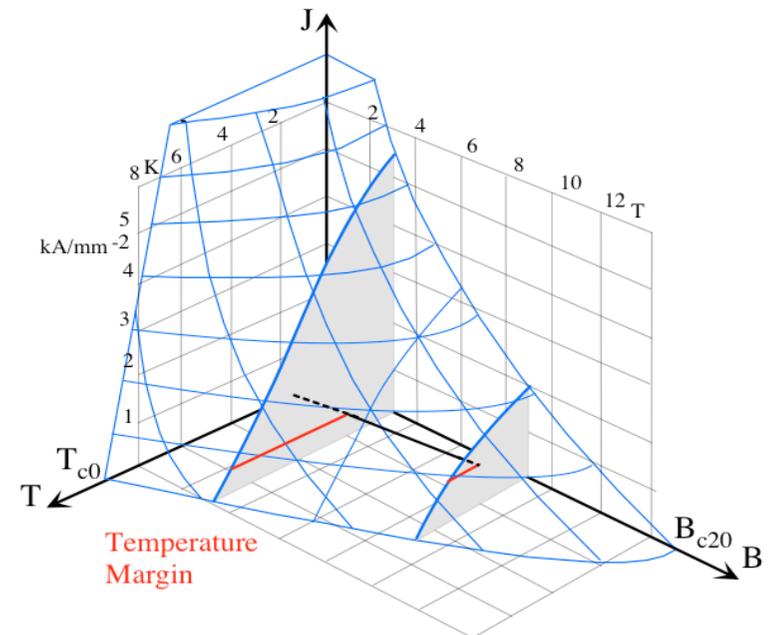
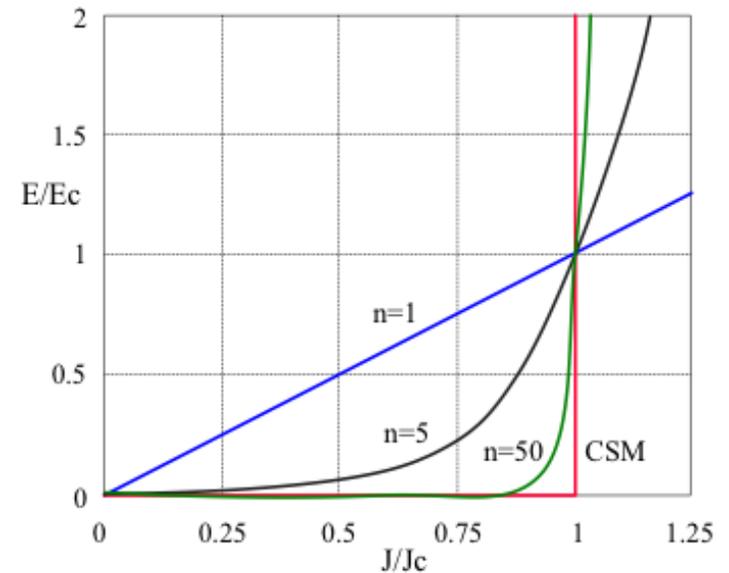
- Magnetic field can penetrate
- Magnetization with hysteresis

→ Critical current density J_c

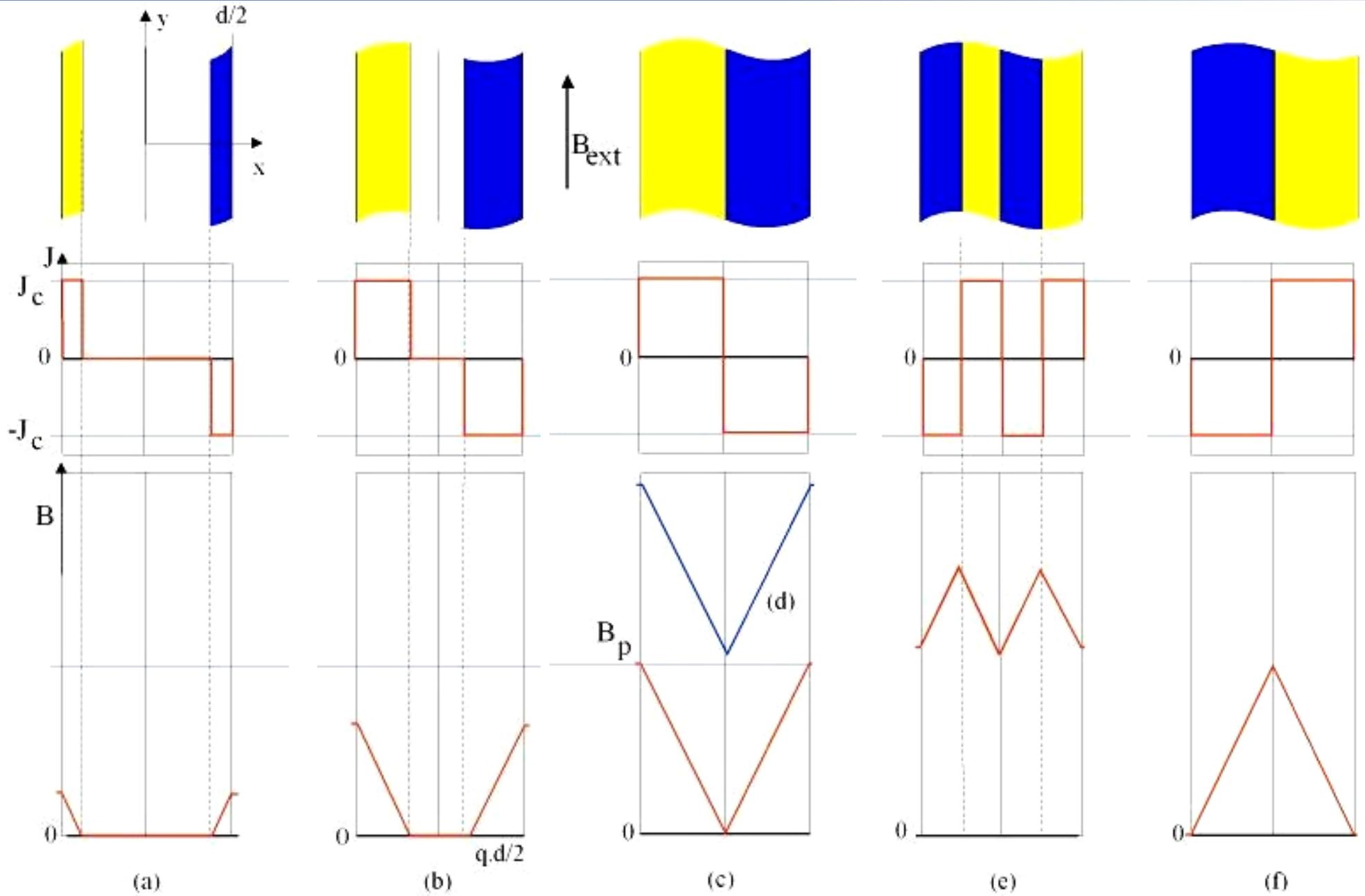
- Current density at spec. electric field ($E_c = 1 \mu\text{V}/\text{cm}$)

