

# Introduction to Transverse Beam Dynamics

## Lecture 1: Magnetic fields and particle trajectories

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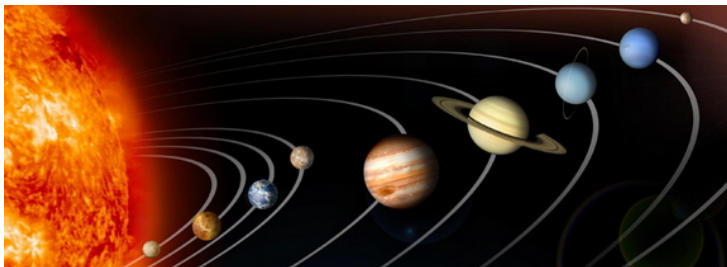
# Luminosity run of a typical storage ring

**In a storage ring:** the protons are accelerated and stored for  $\sim 12$  hours

The distance traveled by particles running at nearly the speed of light,  $v \approx c$ , for 12 hours is

$$d \approx 12 \times 10^{11} \text{ km}$$

→ it's several times the distance from Sun to Pluto and back !



# Introduction and basic ideas

It's a circular machine: we need a transverse deflecting force → the Lorentz force

$$\vec{F} = q \cdot \left( \vec{E} + \vec{v} \wedge \vec{B} \right)$$

where, in high energy machines,  $|\vec{v}| \approx c \approx 3 \cdot 10^8$  m/s. Usually there is no electric field, and the transverse deflection is given by a magnetic field only.

Example

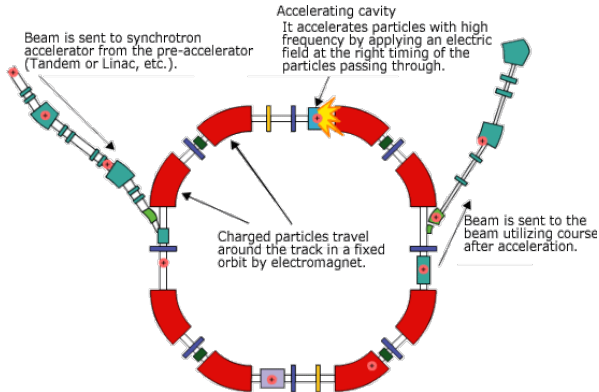
$$\begin{aligned} F &= q \cdot 3 \cdot 10^8 \frac{m}{s} \cdot 1 \text{ T} \\ B = 1 \text{ T} \rightarrow &= q \cdot 3 \cdot 10^8 \frac{m}{s} \cdot 1 \frac{Vs}{m^2} \\ &= q \cdot 300 \frac{MV}{m} \end{aligned}$$

Notice that there is a technical limit for an electric field:

$$E \lesssim 1 \frac{MV}{m}$$

Therefore in an accelerator, use magnetic fields wherever it's possible

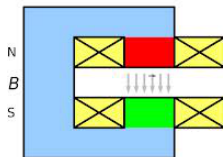
$$\left. \begin{array}{lcl} \text{Lorentz force} & F_L & = evB \\ \text{Centrifugal force} & F_{\text{centr}} & = \frac{\gamma mv^2}{\rho} \\ & & = \frac{\gamma mv^2}{\rho} \end{array} \right\} \begin{array}{l} \frac{p}{q} = B\rho \\ B\rho = \text{"beam rigidity"} \end{array}$$



# Dipole magnets: the magnetic guide

## ► Dipole magnets:

- define the ideal orbit
- in a homogeneous field created by two flat pole shoes,  $B = \frac{\mu_0 n I}{h}$



## ► Normalise magnetic field to momentum:

$$\boxed{\frac{p}{e} = B\rho \quad \Rightarrow \quad \frac{1}{\rho} = \frac{eB}{p}} \quad B = [T]; \quad p = \left[ \frac{\text{GeV}}{c} \right]; \quad 1 \text{ T} = \frac{1 \text{ V} \cdot 1 \text{ s}}{1 \text{ m}^2}$$

## ► Example: the LHC

$$\left. \begin{array}{l} B = 8.3 \text{ T} \\ p = 7000 \frac{\text{GeV}}{c} \end{array} \right\} \quad \frac{1}{\rho} = e \frac{8.3 \frac{\text{Vs}}{\text{m}^2}}{7000 \cdot 10^9 \frac{\text{eV}}{c}} = \frac{8.3 \text{ s} \cdot 3 \cdot 10^8 \frac{\text{m}}{\text{s}}}{7000 \cdot 10^9 \text{ m}^2} =$$

$$= 0.333 \cdot \frac{8.3}{7000} \frac{1}{\text{m}} = \frac{1}{2.53} \frac{1}{\text{km}}$$

