

# Exercises on Space Charge

## Exercise 1

*Compute the transverse space charge forces and the tune shifts for a cylindrical beam in a circular beam pipe, having the following longitudinal distributions: parabolic, sinusoidal modulation, Gaussian*

*parabolic*

$$\lambda(z) = \frac{3Ne}{2l_o} \left[ 1 - \left( \frac{2z}{l_o} \right)^2 \right]$$

*sinusoidal modulation*     $\lambda(z) = \lambda_o + \Delta\lambda \cos(k_z z)$  ;  $k_z = 2\pi/\lambda_w$

*Gaussian*

$$\lambda(z) = \frac{Ne}{\sqrt{2\pi}\sigma_z} \exp\left(-\frac{z^2}{2\sigma_z^2}\right)$$

$$E_r(\boldsymbol{r})=\frac{\lambda(z)}{2\pi~\varepsilon_o}\frac{r}{a^2};~~~B_{\theta}(\boldsymbol{r})=\frac{\lambda(z)\beta}{2\pi\varepsilon_o c}\frac{r}{a^2}$$

$$F_{\perp}(r)=e(E_r-\beta c~B_{\theta})=\frac{e}{\gamma^2}E_r=\frac{e}{\gamma^2}\frac{\lambda(z)}{2\pi~\varepsilon_o}\frac{r}{a^2}$$

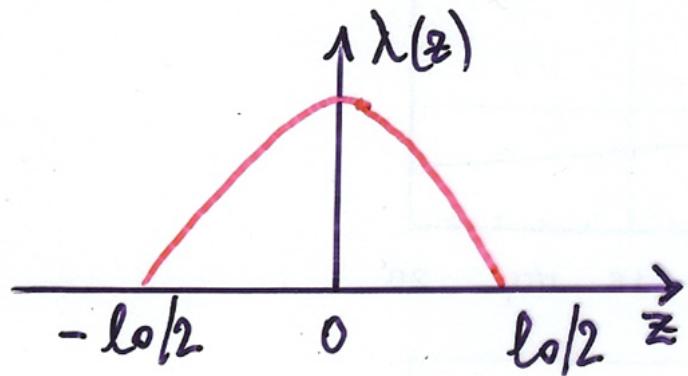
$$\Delta Q_x=-\frac{\rho_x^2}{2\beta^2 E_0Q_{x0}}\Bigg(\frac{\partial~F^{s.c.}_x}{\partial x}\Bigg)=-\frac{\rho_x^2}{2\beta^2 E_0Q_{x0}}\frac{e}{\gamma^2}\frac{\lambda(z)}{2\pi~\varepsilon_o a^2}=-\frac{\rho_x^2\lambda(z)r_{e,p}}{e\beta^2\gamma^3 a^2 Q_{x0}}$$

$$r_{e,p}=\frac{e^2}{4\pi \varepsilon_o m_o c^2}~(electrons: 2.82~10^{-15}m, protons: 1.53~10^{-18}m)$$

$$F_{\perp}(r) = \frac{e}{\gamma^2} \frac{\lambda(z)}{2\pi \epsilon_o} \frac{r}{a^2}$$

$$\Delta Q_x = -\frac{\rho_x^2 \lambda(z) r_{e,p}}{e \beta^2 \gamma^3 a^2 Q_{x0}}$$

### a) Parabolic bunch ( $q_0 = Ne$ )



$$\lambda(z) = \frac{3Ne}{2l_o} \left[ 1 - \left( \frac{2z}{l_o} \right)^2 \right]$$

$$\Delta Q_{\max} (\text{at } z = 0) = -\frac{\rho_x^2 r_{e,p}}{\beta^2 \gamma^3 a^2 Q_{x0}} \frac{3N}{2l_0}$$