→ Critical surface

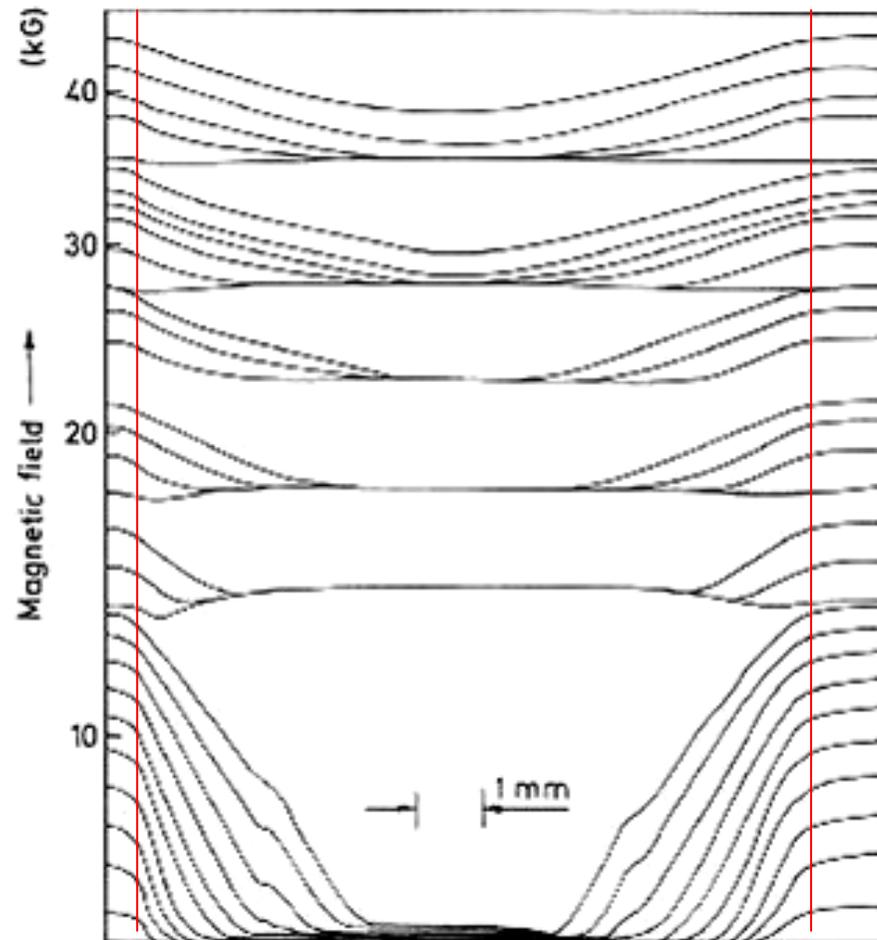
- Dependence of J_c on T and B



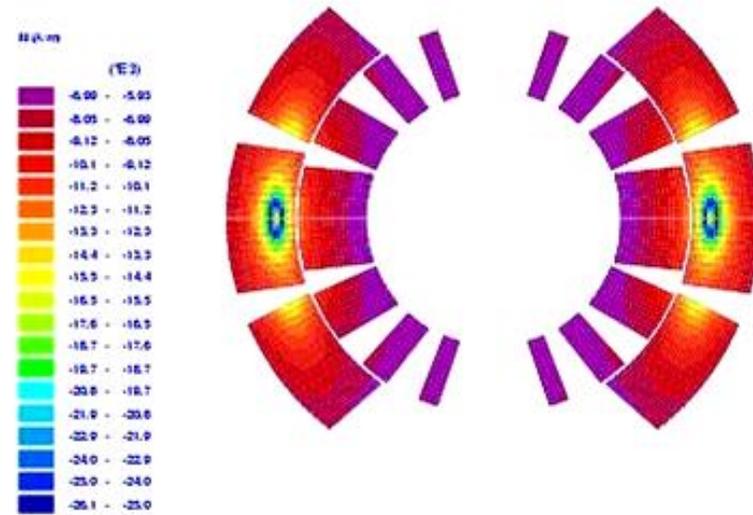
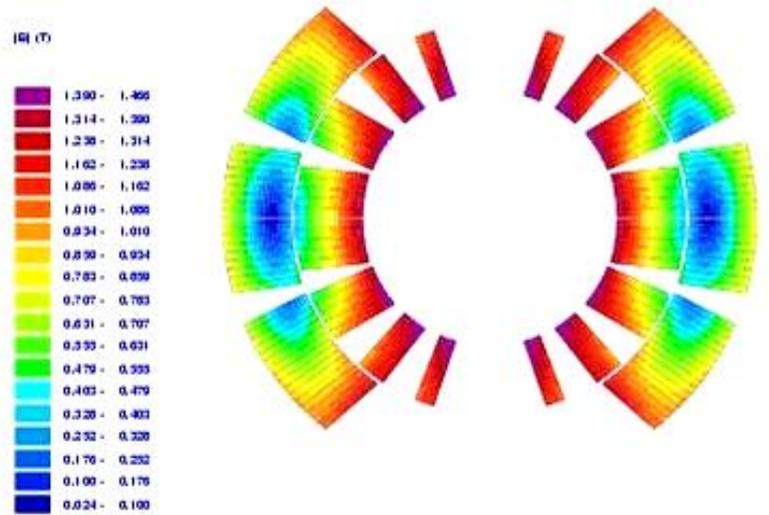
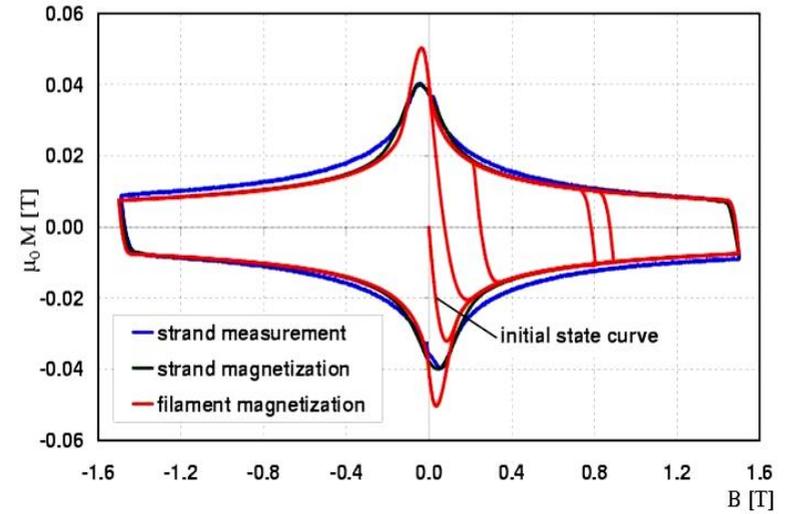
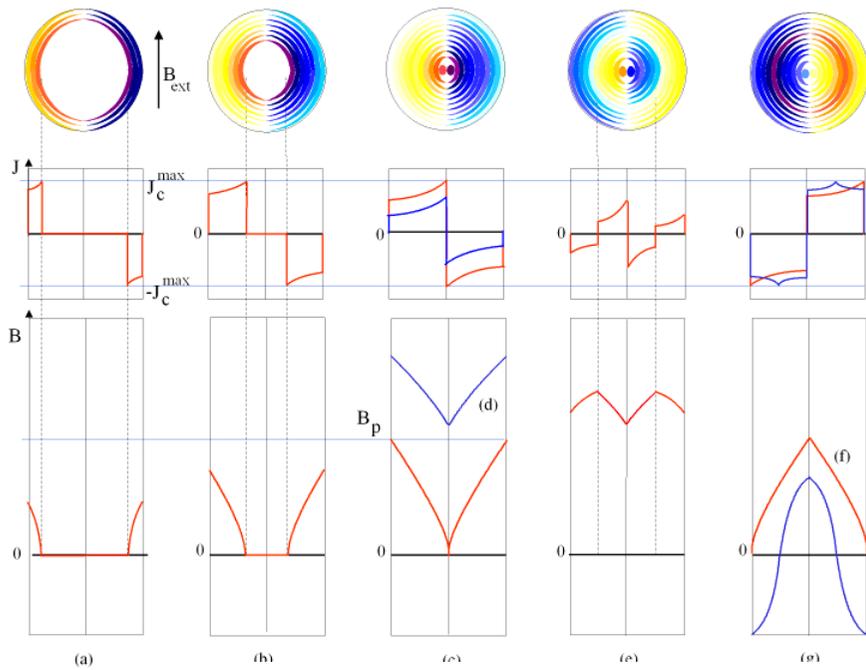
Bean's Critical State Model (CSM)



Screening Field in a Slab



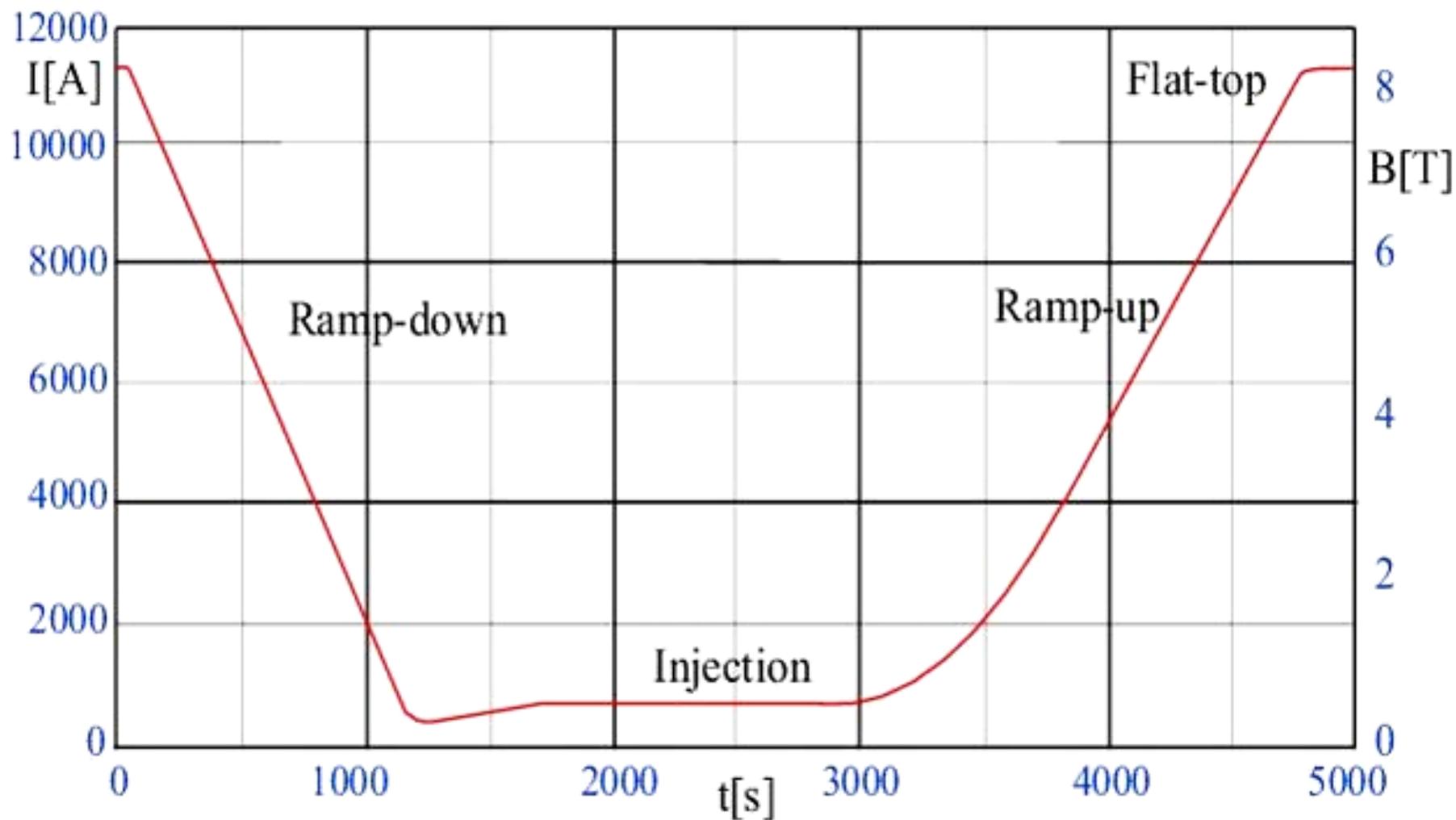
Superconducting Magnetization (Hysteresis Model)