# Dipole magnets: the magnetic guide

Very important rule of thumb:

$$\frac{1}{\rho [m]} \approx 0.3 \frac{B [T]}{p [GeV/c]}$$

In the LHC,  $\rho = 2.53$  km. The circumference  $2\pi\rho = 17.6$  km  $\approx 66\%$  of the entire LHC.

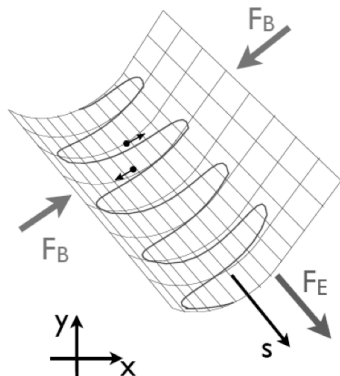
The field  $B$  is  $\approx 1 \dots 8$  T

which is a sort of “normalised bending strength”, normalised to the momentum of the particles.

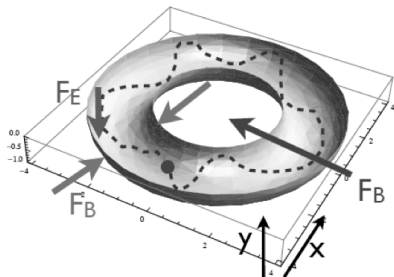
# The focusing force

$$\vec{F} = q \cdot (\vec{E} + \vec{v} \wedge \vec{B})$$

Linear Accelerator



Circular Accelerator



Remember the 1d harmonic oscillator:  $F = -kx$

# Quadrupole magnets: the focusing force

Quadrupole magnets are required to keep the trajectories in vicinity of the ideal orbit

They exert a linearly increasing Lorentz force, thru a linearly increasing magnetic field:

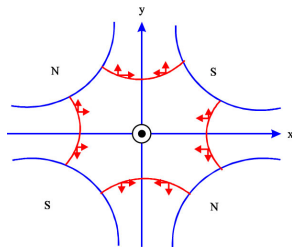
$$\begin{aligned} B_x &= -gy & \Rightarrow & F_x = -evgx \\ B_y &= -gx & \Rightarrow & F_y = evgy \end{aligned}$$

Gradient of a quadrupole magnet:

$$g = \frac{2\mu_0 nI}{r^2} \left[ \frac{T}{m} \right] = \frac{B_{\text{poles}}}{r_{\text{aperture}}} \left[ \frac{T}{m} \right]$$

► LHC main quadrupole magnets:

$$g \approx 25 \dots 220 \text{ T/m}$$



*the arrows show the force exerted on a particle*

Focusing strength:

$$k = \frac{g}{p/e} [m^{-2}]; \quad \Rightarrow \quad g = \left[ \frac{T}{m} \right]; \quad \frac{p}{e} = \left[ \frac{\text{GeV}}{c \cdot e} \right] = \left[ \frac{GV}{c} \right] = [T \cdot m]$$

A simple rule:  $k [m^{-2}] \approx 0.3 \frac{g [T/m]}{p [GeV/c]}.$

# Fringe fields

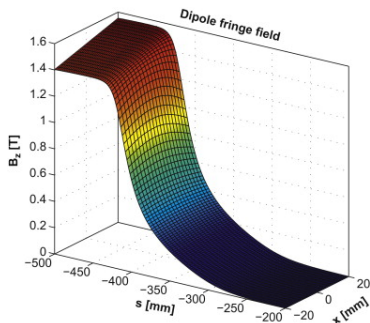
- ▶ Hard edge model:

$$x'' + \left( \frac{1}{\rho^2} - k \right) x = 0$$

this equation is not really correct

- ▶ Bending and focusing forces -even inside a magnet- depend on the position  $s$

$$x''(s) + \left\{ \frac{1}{\rho^2(s)} - k(s) \right\} x(s) = 0$$



*Dipole fringe field. Quadrupole and sextupole field components can be seen by looking at the transverse slope and curvature, respectively.*