$$\Delta Q_{\min} \left( \text{at } z = \pm \frac{l_0}{2} \right) = 0$$

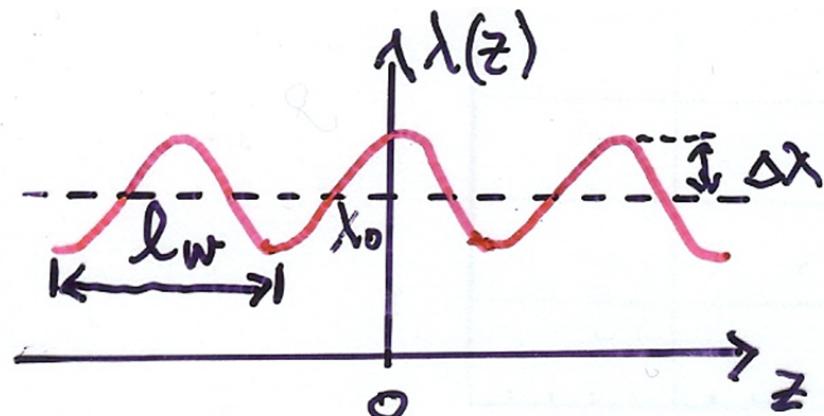
$$\Delta Q_{\text{spread}} = \Delta Q_{\max} - \Delta Q_{\min} = \Delta Q_{\max}$$

$$F_{\perp}(r) = \frac{e}{\gamma^2} \frac{\lambda(z)}{2\pi \epsilon_o} \frac{r}{a^2}$$

$$\Delta Q_x = -\frac{\rho_x^2 \lambda(z) r_{e,p}}{e \beta^2 \gamma^3 a^2 Q_{x0}}$$

## b) Sinusoidal modulation ( $\lambda_0 = N_e / I_0$ )

$$\lambda(z) = \lambda_0 + \Delta\lambda \cos(k_z z) ; k_z = 2\pi / \lambda_w$$



$$\Delta Q_{\max} (\text{at } k_z z = 2n\pi) = -\frac{\rho_x^2 r_{e,p} (\lambda_0 + \Delta\lambda)}{e \beta^2 \gamma^3 a^2 Q_{x0}}$$

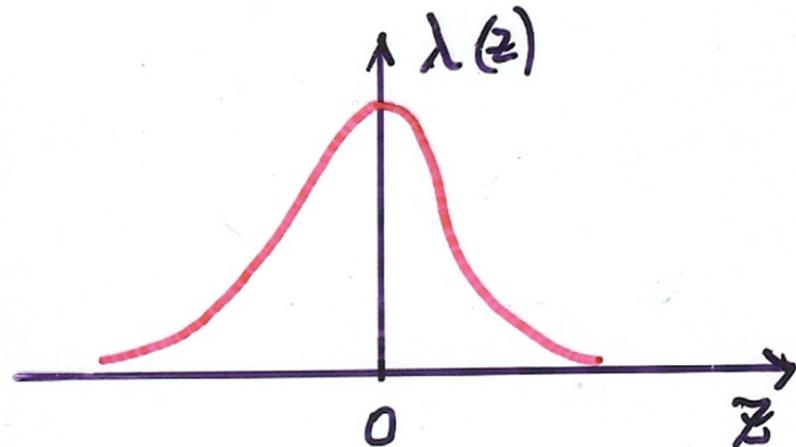
$$\Delta Q_{\min} (\text{at } k_z z = (2n+1)\pi) = -\frac{\rho_x^2 r_{e,p} (\lambda_0 - \Delta\lambda)}{e \beta^2 \gamma^3 a^2 Q_{x0}}$$

$$\Delta Q_{\text{spread}} = \Delta Q_{\max} - \Delta Q_{\min} = -\frac{2\rho_x^2 r_{e,p} \Delta\lambda}{e \beta^2 \gamma^3 a^2 Q_{x0}}$$

$$F_{\perp}(r) = \frac{e}{\gamma^2} \frac{\lambda(z)}{2\pi \epsilon_o} \frac{r}{a^2}$$

$$\Delta Q_x = -\frac{\rho_x^2 \lambda(z) r_{e,p}}{e \beta^2 \gamma^3 a^2 Q_{x0}}$$