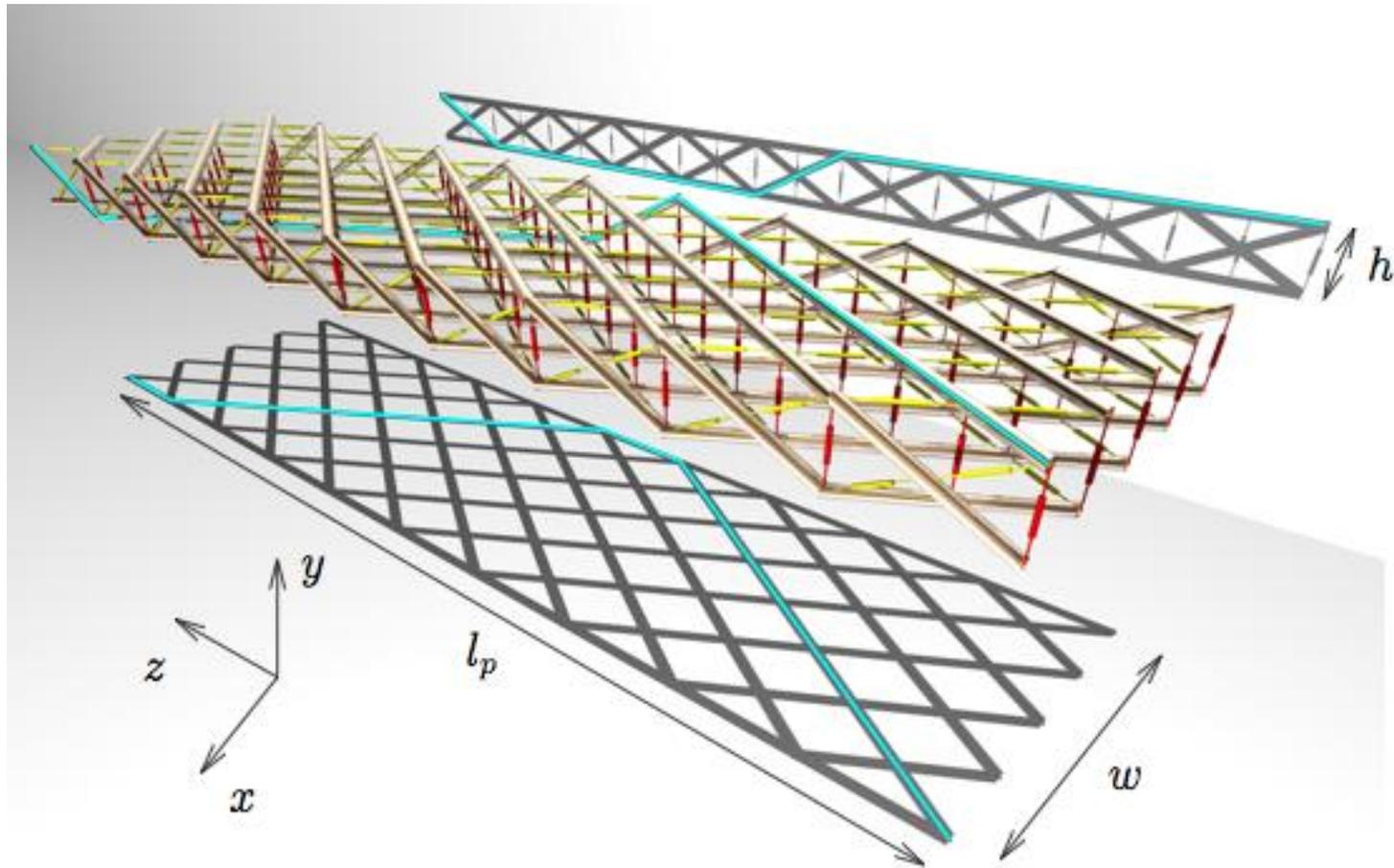
The LHC Excitation Cycle

$$V \approx 2 E / I t$$

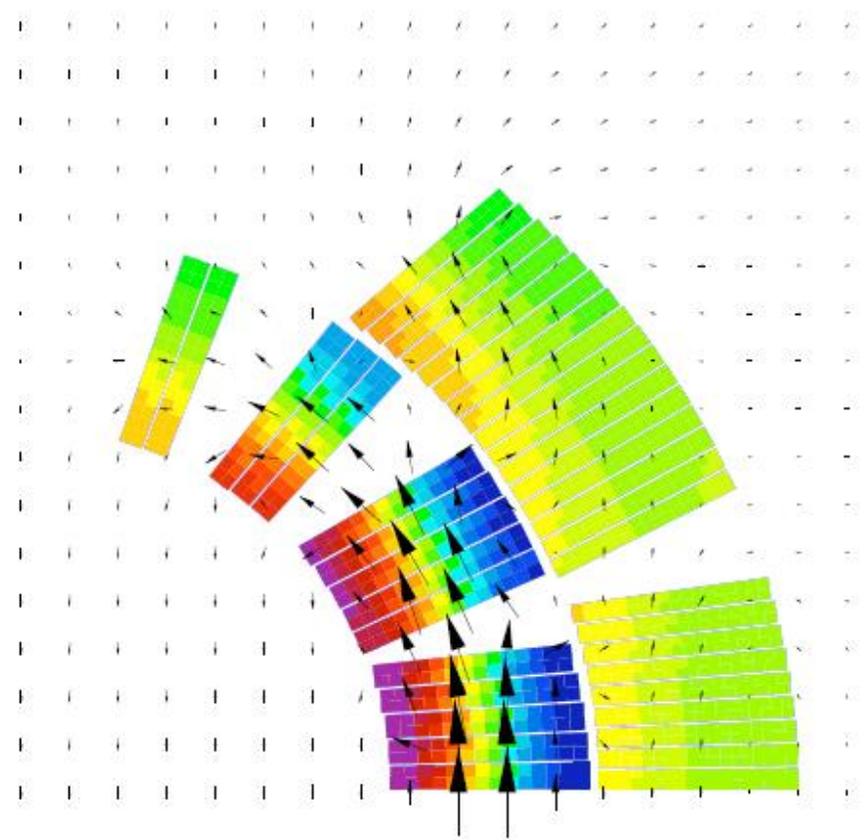
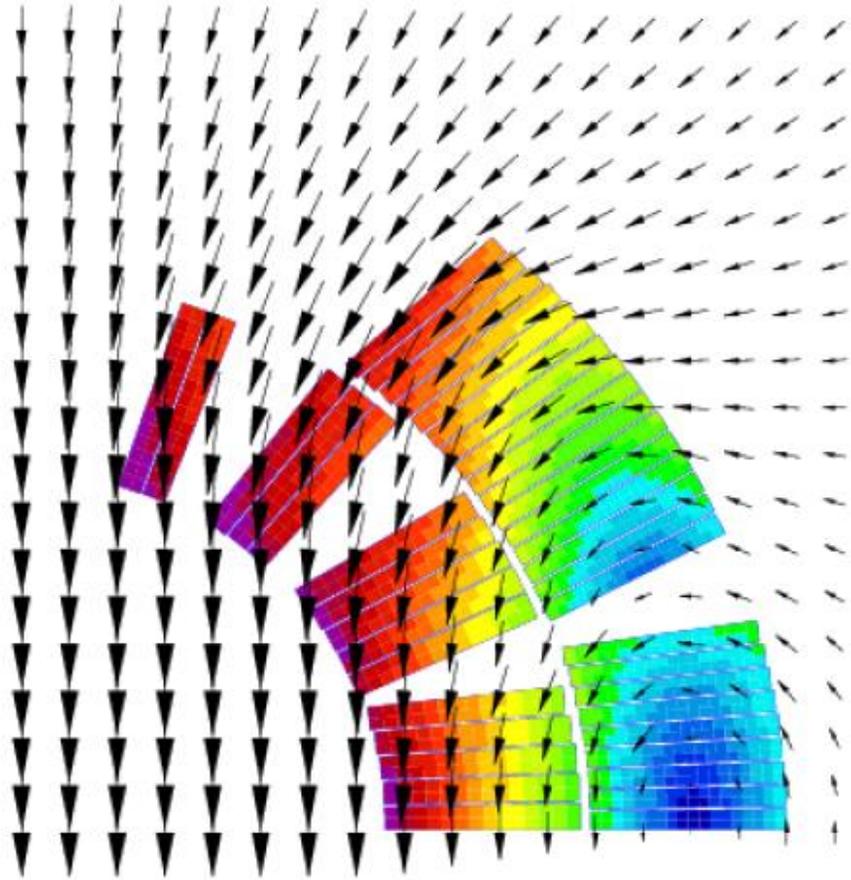
$E = 1.15 \text{ TJ}$ (320 kWh), $I = 11800 \text{ A}$, Ramp rate 10 A/s , 155 V



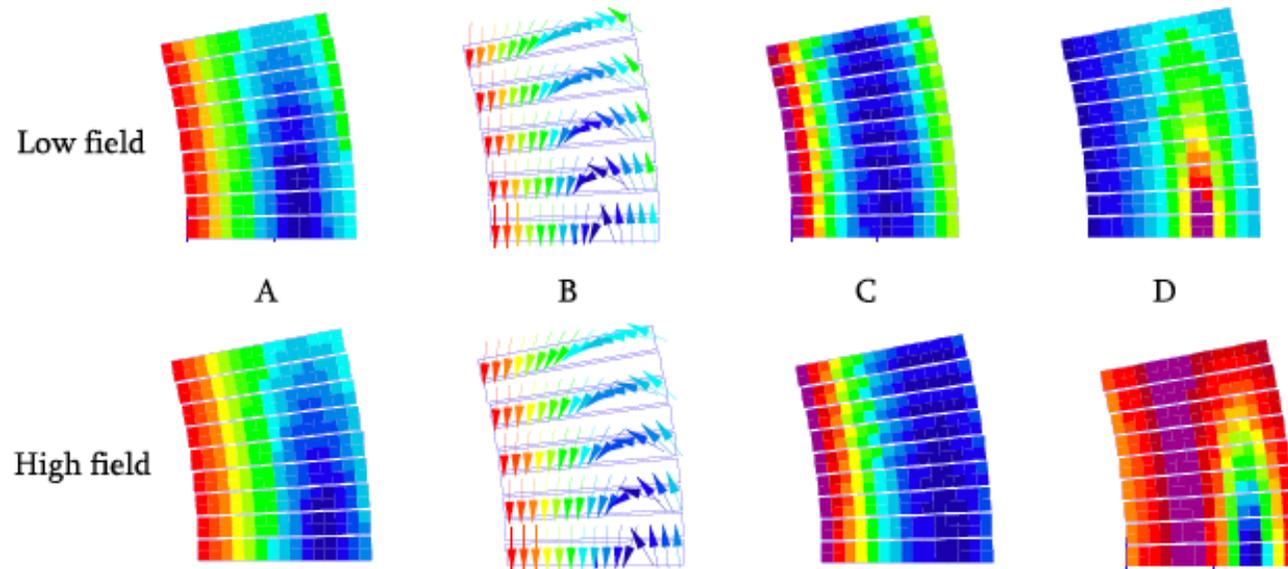
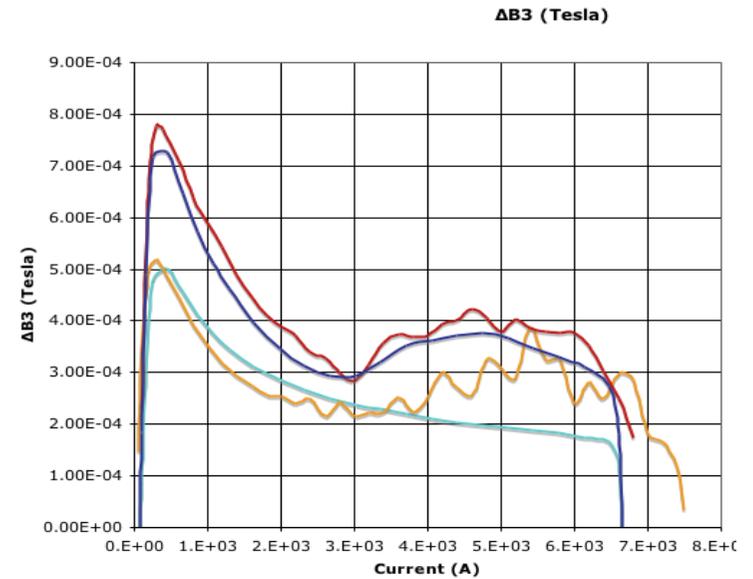
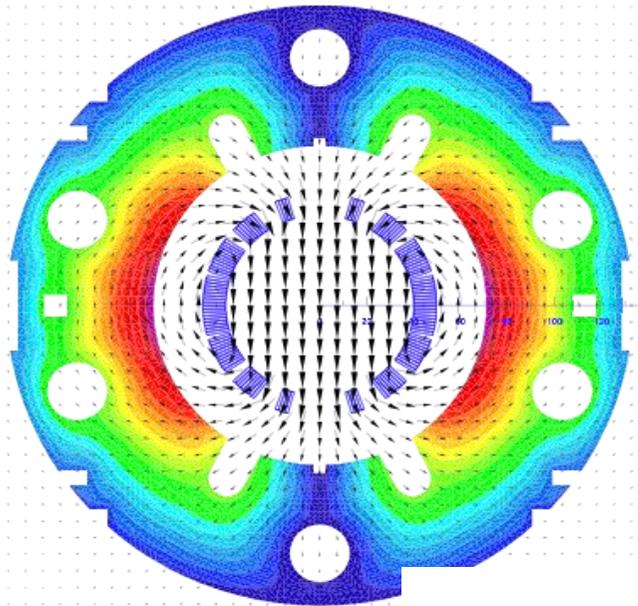
Eddy Currents in Rutherford Cables