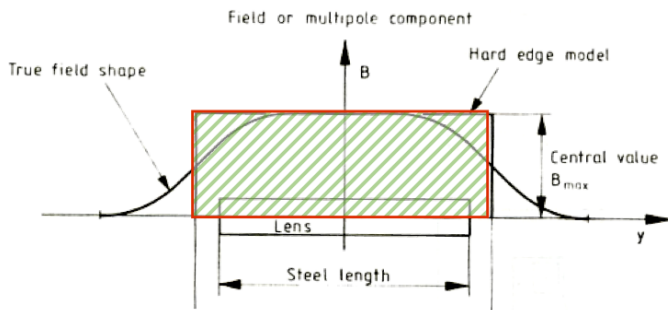
*But still: inside the magnet the focusing properties hold:*

$$\frac{1}{\rho} = \text{const}$$

$$k = \text{const}$$

# Effective length

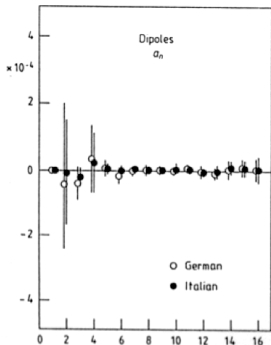
$$B \cdot L_{\text{eff}} = \int_0^{l_{\text{mag}}} B ds$$



# Multipolar moments

Taylor expansion of the  $B$  field:

$$B_y(x) = B_{y0} + \frac{\partial B_y}{\partial x}x + \frac{1}{2} \frac{\partial^2 B_y}{\partial x^2}x^2 + \frac{1}{3!} \frac{\partial^3 B_y}{\partial x^3}x^3 + \dots \quad \text{divide by } B_{y0}$$



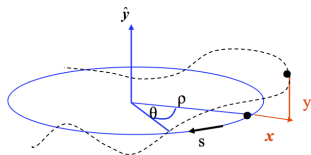
*Multipole coefficients:*

- ▶ *divide by the main field to get the relative error contribution*

# Towards the equation of motion

Linear approximation:

- ▶ the ideal particle  $\Rightarrow$  stays on the **design orbit**
- ▶ any other particle  $\Rightarrow$  has coordinates  $x, y$ 
  - ▶ which are small quantities  $x, y \ll \rho$
- ▶ only linear terms in  $x$  and  $y$  of  $B$  are taken into account



Let's recall some useful relativistic formulæ:

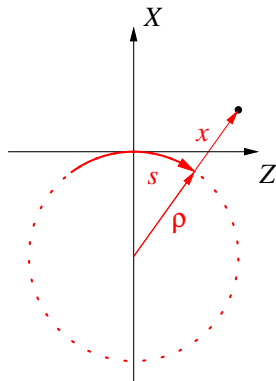
$P_0$	$= m \gamma v$	reference momentum
$\delta$		relative momentum offset
$Pc$	$= P_0 c (1 + \delta)$	total kinetic energy
$E$	$= \sqrt{P^2 c^2 + m^2 c^4}$	total energy
$\beta$	$= \frac{Pc}{E}$	relativistic beta

We assume to be ultra-relativistic...

# Towards the equation of motion

We use a Curved Reference System: the Frenet–Serret rotating frame

Curvilinear $\rightarrow$ Cartesian	Cartesian $\rightarrow$ Curvilinear
$(x, y, z) \rightarrow (X, Y, Z)$	$(X, Y, Z) \rightarrow (x, y, z)$
$z = s + \beta ct$	$s = \rho \arctan \frac{Z}{X+\rho}$
$X = (\rho + x) \cos \frac{s}{\rho} - \rho$	$x = \sqrt{(X + \rho)^2 + Z^2} - \rho$
$Y = y$	$y = Y$
$Z = (\rho + x) \sin \frac{s}{\rho}$	$z = s - \beta ct$
$P_x = P_X \cos \frac{s}{\rho} + P_Z \sin \frac{s}{\rho}$	$P_X = P_x \cos \frac{s}{\rho} - P_z \sin \frac{s}{\rho}$
$P_y = P_Y$	$P_Y = P_y$



The  $y$  and  $Y$  axes are parallel and orthogonal to this page.