### c) Gaussian bunch ( $\mathbf{q}_0 = N\mathbf{e}$ )



$$\lambda(z) = \frac{Ne}{\sqrt{2\pi}\sigma_z} \exp\left(-\frac{z^2}{2\sigma_z^2}\right)$$

$$\Delta Q_{\max} (\text{at } z = 0) = -\frac{\rho_x^2 r_{e,p}}{\beta^2 \gamma^3 a^2 Q_{x0}} \frac{N}{\sqrt{2\pi}\sigma_z}$$

$$\Delta Q_{\min} (\text{at } z \rightarrow \pm\infty) = 0$$

$$\Delta Q_{\text{spread}} = \Delta Q_{\max} - \Delta Q_{\min} = \Delta Q_{\max}$$

## Exercise 2

*Compute the transverse space charge force and the tune shift for a cylindrical beam in a circular beam pipe, having a bi-Gaussian longitudinal and transverse distribution.*

*bi - Gaussian*

$$\lambda(z) = \frac{Ne}{\sqrt{2\pi}\sigma_z} \exp\left(-\frac{z^2}{2\sigma_z^2}\right)$$

$$\rho(r, z) = \frac{\lambda(z)}{2\pi\sigma_r^2} \exp\left(\frac{-r^2}{2\sigma_r^2}\right)$$

*bi-Gaussian*

$$\lambda(z) = \frac{Ne}{\sqrt{2\pi}\sigma_z} \exp\left(-\frac{z^2}{2\sigma_z^2}\right)$$

$$\rho(r, z) = \frac{\lambda(z)}{2\pi\sigma_r^2} \exp\left(\frac{-r^2}{2\sigma_r^2}\right)$$

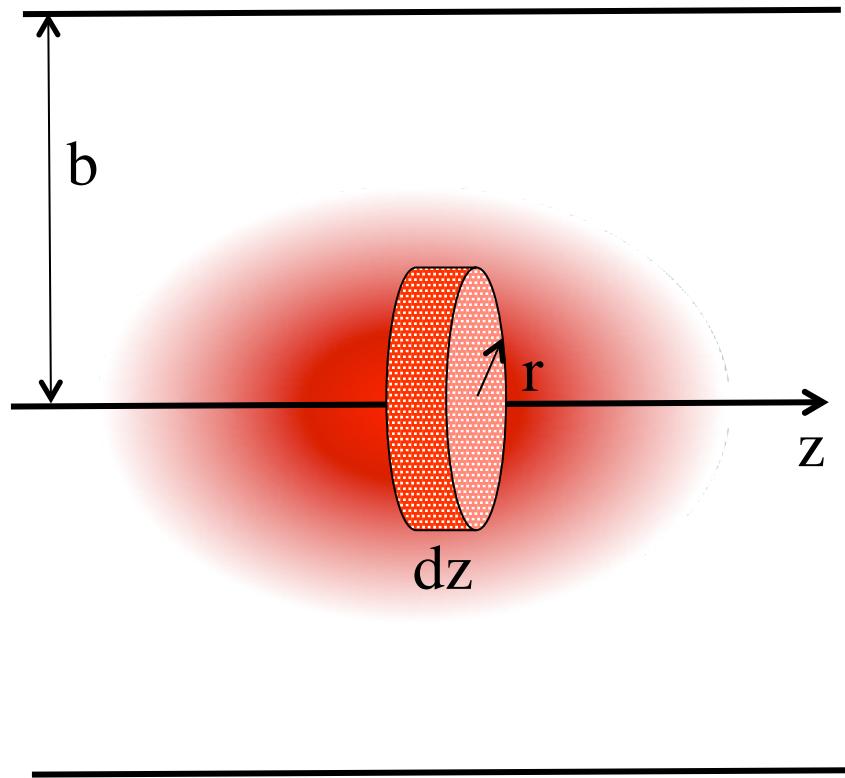
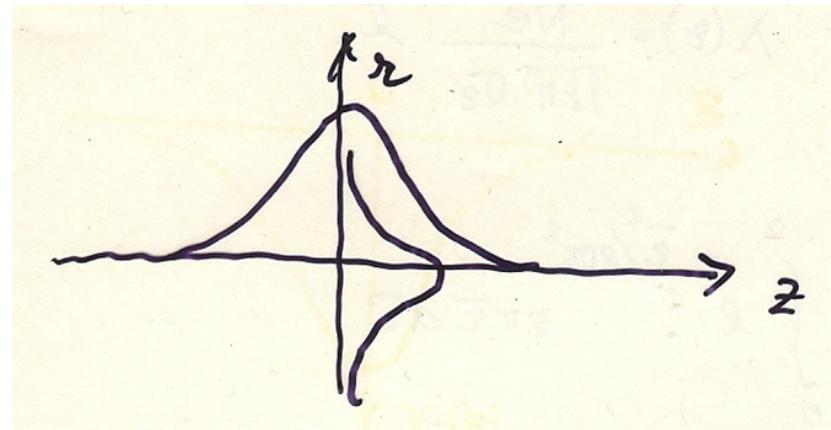
$$\int \vec{E} \cdot \hat{n} \, dS = \frac{q(r)}{\epsilon_0} \quad (\gamma \gg 1 \quad E_z \approx 0)$$