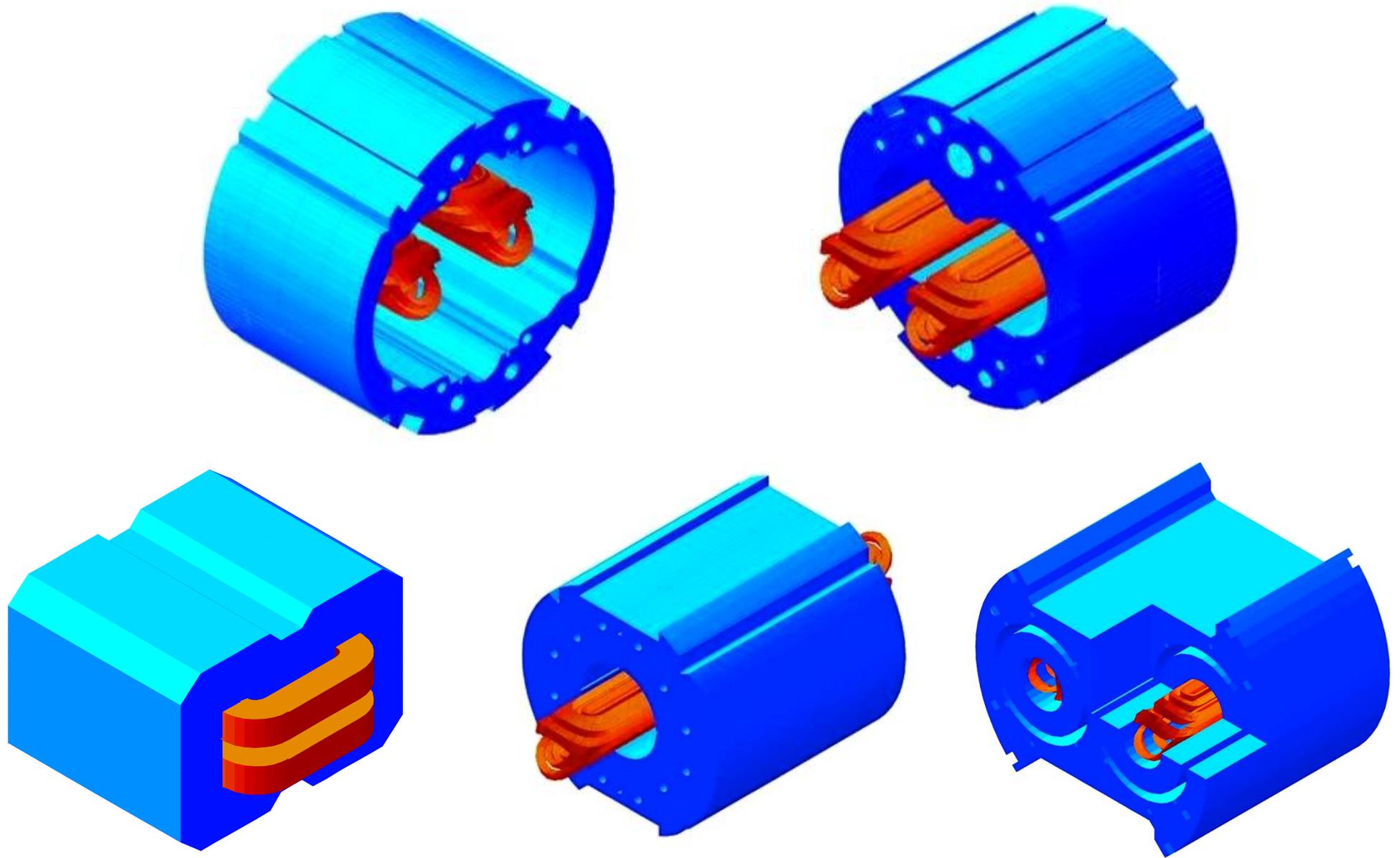
Field Generated by ISCC



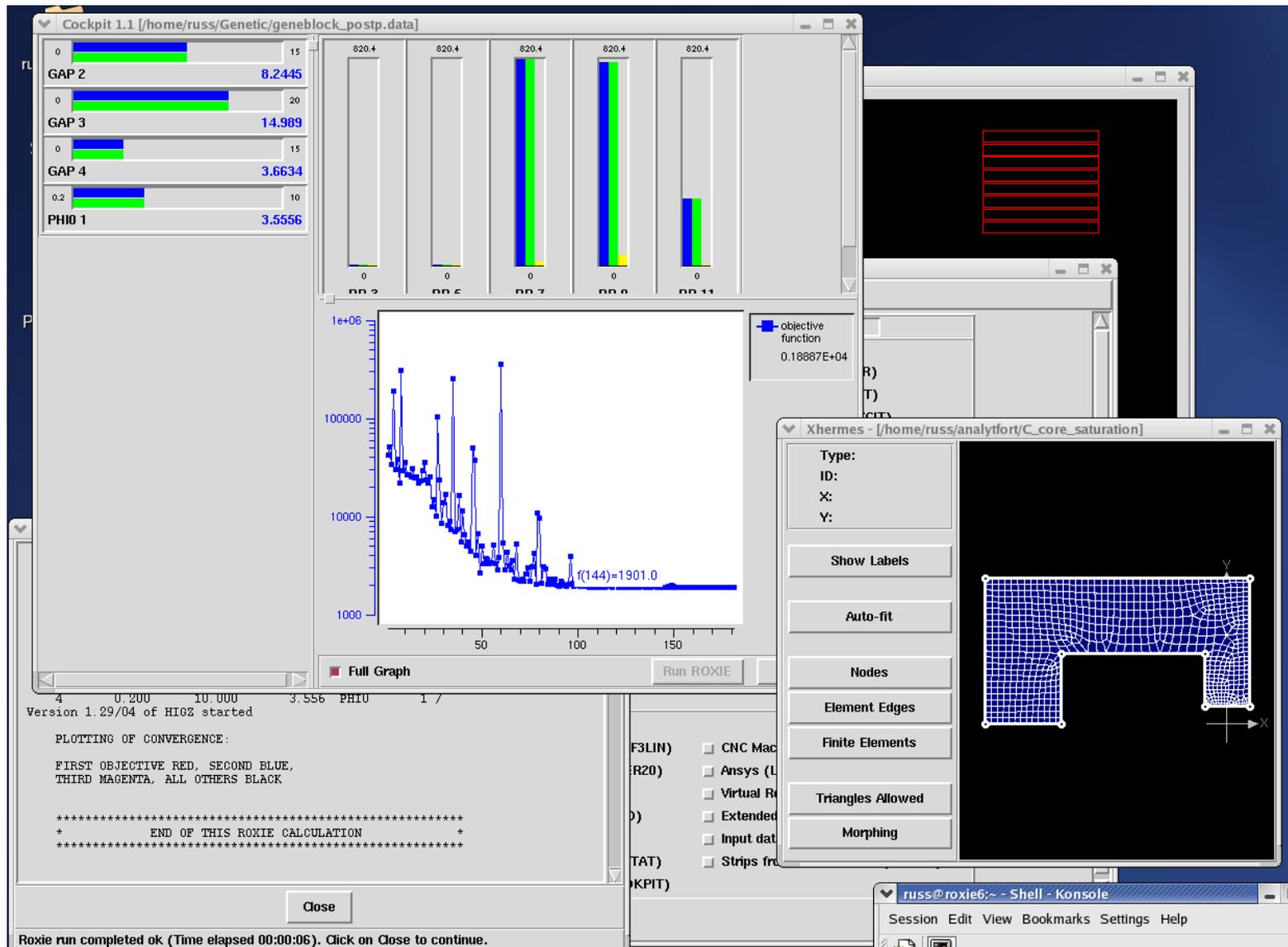
2-D Transient Field Computation for GSI-001



Magnet Extremities



The CERN Field Computation Program ROXIE



Objectives for the ROXIE Development

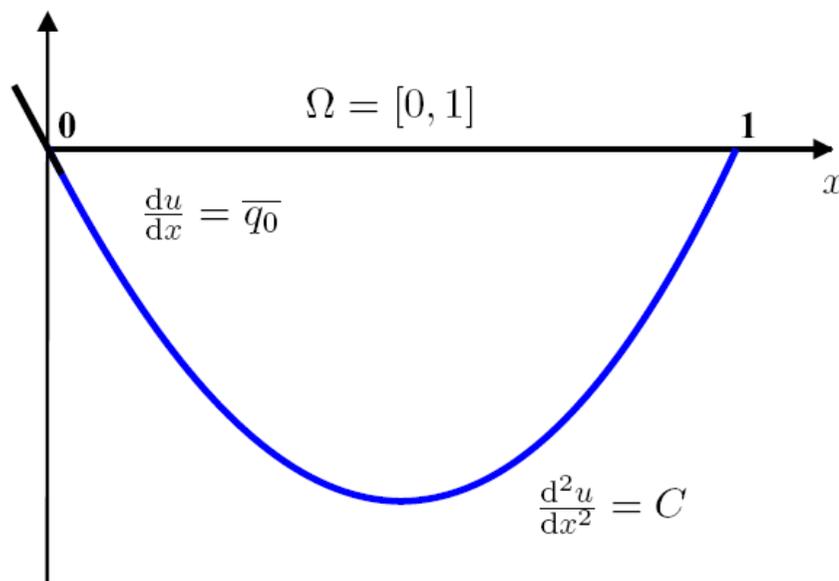
- ➔ Automatic generation of coil and yoke geometries
 - Features: Layers, coil-blocks, conductors, strands, holes, keys
- ➔ Field computation specially suited for magnet design (Ar, BEM-FEM)
 - No meshing of the coil
 - No artificial boundary conditions
 - Higher order quadrilateral meshes, Parametric mesh generator
 - Modeling of superconductor magnetization
- ➔ Mathematical optimization techniques
 - Genetic optimization, Pareto optimization, Search algorithms
- ➔ CAD/CAM interfaces
 - Drawings, End-spacer design and manufacture

- ➔ Principles of numerical field computation
 - Formulation of the Problem
 - Weighted residual
 - Weak form
 - Discretization
 - Numerical example
- ➔ Total vector potential formulation
 - Weak form in 3-D
- ➔ Element shape functions
 - Global shape functions
 - Barycentric coordinates
- ➔ Mesh generation

The Model Problem (1-D)

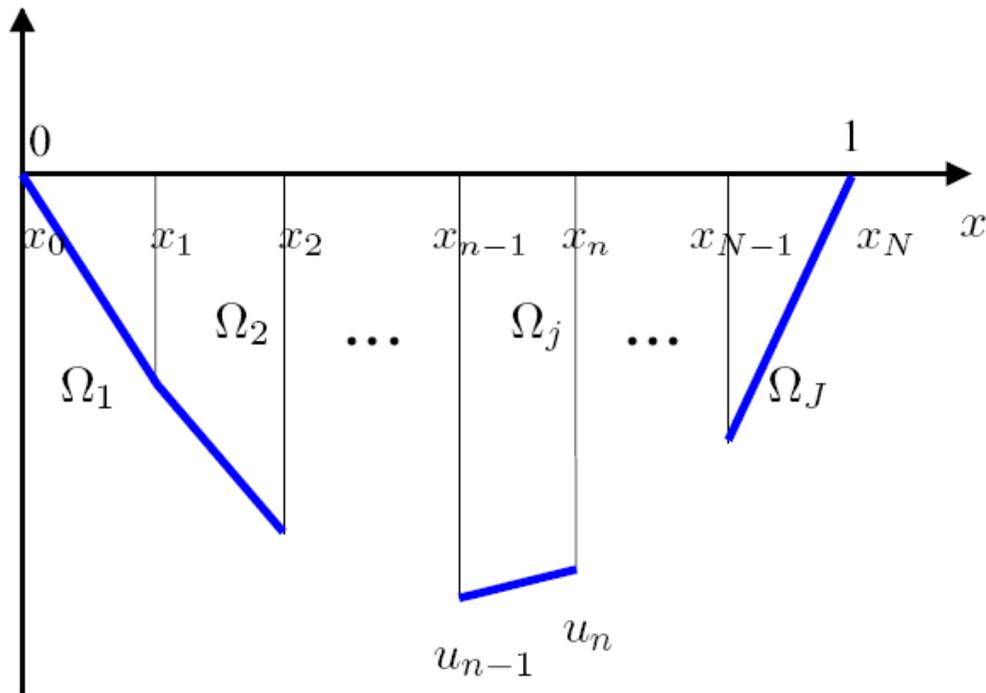
$$\frac{d^2 u(x)}{dx^2} = f(x), \quad x \in \Omega$$

$$u(x)|_{x=0} = \bar{u}_0 \quad u(x)|_{x=1} = \bar{u}_1 \quad \text{or} \quad \left. \frac{du}{dx} \right|_{x=1} = \bar{q}_1$$