# Summary of momenta and angles definitions

$$P = P_0(1 + \delta) \quad \text{total w.r.t. reference momentum}$$

$$P = \sqrt{P_x^2 + P_y^2 + P_z^2}$$

- General convention: lower case momenta: normalised to  $P_0$

$$p = \frac{P}{P_0} = 1 + \delta$$

$$p_x = \frac{P_x}{P_0}$$

$$p_y = \frac{P_y}{P_0}$$

$$p_z = \frac{P_z}{P_0} = \frac{\sqrt{P^2 - P_x^2 - P_y^2}}{P_0} = \sqrt{(1 + \delta)^2 - p_x^2 - p_y^2} \approx$$

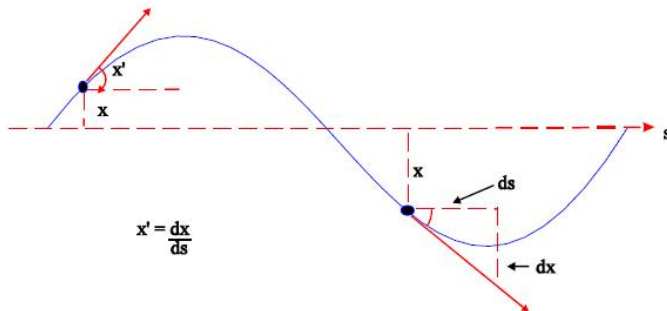
$$\approx (1 + \delta) \left( 1 - \frac{1}{2} \frac{p_x^2 + p_y^2}{(1 + \delta)^2} \right) =$$

$$= 1 + \delta - \frac{1}{2} \frac{p_x^2 + p_y^2}{1 + \delta} \approx 1 + \delta \text{ for small } p_x \text{ and } p_y$$

$$x' = \frac{dx}{ds} = \frac{P_x}{P_z} = \frac{p_x}{p_z} \approx \frac{p_x}{1 + \delta}$$

$$y' = \frac{dy}{ds} = \frac{P_y}{P_z} = \frac{p_y}{p_z} \approx \frac{p_y}{1 + \delta}$$

# Representation of the transverse coordinates



With

$$x' = \frac{dx}{ds} = \frac{P_x}{P_z} \approx \frac{P_x}{P_0(1 + \delta)}; \quad y' = \frac{dy}{ds} = \frac{P_y}{P_z} \approx \frac{P_y}{P_0(1 + \delta)}$$

The state of a particle (the phase space) is represented with a 6-dimensional vector:

$$(x, x', y, y', z = s - \beta ct, \delta)$$

# Towards the equation of motion

Taylor expansion of the  $B$  field:

$$B_y(x) = B_{y0} + \frac{\partial B_y}{\partial x}x + \frac{1}{2} \frac{\partial^2 B_y}{\partial x^2}x^2 + \frac{1}{3!} \frac{\partial^3 B_y}{\partial x^3}x^3 + \dots$$

if we normalise to the momentum  $p/e = B\rho$

$$\begin{aligned} \frac{B(x)}{p/e} &= \frac{B_0}{B_0\rho} + \frac{g}{p/e}x + \frac{1}{2} \frac{eg'}{p/e}x^2 + \frac{1}{3!} \frac{eg''}{p/e}x^3 + \dots \\ &= \frac{1}{\rho} + kx + \frac{1}{2}mx^2 + \frac{1}{3!}nx^3 + \dots \end{aligned}$$

In the linear approximation, only the terms linear in  $x$  and  $y$  are taken into account:

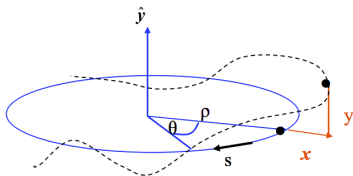
- ▶ dipole fields
- ▶ quadrupole fields

It is more practical to use “separate function” machines:

- ▶ split the magnets and optimise them regarding their function
  - ▶ bending
  - ▶ focusing, etc.

# The equation of motion in radial coordinates

Let's consider a local segment of one particle's trajectory:



the radial acceleration is  $a_r = \frac{d^2\rho}{dt^2} - \rho \left( \frac{d\theta}{dt} \right)^2 = \frac{d^2\rho}{dt^2} - \rho\omega^2$ . In our case, for the

ideal orbit:  $\rho = \text{const} \Rightarrow \frac{d\rho}{dt} = 0$

$\Rightarrow$  the force is  $F = m\rho \left( \frac{d\theta}{dt} \right)^2 = m\rho\omega^2$

$$F = mv^2/\rho$$

For a general trajectory:

$$\rho \rightarrow \rho + x : \quad F = m a_r \quad \Rightarrow \quad m \left[ \frac{d^2}{dt^2} (\rho + x) - \frac{v^2}{\rho + x} \right] = eB_y v$$

$$F = \underbrace{m \frac{d^2}{dt^2} (\rho + x)}_{\text{term 1}} - \underbrace{\frac{mv^2}{\rho + x}}_{\text{term 2}} = eB_y v$$