$$E_r(r) 2\pi r dz = \frac{dz}{\epsilon_0} \int_0^r \rho(r', z) 2\pi r' dr'$$

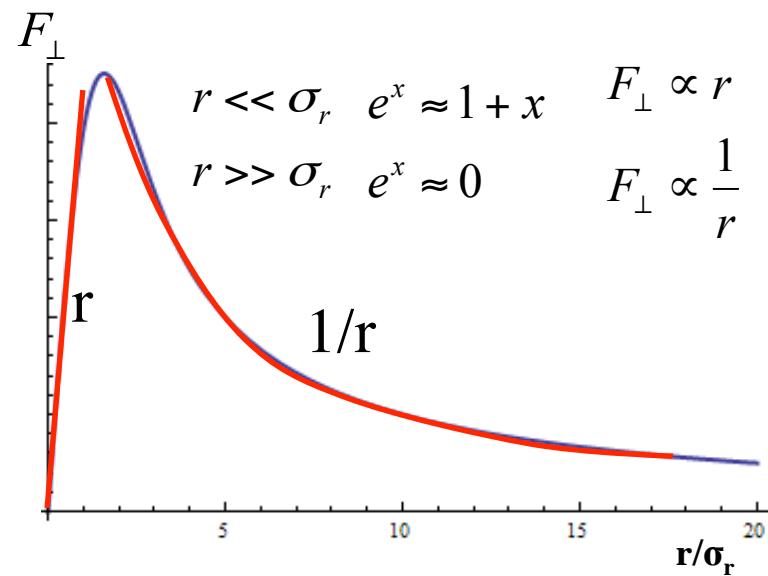
$$E_r(r) = \frac{\lambda(z)}{2\pi\epsilon_0\sigma_r^2 r} \int_0^r \exp\left(\frac{-r'^2}{2\sigma_r^2}\right) r' dr'$$

$$E_r(r) = \frac{\lambda(z)}{2\pi\epsilon_0 r} \left[ -\exp\left(\frac{-r'^2}{2\sigma_r^2}\right) \right]_0^r = \frac{\lambda(z)}{2\pi\epsilon_0} \left[ \frac{1 - \exp\left(\frac{-r^2}{2\sigma_r^2}\right)}{r} \right]$$

$$F_\perp(r) = \frac{e}{\gamma^2} E_r = \frac{e\lambda(z)}{2\pi\epsilon_0\gamma^2} \left[ \frac{1 - \exp\left(\frac{-r^2}{2\sigma_r^2}\right)}{r} \right]$$



$$F_{\perp}(r) = \frac{e\lambda(z)}{2\pi\varepsilon_0\gamma^2} \left[ \frac{1 - \exp\left(\frac{-r^2}{2\sigma_r^2}\right)}{r} \right] \approx \frac{e\lambda(z)}{2\pi\varepsilon_0\gamma^2} \frac{r}{2\sigma_r^2} \quad (r \ll \sigma_r)$$