$$u(x) = \frac{C}{2} (x^2 - x)$$

Shape Functions



$$\Omega = \bigcup_{j=1}^J \Omega_j$$

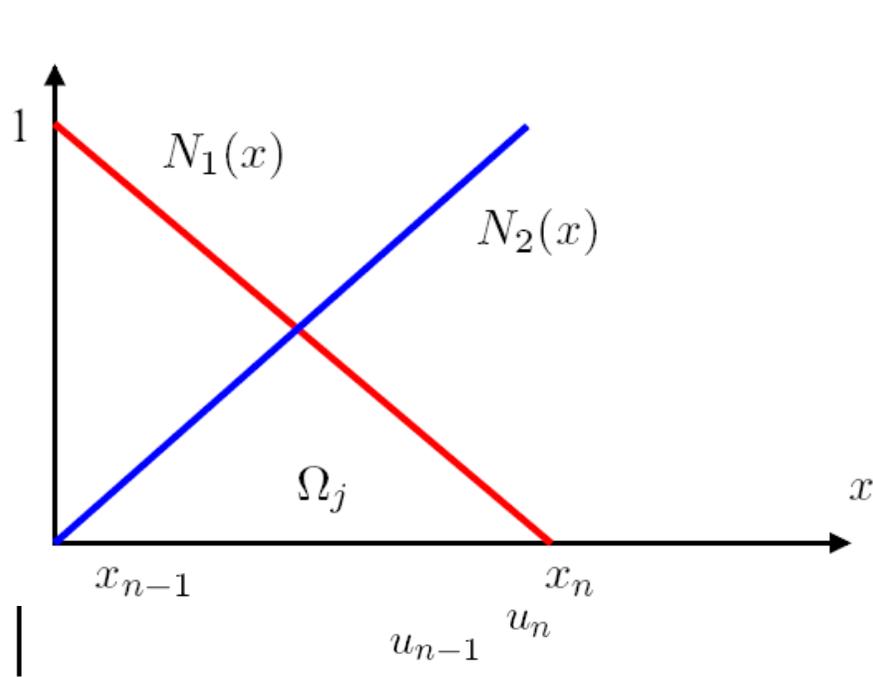
$$\Omega_j = [x_{n-1}, x_n]$$

$$u_j(x) = \alpha_{j1} + \alpha_{j2}x \quad x \in \Omega_j$$

$$u_{n-1} = \alpha_{j1} + \alpha_{j2}x_{n-1}$$

$$u_n = \alpha_{j1} + \alpha_{j2}x_n$$

Shape Functions



$$\alpha_{j1} = \frac{\begin{vmatrix} u_{n-1} & x_{n-1} \\ u_n & x_n \end{vmatrix}}{\begin{vmatrix} 1 & x_{n-1} \\ 1 & x_n \end{vmatrix}} \quad \text{Cramer's rule}$$

$$\alpha_{j1} = \frac{x_n u_{n-1} - x_{n-1} u_n}{x_n - x_{n-1}}$$

$$\alpha_{j2} = \frac{u_n - u_{n-1}}{x_n - x_{n-1}}$$

$$u_j(x) = \alpha_{j1} + \alpha_{j2}x = \frac{x_n - x}{x_n - x_{n-1}}u_{n-1} + \frac{-x_{n-1} + x}{x_n - x_{n-1}}u_n$$

What have we won? We can express the field in the element as a function of the node potentials using known polynomials in the spatial coordinates

The Weighted Residual

$$R(x) := \frac{d^2 u(x)}{dx^2} - f(x)$$

$$\int_{\Omega} w(x) R(x) d\Omega = \int_{\Omega} w(x) \frac{d^2 u(x)}{dx^2} d\Omega - \int_{\Omega} w(x) f(x) d\Omega = 0.$$

$$\int_a^b \phi \psi' dx = [\phi \psi]_a^b - \int_a^b \phi' \psi dx \quad w(x) = \phi \quad \frac{du(x)}{dx} = \psi$$

$$- \int_{\Omega} \frac{dw(x)}{dx} \frac{du(x)}{dx} d\Omega + \left[w(x) \frac{du(x)}{dx} \right]_0^1 - \int_{\Omega} w(x) f(x) d\Omega = 0$$

What have we won? Removal of the second derivative, a way to incorporate Neumann boundary conditions

$$\int_{\Omega} \frac{dw(x)}{dx} \frac{du(x)}{dx} d\Omega = - \int_{\Omega} w(x) f(x) d\Omega$$

$$\int_{\Omega_j} \frac{dw_l(x)}{dx} \sum_{k=1,2} \frac{dN_{jk}(x)}{dx} u^{(k)} d\Omega_j = - \int_{\Omega_j} w_l(x) f(x) d\Omega_j, \quad l = 1, 2.$$

$$\int_{\Omega_j} \frac{dN_{jl}(x)}{dx} \sum_{k=1,2} \frac{dN_{jk}(x)}{dx} u^{(k)} d\Omega_j = - \int_{\Omega_j} N_{jk}(x) f(x) d\Omega_j, \quad l = 1, 2$$

$$\int_{x_{n-1}}^{x_n} \left(\frac{dN_{j1}}{dx} \frac{dN_{j1}}{dx} u_{n-1} + \frac{dN_{j1}}{dx} \frac{dN_{j2}}{dx} u_n \right) dx = - \int_{x_{n-1}}^{x_n} N_{j1} f(x) dx$$

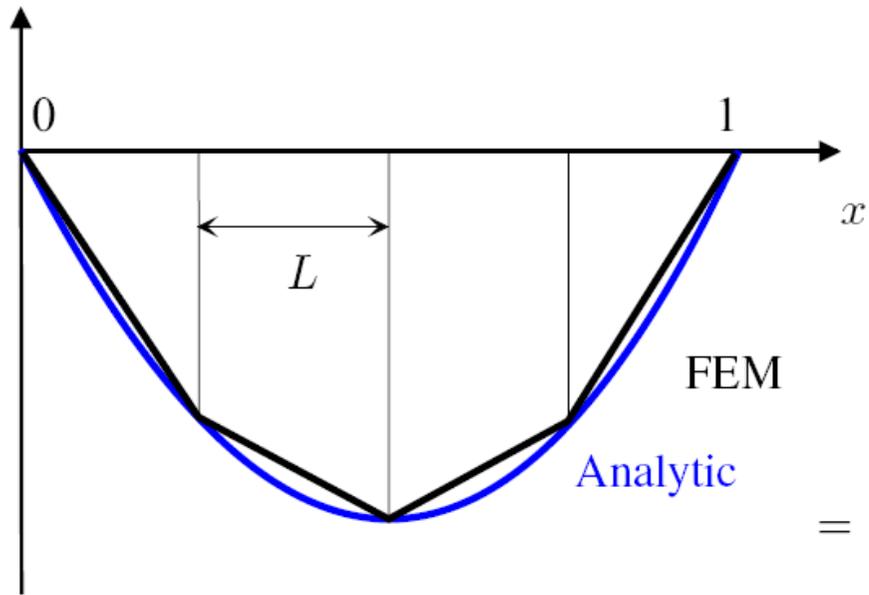
$$\int_{x_{n-1}}^{x_n} \left(\frac{dN_{j2}}{dx} \frac{dN_{j1}}{dx} u_{n-1} + \frac{dN_{j2}}{dx} \frac{dN_{j2}}{dx} u_n \right) dx = - \int_{x_{n-1}}^{x_n} N_{j2} f(x) dx$$

$$[k_j] \{u_j\} = \{f_j\}$$

Linear equation system for the node potentials

Numerical Example

4 finite elements $\Omega_j, j = 1, \dots, 4$ of equidistant length L



$$[k_j] = \int_{x_{n-1}}^{x_n} \begin{pmatrix} \frac{dN_{j1}}{dx} & \frac{dN_{j1}}{dx} & \frac{dN_{j1}}{dx} & \frac{dN_{j2}}{dx} \\ \frac{dN_{j2}}{dx} & \frac{dN_{j1}}{dx} & \frac{dN_{j2}}{dx} & \frac{dN_{j2}}{dx} \end{pmatrix} dx$$

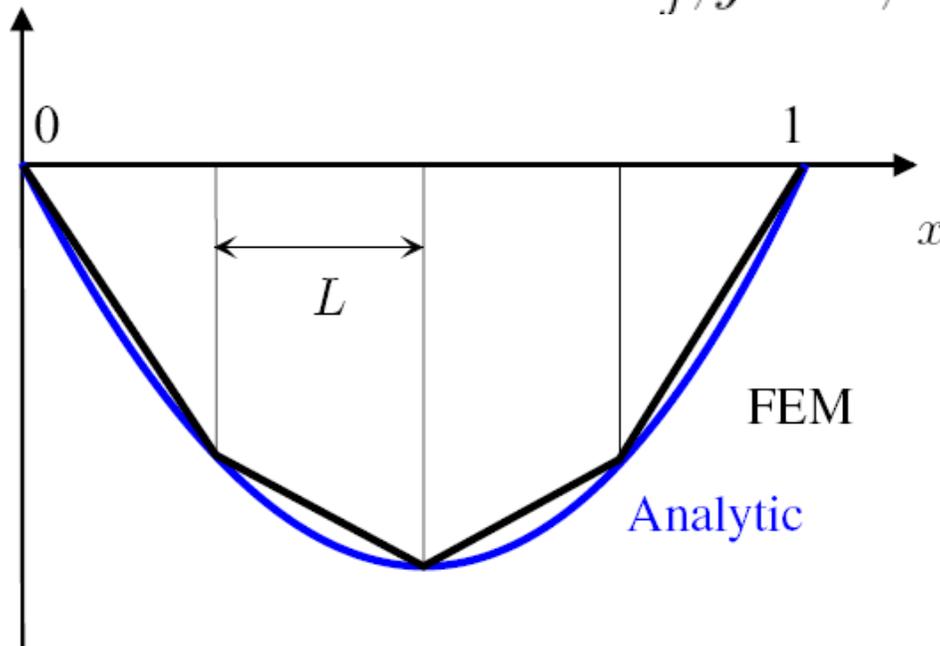
$$= \int_{x_{n-1}}^{x_n} \begin{pmatrix} \frac{1}{(x_n - x_{n-1})^2} & \frac{-1}{(x_n - x_{n-1})^2} \\ \frac{-1}{(x_n - x_{n-1})^2} & \frac{1}{(x_n - x_{n-1})^2} \end{pmatrix} dx = \begin{pmatrix} \frac{1}{L} & \frac{-1}{L} \\ \frac{-1}{L} & \frac{1}{L} \end{pmatrix}$$

$$\{f_j\} = - \int_{x_{n-1}}^{x_n} \begin{pmatrix} N_{j1} \\ N_{j2} \end{pmatrix} C dx = -C \int_{x_{n-1}}^{x_n} \begin{pmatrix} \frac{x_n - x}{x_n - x_{n-1}} \\ \frac{-x_{n-1} + x}{x_n - x_{n-1}} \end{pmatrix} dx$$

$$= -\frac{C}{2L} \begin{pmatrix} 2x_n x - x^2 \\ -2x_{n-1} x + x^2 \end{pmatrix} \Big|_{x_{n-1}}^{x_n} = -\frac{C}{2L} \begin{pmatrix} (x_n - x_{n-1})^2 \\ (x_{n-1} - x_n)^2 \end{pmatrix} = - \begin{pmatrix} 0.5 CL \\ 0.5 CL \end{pmatrix}$$