- Term 1. As  $\rho = \text{const} \dots$

$$m \frac{d^2}{dt^2} (\rho + x) = \frac{d^2}{dt^2} x$$

- Term 2. Remember:  $x \approx \text{mm}$  whereas  $\rho \approx \text{m} \rightarrow$  we develop for small  $x$

$$\frac{1}{\rho + x} \overset{\text{remember}}{\approx} \frac{1}{\rho} \left( 1 - \frac{x}{\rho} \right)$$

Taylor expansion:

$$f(x) = f(x_0) + (x - x_0) f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0) + \dots$$

$$m \frac{d^2 x}{dt^2} - \frac{mv^2}{\rho} \left( 1 - \frac{x}{\rho} \right) = eB_y v$$

The guide field in linear approximation  $B_y = B_0 + x \frac{\partial B_y}{\partial x}$

$$m \frac{d^2 x}{dt^2} - \frac{mv^2}{\rho} \left(1 - \frac{x}{\rho}\right) = ev \left\{ B_0 + x \frac{\partial B_y}{\partial x} \right\} \quad \text{let's divide by } m$$

$$\frac{d^2 x}{dt^2} - \frac{v^2}{\rho} \left(1 - \frac{x}{\rho}\right) = \frac{evB_0}{m} + x \frac{evg}{m}$$

Independent variable:  $t \rightarrow s$

$$\frac{dx}{dt} = \frac{dx}{ds} \frac{ds}{dt} = x' v$$

$$\frac{d^2 x}{dt^2} = \frac{d}{dt} \frac{dx}{dt} = \frac{d}{dt} \left( \underbrace{\frac{dx}{ds}}_{x'} \underbrace{\frac{ds}{dt}}_v \right) = \frac{d}{dt} (x' v) =$$

$$= \frac{d}{ds} \underbrace{\frac{ds}{dt}}_v (x' v) = \frac{d}{ds} (x' v^2) = x'' v^2 + x' 2v \cancel{\frac{dv}{ds}}$$

$$x'' v^2 - \frac{v^2}{\rho} \left(1 - \frac{x}{\rho}\right) = \frac{evB_0}{m} + x \frac{evg}{m} \quad \text{let's divide by } v^2$$

$$x'' - \frac{1}{\rho} \left( 1 - \frac{x}{\rho} \right) = \frac{eB_0}{mv} + x \frac{eg}{mv}$$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} = \frac{B_0}{p/e} + \frac{xg}{p/e}$$

$$x'' - \cancel{\frac{1}{\rho}} + \frac{x}{\rho^2} = \cancel{\frac{1}{\rho}} + kx$$

Remember:

$$mv = p$$

Normalise to the momentum of the particle:

$$\frac{g}{p/e} \frac{B_0}{p/e} = -\frac{1}{\rho}; \quad \frac{g}{p/e} = k.$$

$$x'' + x \left( \frac{1}{\rho^2} - k \right) = 0$$

Equation for the vertical motion

- ▶  $\frac{1}{\rho^2} = 0$  usually there are not vertical bends
- ▶  $k \longleftrightarrow -k$  quadrupole field changes sign

$$y'' + ky = 0$$

## Remarks

- ▶ Weak focusing:

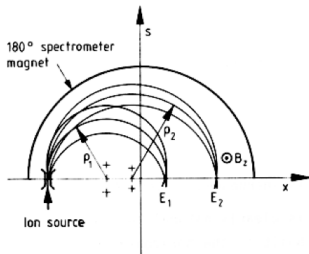
$$x'' + \left( \frac{1}{\rho^2} - k \right) x = 0$$

there is a focusing force even without a quadrupole gradient

$$k = 0 \quad \Rightarrow \quad x'' = -\frac{1}{\rho^2} x$$

even without quadrupoles there is retrieving force (focusing) in the bending plane of the dipole magnets

- ▶ In large machine this effect is very weak...