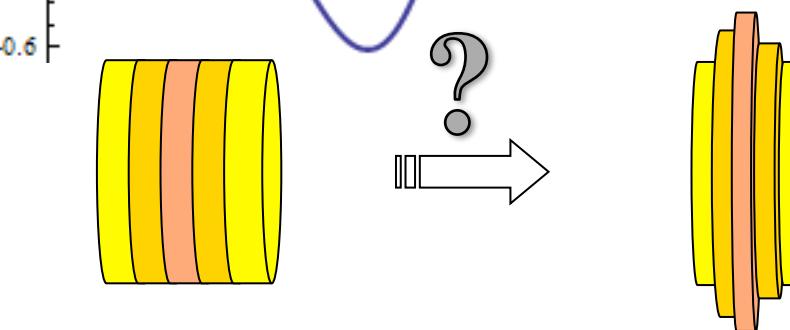
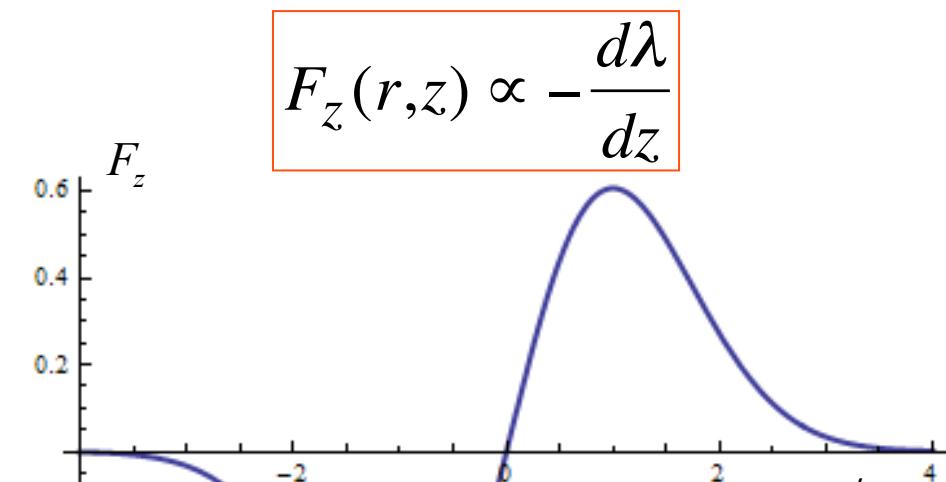
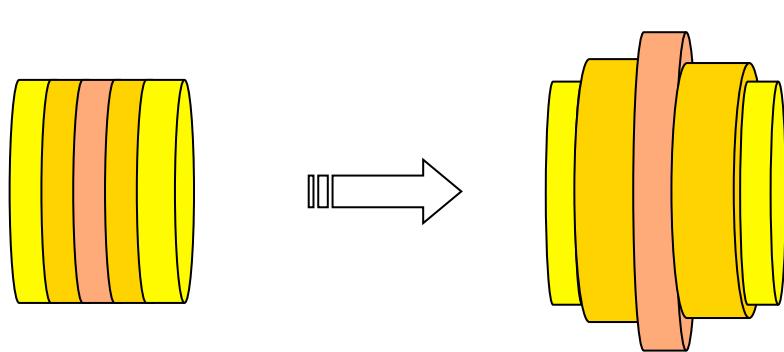
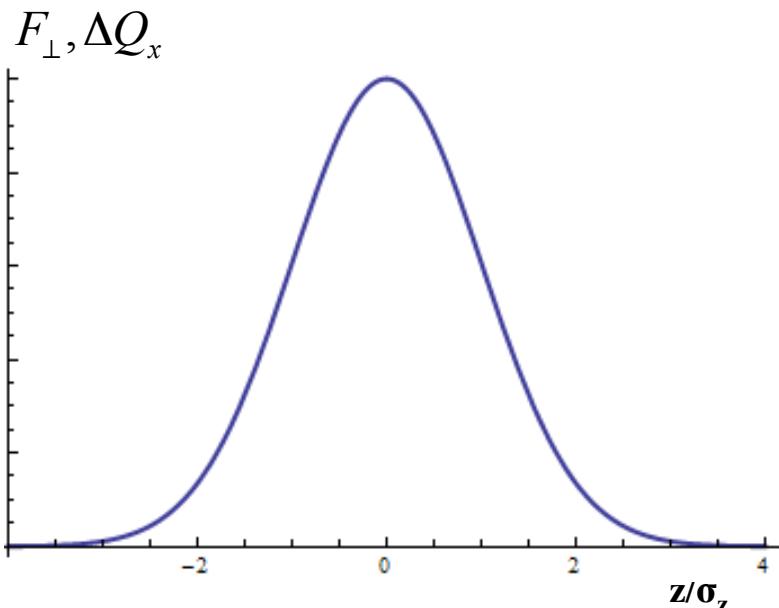
$(r \ll \sigma_r)$