Numerical Example

4 finite elements $\Omega_j, j = 1, \dots, 4$ of equidistant length L



$$\begin{pmatrix} \frac{1}{L} & -\frac{1}{L} & 0 & 0 & 0 \\ \frac{1}{L} & \frac{2}{L} & -\frac{1}{L} & 0 & 0 \\ 0 & -\frac{1}{L} & \frac{2}{L} & -\frac{1}{L} & 0 \\ 0 & 0 & -\frac{1}{L} & \frac{2}{L} & -\frac{1}{L} \\ 0 & 0 & 0 & -\frac{1}{L} & \frac{1}{L} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} = - \begin{pmatrix} 0.5CL \\ CL \\ CL \\ CL \\ 0.5CL \end{pmatrix}$$

Essential boundary conditions (Dirichlet)

$$\begin{pmatrix} \frac{2}{L} & -\frac{1}{L} & 0 \\ -\frac{1}{L} & \frac{2}{L} & -\frac{1}{L} \\ 0 & -\frac{1}{L} & \frac{2}{L} \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix} = - \begin{pmatrix} CL \\ CL \\ CL \end{pmatrix}$$

$$\begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix} = - \begin{pmatrix} \frac{3L}{4} & \frac{L}{2} & \frac{L}{4} \\ \frac{L}{2} & L & \frac{L}{2} \\ \frac{L}{4} & \frac{L}{2} & \frac{2L}{4} \end{pmatrix} \begin{pmatrix} CL \\ CL \\ CL \end{pmatrix} = \begin{pmatrix} -0.375 \\ -0.5 \\ -0.375 \end{pmatrix}$$

Higher order elements

$$u^{(1)} = \alpha_{j1} + \alpha_{j2}x_1 + \alpha_{j3}x_1^2$$

$$u^{(2)} = \alpha_{j1} + \alpha_{j2}x_2 + \alpha_{j3}x_2^2$$

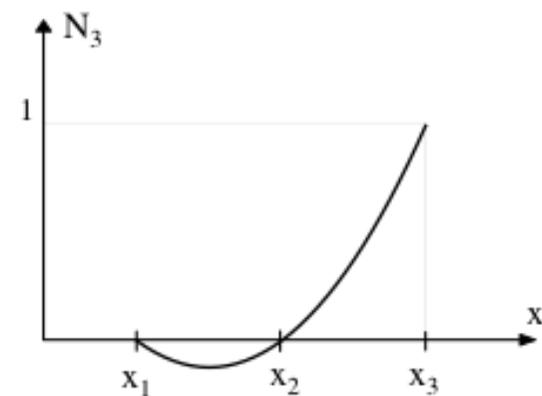
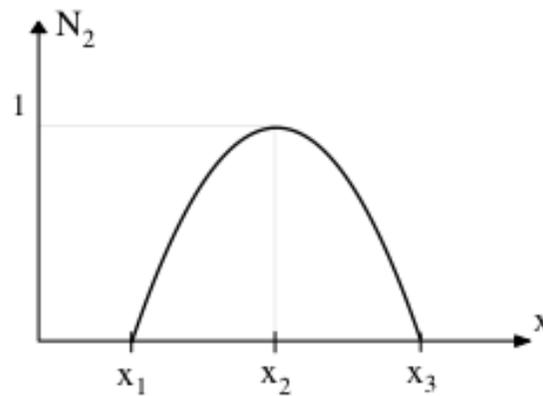
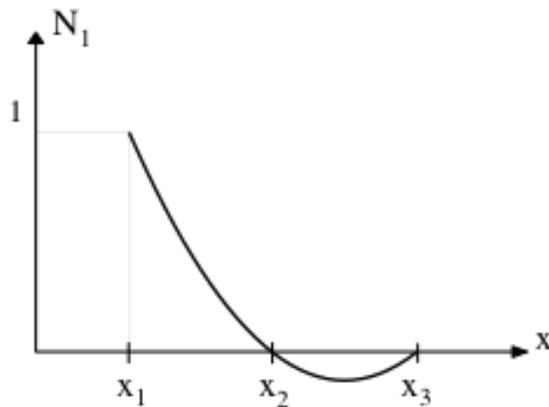
$$u^{(3)} = \alpha_{j1} + \alpha_{j2}x_3 + \alpha_{j3}x_3^2$$

$$u_j(x) = \sum_{k=1}^3 N_{jk}(x)u^{(k)}$$

$$N_{j1}(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)},$$

$$N_{j2}(x) = \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)}$$

$$N_{j3}(x) = \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}.$$



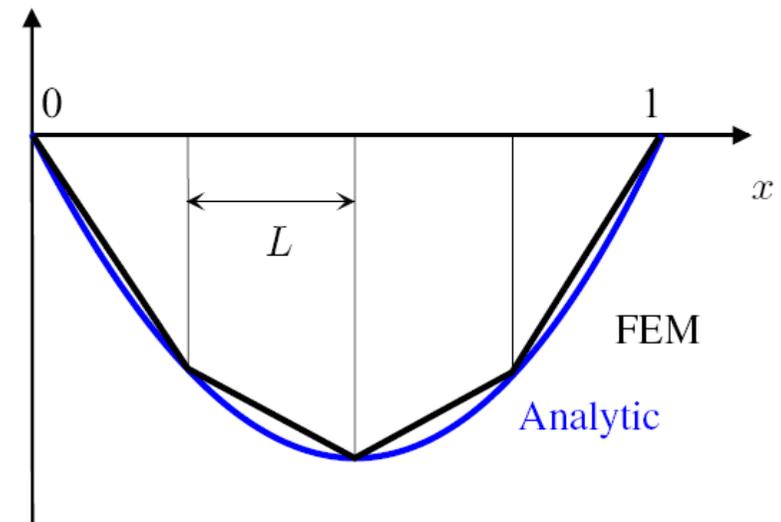
Two Quadratic Elements ($l = 2L$)

$$[k_j] = \begin{pmatrix} \frac{7}{6l} & \frac{-8}{6l} & \frac{1}{6l} \\ \frac{-8}{6l} & \frac{16}{6l} & \frac{-8}{6l} \\ \frac{1}{6l} & \frac{-8}{6l} & \frac{7}{6l} \end{pmatrix}$$

$$\{f_j\} = -\frac{1}{3}c \begin{pmatrix} l \\ 4l \\ l \end{pmatrix}$$

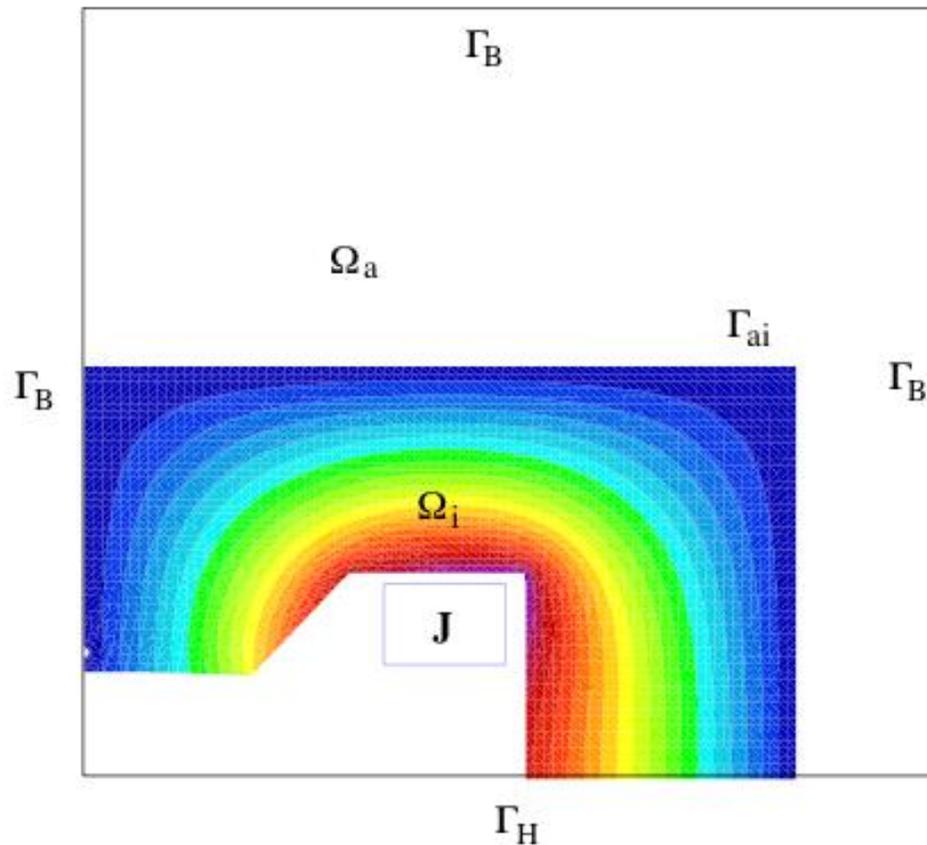
$$\begin{pmatrix} \frac{2}{l} & \frac{-1}{l} & 0 \\ \frac{-1}{l} & \frac{2}{l} & \frac{-1}{l} \\ 0 & \frac{-1}{l} & \frac{2}{l} \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix} = - \begin{pmatrix} cl \\ cl \\ cl \end{pmatrix}$$

$$\begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix} = - \begin{pmatrix} \frac{3l}{4} & \frac{l}{2} & \frac{l}{4} \\ \frac{l}{2} & l & \frac{l}{2} \\ \frac{l}{4} & \frac{l}{2} & \frac{3l}{4} \end{pmatrix} \begin{pmatrix} cl \\ cl \\ cl \end{pmatrix} = \begin{pmatrix} -0.375 \\ -0.5 \\ -0.375 \end{pmatrix}$$



Weak Form in the FEM Problem

$$\operatorname{curl} \frac{1}{\mu} \operatorname{curl} \mathbf{A} - \operatorname{grad} \frac{1}{\mu} \operatorname{div} \mathbf{A} = \mathbf{J} \quad \text{in } \Omega$$



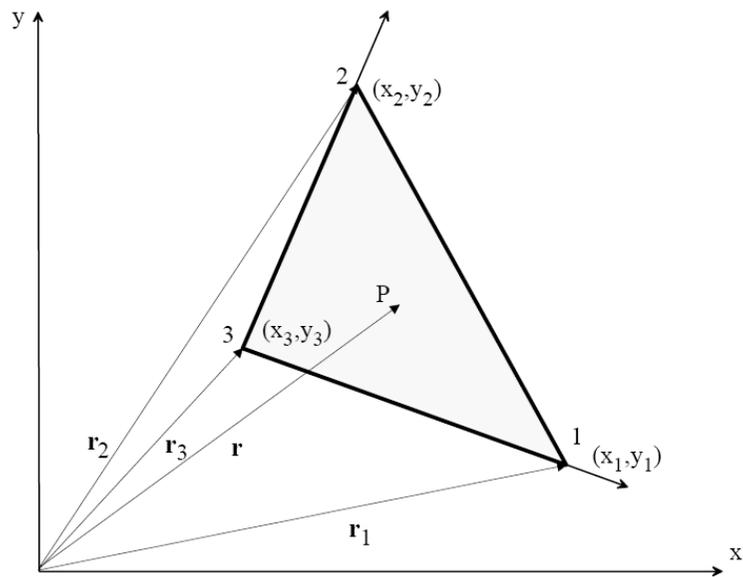
$$\begin{aligned} \mathbf{A} \cdot \mathbf{n} &= 0 && \text{on } \Gamma_H, \\ \frac{1}{\mu} \operatorname{div} \mathbf{A} &= 0 && \text{on } \Gamma_B, \\ \mathbf{n} \times (\mathbf{A} \times \mathbf{n}) &= \mathbf{0} && \text{on } \Gamma_B, \\ \mathbf{n} \times \left(\frac{1}{\mu} (\operatorname{curl} \mathbf{A}) \times \mathbf{n} \right) &= \mathbf{0} && \text{on } \Gamma_H, \\ \left[\frac{1}{\mu} \operatorname{div} \mathbf{A} \right]_{\text{ai}} &= 0 && \text{on } \Gamma_{\text{ai}}, \\ \left[\frac{1}{\mu} (\operatorname{curl} \mathbf{A}) \times \mathbf{n} \right]_{\text{ai}} &= \mathbf{0} && \text{on } \Gamma_{\text{ai}}, \\ [\mathbf{A}]_{\text{ai}} &= \mathbf{0} && \text{on } \Gamma_{\text{ai}}. \end{aligned}$$