*Mass spectrometer: particles are separated according to their energy and focused due to the  $1/\rho$  effect of the dipole*

# Solution of the trajectory equations: focusing quadrupole

Definition:

$$\left. \begin{array}{l} \text{horizontal plane } K = 1/\rho^2 - k \\ \text{vertical plane } K = k \end{array} \right\} x'' + Kx = 0$$

This is the differential equation of a harmonic oscillator ... with spring constant  $K$ . We make an ansatz:

$$x(s) = a_1 \cos(\omega s) + a_2 \sin(\omega s)$$

General solution: a linear combination of two independent solutions:

$$x'(s) = -a_1 \omega \sin(\omega s) + a_2 \omega \cos(\omega s)$$

$$x''(s) = -a_1 \omega^2 \cos(\omega s) + a_2 \omega^2 \sin(\omega s) = -\omega^2 x(s) \rightarrow \omega = \sqrt{K}$$

General solution, for  $K > 0$ :

$$x(s) = a_1 \cos(\sqrt{K}s) + a_2 \sin(\sqrt{K}s)$$

We determine  $a_1$ ,  $a_2$  by imposing the following boundary conditions:

$$s = 0 \rightarrow \begin{cases} x(0) = x_0, & a_1 = x_0 \\ x'(0) = x'_0, & a_2 = \frac{x'_0}{\sqrt{K}} \end{cases}$$

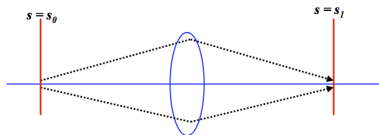
Horizontal focusing quadrupole,  $K > 0$ :

$$x(s) = x_0 \cos(\sqrt{K}s) + x'_0 \frac{1}{\sqrt{K}} \sin(\sqrt{K}s)$$

$$x'(s) = -x_0 \sqrt{K} \sin(\sqrt{K}s) + x'_0 \cos(\sqrt{K}s)$$

For convenience we can use a matrix formalism:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_1} = M_{\text{foc}} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}_{s_0}$$



Where:

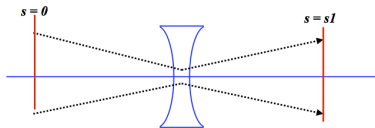
$$M_{\text{foc}} = \begin{pmatrix} \cos(\sqrt{K}s) & \frac{1}{\sqrt{K}} \sin(\sqrt{K}s) \\ -\sqrt{K} \sin(\sqrt{K}s) & \cos(\sqrt{K}s) \end{pmatrix}$$

# Defocusing quadrupole

The equation of motion is

$$x'' + Kx = 0$$

with  $K < 0$



Remember:

$$f(s) = \cosh(s)$$

$$f'(s) = \sinh(s)$$

Now the solution is in the form:

$$x(s) = a_1 \cosh(\omega s) + a_2 \sinh(\omega s)$$

and the transfer matrix:

$$M_{\text{defoc}} = \begin{pmatrix} \cosh(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}} \sinh(\sqrt{|K|}s) \\ \sqrt{|K|} \sinh(\sqrt{|K|}s) & \cosh(\sqrt{|K|}s) \end{pmatrix}$$

Notice that for a drift space, when  $K = 0 \rightarrow M_{\text{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$

# Summary of the transfer matrices

- ▶ Focusing quad,  $K > 0$

$$M_{\text{foc}} = \begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}} \sin(\sqrt{K}L) \\ -\sqrt{K} \sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix}$$

- ▶ Defocusing quad,  $K < 0$

$$M_{\text{defoc}} = \begin{pmatrix} \cosh(\sqrt{|K|}L) & \frac{1}{\sqrt{|K|}} \sinh(\sqrt{|K|}L) \\ \sqrt{|K|} \sinh(\sqrt{|K|}L) & \cosh(\sqrt{|K|}L) \end{pmatrix}$$

- ▶ Drift space,  $K = 0$

$$M_{\text{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

With the assumptions we have made, the motion in the horizontal and vertical planes is independent: “... the particle motion in  $x$  and  $y$  is uncoupled”

# Thin-lens approximation

When the focal length,  $f$ , of the lens is much bigger than the length of the magnet  $L$

$$f = \frac{1}{K \cdot L} \gg L$$

we can derive the limit for  $L \rightarrow 0$  while we keep  $K \cdot L = \text{const.}$

The transfer matrices are

$$M_x = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \quad M_y = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

focusing, and defocusing respectively.