$$\Delta Q_x = -\frac{\rho_x^2}{2\beta^2 E_0 Q_{x0}} \left( \frac{\partial F_x^{s.c.}}{\partial x} \right) = -\frac{\rho_x^2}{2\beta^2 E_0 Q_{x0}} \frac{e}{\gamma^2} \frac{\lambda(z)}{2\pi \varepsilon_0} \frac{1}{2\sigma_x^2}$$

## Longitudinal coordinate ( $r$ fixed)

$$F_{\perp}, \Delta Q_x \propto \lambda(z)$$

$$F_z(r, z) = -\frac{e}{\gamma^2} \frac{\partial}{\partial z} \int_r^b E_r(r', z) dr'$$



### Exercise 3

*Compute the longitudinal space charge force of a transverse uniform cylindrical beam in a circular perfectly conducting beam pipe*

$$E_z(r,z) = -\frac{1}{\gamma^2} \frac{\partial}{\partial z} \int_r^b E_r(r,z) dr$$

$$F_z(r,z) = -\frac{e}{\gamma^2} \frac{\partial}{\partial z} \int_r^b E_r(r',z) dr'$$

$$E_r(r \leq a) = \frac{\lambda(z)}{2\pi \epsilon_0} \frac{r}{a^2}$$

$$E_r(r \geq a) = \frac{\lambda(z)}{2\pi \epsilon_0 r}$$

$$F_z(r,z) = -\frac{e}{2\pi \epsilon_0 \gamma^2} \left[ \int_r^a \frac{r'}{a^2} dr' + \int_a^b \frac{1}{r'} dr' \right] \frac{\partial \lambda(z)}{\partial z}$$

$$F_z(r,z) = -\frac{e}{4\pi \epsilon_0 \gamma^2} \left( 1 - \frac{r^2}{a^2} + 2 \ln \frac{b}{a} \right) \frac{\partial \lambda(z)}{\partial z}$$

## Exercise 4

*Compute the longitudinal space charge forces for a cylindrical beam in a circular beam pipe, having the following longitudinal distributions: parabolic, sinusoidal modulation, Gaussian*

*parabolic*

$$\lambda(z) = \frac{3Ne}{2l_o} \left[ 1 - \left( \frac{2z}{l_o} \right)^2 \right]$$

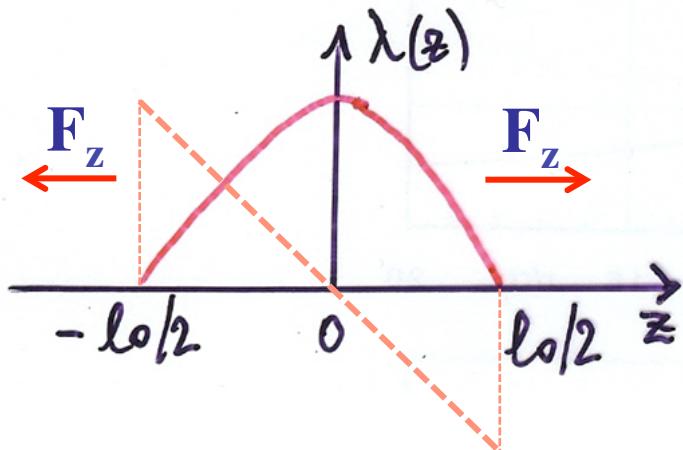
*sinusoidal modulation*    $\lambda(z) = \lambda_o + \Delta\lambda \cos(k_z z)$  ;  $k_z = 2\pi/l_w$