Shape Functions

$$A_j(\mathbf{x}) = \alpha_1 + \alpha_2 x + \alpha_3 y,$$

$$\mathbf{x} \in \Omega_j$$

$$A_j(\mathbf{x}) = A_{z_j}(x, y)$$



$$A^{(1)} = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1$$

$$A^{(2)} = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2$$

$$A^{(3)} = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3$$

$$\begin{pmatrix} A^{(1)} \\ A^{(2)} \\ A^{(3)} \end{pmatrix} = \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

$$\{\alpha\} = [C]^{-1}\{A\}$$



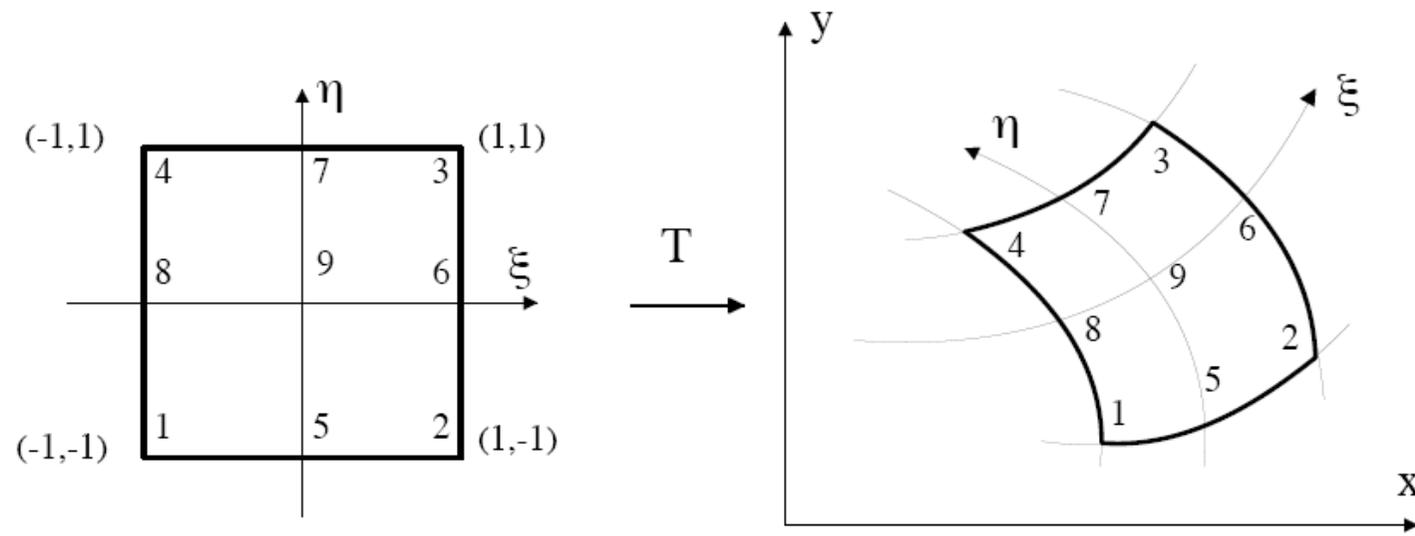
$$\{A\} = [C]\{\alpha\}$$

Mapped Elements (Iso-parametric quadrilateral)

$$x = x(\xi, \eta, \zeta),$$

$$y = y(\xi, \eta, \zeta),$$

$$z = z(\xi, \eta, \zeta)$$



$$A_j(\boldsymbol{\xi}) = \sum_{k=1}^K N_k(\boldsymbol{\xi}) A^{(k)} \quad x_j(\boldsymbol{\xi}) = \sum_{k=1}^K N_k(\boldsymbol{\xi}) x^{(k)} \quad y_j(\boldsymbol{\xi}) = \sum_{k=1}^K N_k(\boldsymbol{\xi}) y^{(k)}$$

Use of the same shape functions for the transformation of the elements

Weak Form in the FEM Problem

$$\operatorname{curl} \frac{1}{\mu} \operatorname{curl} \mathbf{A} - \operatorname{grad} \frac{1}{\mu} \operatorname{div} \mathbf{A} = \mathbf{J} \quad \text{in } \Omega$$

$$\operatorname{curl} \frac{1}{\mu} \operatorname{curl} \mathbf{A} - \operatorname{grad} \frac{1}{\mu} \operatorname{div} \mathbf{A} - \mathbf{J} = \mathbf{R}$$

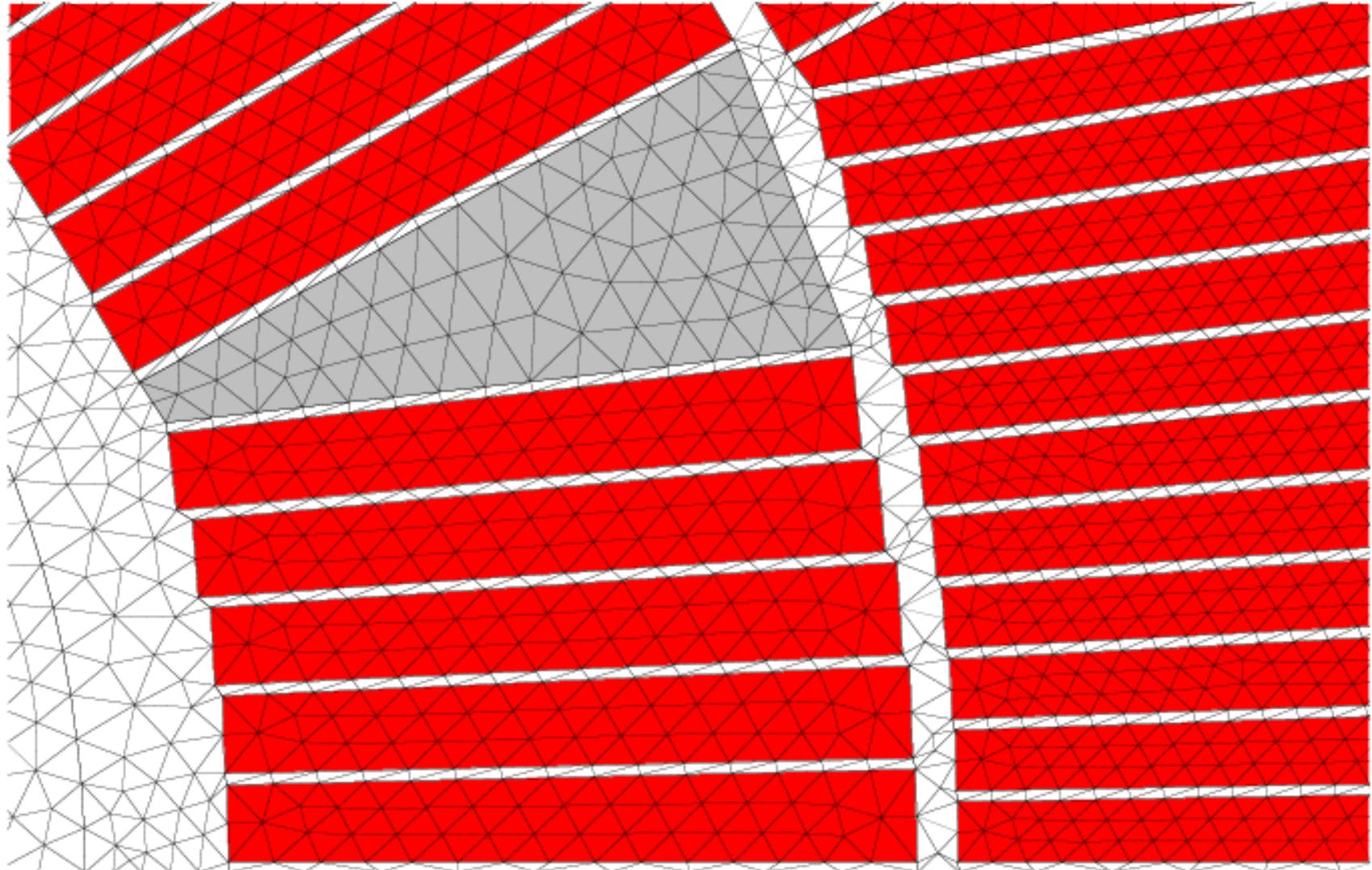
$$\int_{\Omega} \mathbf{w}_a \cdot \left(\operatorname{curl} \frac{1}{\mu} \operatorname{curl} \mathbf{A} - \operatorname{grad} \frac{1}{\mu} \operatorname{div} \mathbf{A} \right) d\Omega = \int_{\Omega} \mathbf{w}_a \cdot \mathbf{J} d\Omega, \quad a = 1, 2, 3.$$

Integration by parts

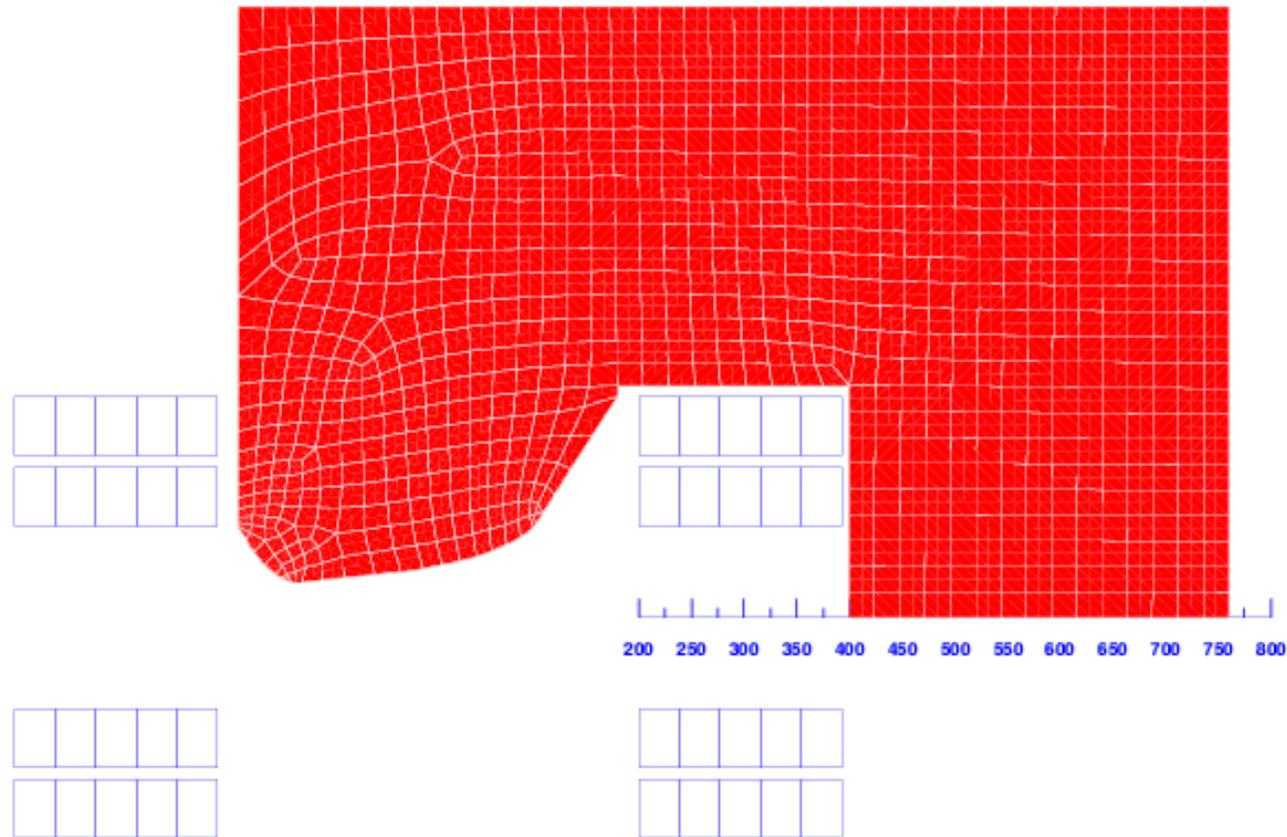
$$\int_{\Omega} \frac{1}{\mu} \operatorname{curl} \mathbf{w}_a \cdot \operatorname{curl} \mathbf{A} d\Omega + \int_{\Omega} \frac{1}{\mu} \operatorname{div} \mathbf{w}_a \operatorname{div} \mathbf{A} d\Omega = \int_{\Omega} \mathbf{w}_a \cdot \mathbf{J} d\Omega$$