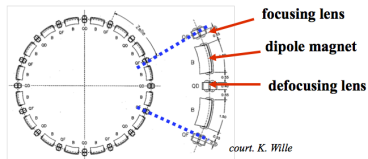
This approximation (yet quite accurate, in large machines) is useful for fast calculations... and for the guided studies !

# Transformation through a system of lattice elements

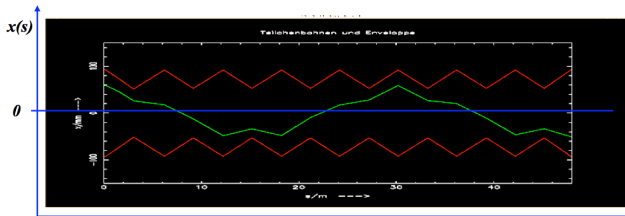
One can compute the solution of a system of elements, by multiplying the matrices of each single element:

$$M_{\text{total}} = M_{\text{QF}} \cdot M_{\text{D}} \cdot M_{\text{Bend}} \cdot M_{\text{D}} \cdot M_{\text{QD}} \cdot \dots$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_2} = M_{s_1 \rightarrow s_2} \cdot M_{s_0 \rightarrow s_1} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$



In each accelerator element the particle trajectory corresponds to the movement of a harmonic oscillator.



...typical values are:

$$x \approx \text{mm}$$

$$x' \leq \text{mrad}$$

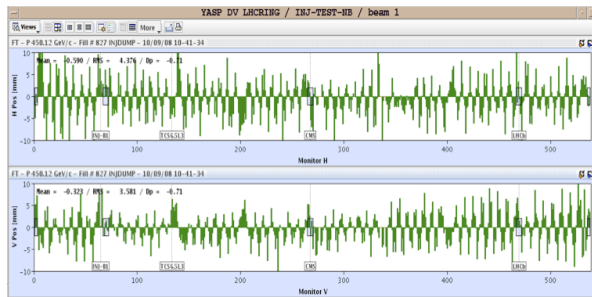
# Orbit and Tune

Tune: the number of oscillations per turn.

Example:

64.31

59.32

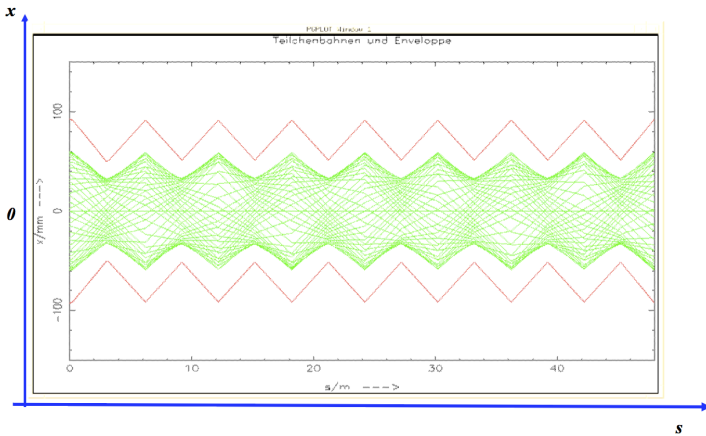


Relevant for beam stability studies is : the non-integer part

# Envelope

Question: what will happen, if the particle performs a second turn ?

- ... or a third one or ...  $10^{10}$  turns ...



# Summary

beam rigidity:  $B\rho = \frac{p}{q}$

bending strength of a dipole:  $\frac{1}{\rho} [m^{-1}] = \frac{0.2998 \cdot B_0 [T]}{p [\text{GeV}/c]}$

focusing strength of a quadrupole:  $k [m^{-2}] = \frac{0.2998 \cdot g}{p [\text{GeV}/c]}$

focal length of a quadrupole:  $f = \frac{1}{k \cdot L_Q}$

equation of motion:  $x'' + Kx = \frac{1}{\rho} \frac{\Delta p}{p}$

transfer matrix of a foc. quad:  $x_{s2} = M \cdot x_{s1}$

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}} \sin(\sqrt{K}L) \\ -\sqrt{K} \sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix}$$

$$M_{QD} = \begin{pmatrix} \cosh(\sqrt{|K|}L) & \frac{1}{\sqrt{|K|}} \sinh(\sqrt{|K|}L) \\ \sqrt{|K|} \sinh(\sqrt{|K|}L) & \cosh(\sqrt{|K|}L) \end{pmatrix} \quad M_D = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

# Bibliography

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