*Gaussian*

$$\lambda(z) = \frac{Ne}{\sqrt{2\pi}\sigma_z} \exp\left(-\frac{z^2}{2\sigma_z^2}\right)$$

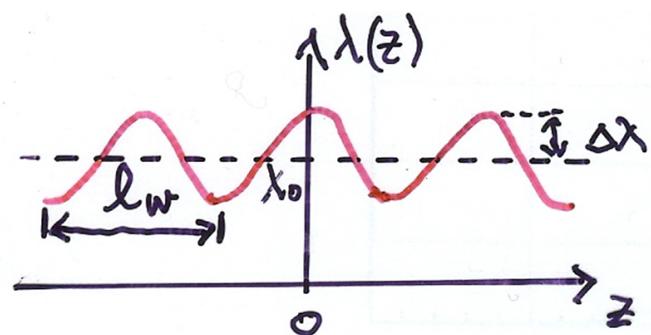
$$F_z(r, z) = -\frac{e}{\gamma^2} \frac{\partial}{\partial z} \int_r^b E_r(r', z) dr'$$

$$F_z(r, z) = -\frac{e}{4\pi\epsilon_0\gamma^2} \left( 1 - \frac{r^2}{a^2} + 2 \ln \frac{b}{a} \right) \frac{\partial \lambda(z)}{\partial z}$$



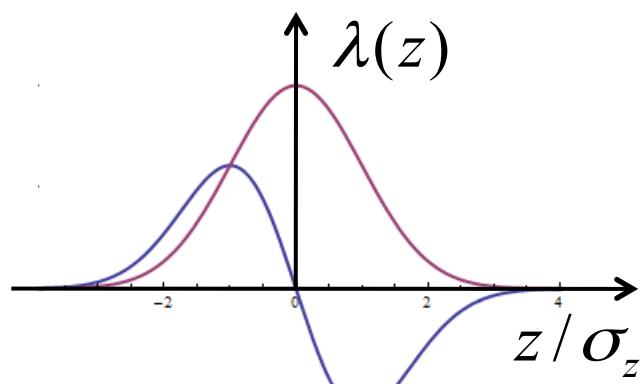
$$\lambda(z) = \frac{3Ne}{2l_o} \left[ 1 - \left( \frac{2z}{l_o} \right)^2 \right]$$

$$\frac{d\lambda(z)}{dz} = -\frac{12Ne}{l_0^3} z$$



$$\lambda(z) = \lambda_o + \Delta\lambda \cos(k_z z) ; k_z = 2\pi/l_w$$

$$\frac{d\lambda(z)}{dz} = -\Delta\lambda k_z \sin(k_z z)$$



$$\lambda(z) = \frac{Ne}{\sqrt{2\pi}\sigma_z} \exp\left(-\frac{z^2}{2\sigma_z^2}\right)$$

$$\frac{d\lambda(z)}{dz} = -\frac{Ne}{\sqrt{2\pi}\sigma_z^3} z \exp\left(-\frac{z^2}{2\sigma_z^2}\right)$$

## Exercise 5

*Compute the incoherent betatron tune shift of a uniform proton beam inside two perfectly conducting parallel plates*

$$\Delta Q_x = -\frac{\rho_x^2}{2\beta^2 E_0 Q_{x0}} \left( \frac{\partial F_x^{s.c.}}{\partial x} \right)$$

$$F_x(z, x) = \frac{e\lambda_0 x}{\pi \epsilon_0} \left( \frac{1}{2a^2 \gamma^2} - \frac{\pi^2}{48h^2} \right)$$

$$\Delta Q_x = -\frac{\rho_x^2 e \lambda_0}{2\pi \epsilon_0 \beta^2 E_0 Q_{x0}} \left( \frac{1}{2a^2 \gamma^2} - \frac{\pi^2}{48h^2} \right)$$