Conclusion: 3-D is more complicated than the addition of just one dimension in space; it's a different mathematics, and thus often a separate software package

Meshing the Coil



Higher Order Elements



Higher accuracy of the field solution, but also better modeling of the iron contour

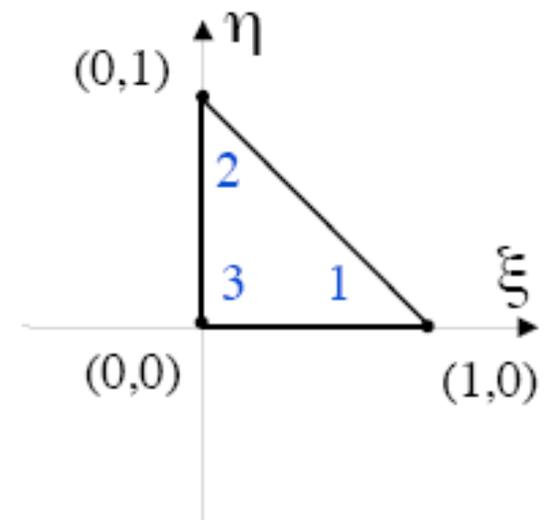
Transformation of Differential Operators

$$\frac{\partial N_k}{\partial x} = \frac{\partial N_k}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial N_k}{\partial \eta} \frac{\partial \eta}{\partial x}$$

Complicated

Easy

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} N_k = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{pmatrix} N_k = [J]_{T^{-1}} \begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{pmatrix} N_k$$

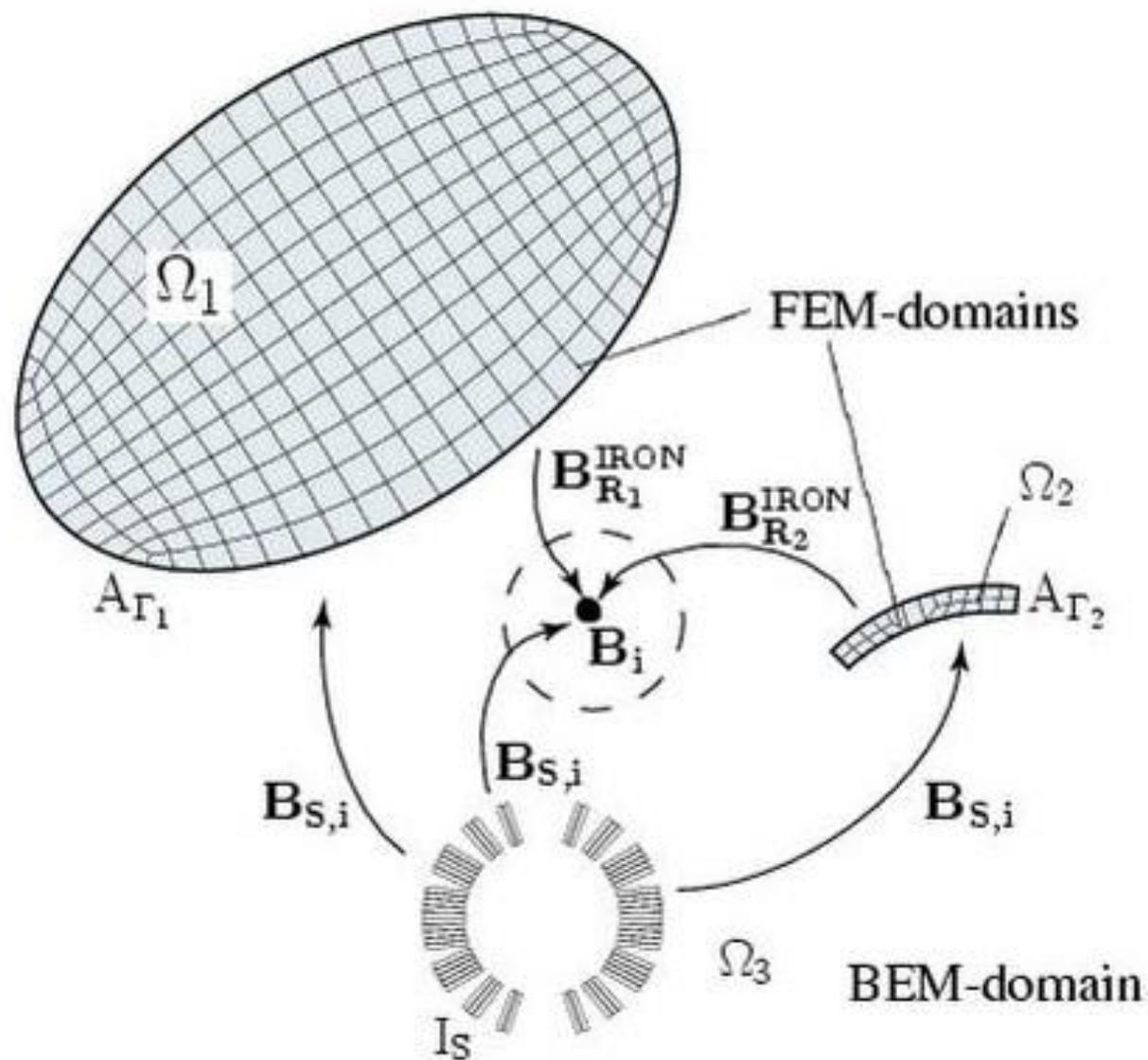


$$[J]_{T^{-1}} = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix}^{-1} = [J]_T^{-1}$$

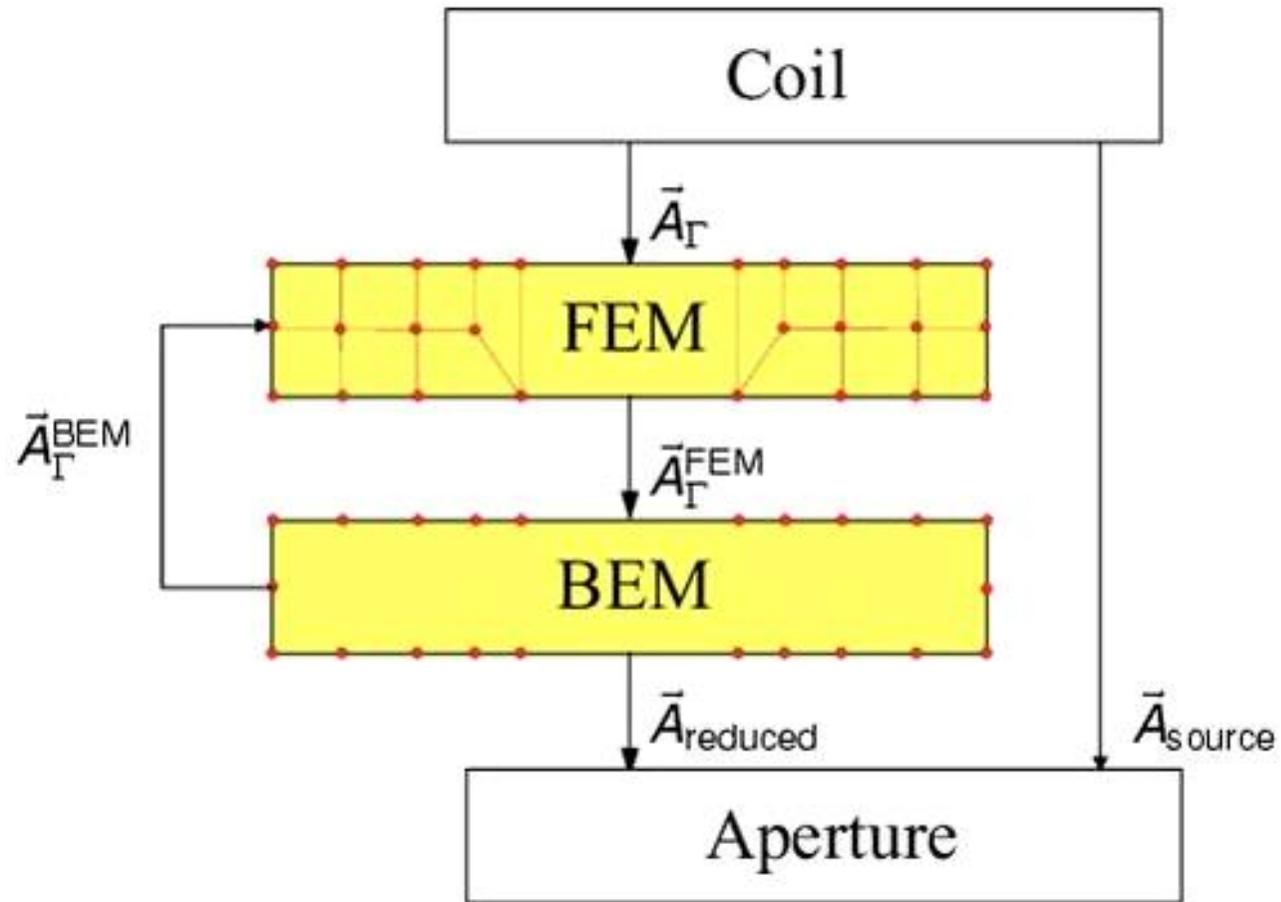
$$[J]_T = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^K \frac{\partial N_k}{\partial \xi} x^{(k)} & \sum_{k=1}^K \frac{\partial N_k}{\partial \xi} y^{(k)} \\ \sum_{k=1}^K \frac{\partial N_k}{\partial \eta} x^{(k)} & \sum_{k=1}^K \frac{\partial N_k}{\partial \eta} y^{(k)} \end{pmatrix} = \begin{pmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \dots & \frac{\partial N_K}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \dots & \frac{\partial N_K}{\partial \eta} \end{pmatrix} \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_k & y_k \end{pmatrix}$$

Note: Bad meshing is not a trivial offence

BEM-FEM Coupling (Elementary Model Problem)



BEM-FEM Coupling



Forces (N) in the Connection Ends of the LHC Main Dipole

I	F _x	F _y	F _z
1	-39.7	-44.0	-45.4
2	-6.5	3.7	-41.7
3	-6.1	88.3	-38.2
4	1.25	3.9	-28.5
5	48.1	-46.7	-48.5
Su m	-2.95	5.2	-202.3

