$\eta - \pi$ isospin violating form factors

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*** Light Meson Dynamics ***

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Motivations

- Light mesons (P, also V, A, S)

  Probe spont. symm. breaking in QCD

  QCD $\longrightarrow$ Effective theory, *weakly coupled* (m_u, m_d, m_s, p_i)

- Isospin breaking phenomena

  Small because $m_u - m_d << m_s << 1$ GeV

  Let $$Q^2 = \frac{m_s^2 - m_{ud}^2}{m_d^2 - m_u^2}$$ (Leutwyler's ratio)

  $Q \simeq 24.2$ [$m_{K^+}^2 - m_{K^0}^2$, +Dashen's LET]
  $Q = 22.6(7)(6)$ [LQCD, FLAG 1310.8555]
  But
  $Q \simeq 20.9$ [$\eta \to 3\pi$, +NLO ChPT]

- Chiral eff. theory slightly *inefficient* for final-state interaction
  → Combine with dispersion relations

- Isospin breaking in form factors (KLOE, KTeV, ISTRA+, NA48...)

  $$K^+_{l3}: \quad \langle K^+ | j^{ud}_\mu | \pi^0 \rangle$$
  $$K^0_{l3}: \quad \langle K^0 | j^{ud}_\mu | \pi^- \rangle$$

  Then

  $Q \simeq 20.7 \pm 1.2$ [Kastner, Neufeld (2008)]

- Consider here: η_{l3}: $\langle\eta|j_\mu^{ud}|\pi^-\rangle$
  - → Two form factors vanish in isospin limit (*second class* [Weinberg (1958)])
  - → Could be measured in tau-charm factories: via $\tau \to \eta\pi\nu$
  - → Probe influence of Q
  - → But also: $\eta\pi \to \pi\pi$ amplitude (via unitarity relations)
  - → ChPT calc. NLO ([Neufeld, Rupertsberger (1995)]), supplemented with unitarity/analyticity

Unitarity relations

Kinematics:

$$\langle \eta\pi^+ | j_\mu^{ud} | 0 \rangle = -\sqrt{2}\,[\, f_+^{\eta\pi}(s)\,(p_\eta - p_\pi)^\mu + f_-^{\eta\pi}(s)\,(p_\eta + p_\pi)^\mu]$$

with: $j_\mu^{ud} = \bar{u}\gamma^\mu d$, $s = (p_\eta + p_\pi)^2$

- Vector form factor: $f_+^{\eta\pi}(s)$
- Scalar form factor:

$$f_0^{\eta\pi}(s) = f_+^{\eta\pi}(s) + \frac{s}{\Delta_{\eta\pi}} f_-^{\eta\pi}(s), \qquad \Delta_{PQ} = m_P^2 - m_Q^2$$

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- Differential decay rate of $\tau \to \eta\pi\nu$

$$\frac{d\Gamma_{\tau\to\eta\pi\nu_\tau}}{ds} = \frac{G_F^2 V_{ud}^2 S_{EW}\, m_\tau^3}{384\,\pi^3} \frac{\sqrt{\lambda_{\eta\pi}(s)}}{s^3} \left(1 - \frac{s}{m_\tau^2}\right)^2 \times \left\{ |f_+^{\eta\pi}(s)|^2 \lambda_{\eta\pi}(s) \left(1 + \frac{2s}{m_\tau^2}\right) + 3|f_0^{\eta\pi}(s)|^2 \Delta_{\eta\pi}^2 \right\}$$

- $f_0^{\eta\pi}(s)$ associated with divergence

$$\langle \eta(p_\eta)\pi(p_\pi)|i\partial^\mu j_\mu^{ud}(0)|0\rangle = \sqrt{2}\Delta_{\eta\pi}\, f_0^{\eta\pi}(s)\ ,$$

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- Obeys Ward identity

$$\boxed{i\partial_\mu \bar{u}\gamma^\mu d = (m_d - m_u)\bar{u}d - eA_\mu \bar{u}\gamma^\mu d}$$

→ Vector current not conserved: two sources of isospin violation

→ Possibility of experimental measurement of $a_0(980)$ coupling to $\bar{u}d$ operator

Unitarity relations for $f_+^{\eta\pi}$

- General expression:

$$-2\sqrt{2}q_{\eta\pi}(s)\,\mathrm{Im}\,[f_+^{\eta\pi}(s)] = \frac{1}{2}\sum_n T^*_{\eta\pi^+\to n} \times \langle n|j_3^{ud}|0\rangle$$

Note: Derived for $m_\eta < 3m_\pi$;
When m_η physical: $\mathrm{Im}\,[f_+^{\eta\pi}] \to \mathrm{disc}[f_+^{\eta\pi}(s)]$
[used in the disp. rels.]

- Successive contributions

  → $n = \pi^0\pi^+$

$$\mathrm{Im}\,[f_+^{\eta\pi}(s)]_n = \frac{\theta(s-m_n^2)(4m_\pi^2-s)}{32\pi\sqrt{\lambda_{\eta\pi}(s)}} \times F_V^\pi(s) \times \int_{-1}^{1} dz\, z T^*_{\eta\pi^+\to\pi\pi}(s,t)$$

$F_V^\pi(s)$ ⇓ Pion vector form factor

$T^*_{\eta\pi^+\to\pi\pi}$ ⇓ $\eta \to 3\pi$ amplitude

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⇓ P-wave suppressed: $J^{PC} = 1^{-+}$, exotic

→ $n = K^0K^+,\ 4\pi, \cdots$: Unimportant below 1 GeV

Unitarity relations for $f_0^{\eta\pi}$

$\rightarrow$ $n = \pi^0\pi^+$

$$\mathrm{Im}\,[f_0^{\eta\pi}(s)]_n = \frac{\theta(s-m_n^2)\sqrt{s-4m_\pi^2}}{32\pi\sqrt{s}}\,\frac{\Delta_{\pi^0\pi^+}}{\Delta_{\eta\pi^+}}\,f_0^{\pi 0\pi^+}(s)\times\int_{-1}^{1}dz\;T^*_{\eta\pi^+\to\pi\pi}(s,t)$$

$\Downarrow$

Pion (I=2) scalar form factor

[*Double isospin suppression*]

$\rightarrow$ $n = \eta\pi^+$

$$\mathrm{Im}\,[f_0^{\eta\pi}(s)]_n = \frac{\theta(s-m_n^2)\sqrt{\lambda_{\eta\pi}(s)}}{32\pi s} f_0^{\eta\pi}(s) \times \int_{-1}^{1} dz\; T^*_{\eta\pi^+\to\eta\pi^+}(s,t)$$

[*Dominant contribution*]

$\rightarrow$ $n = \bar{K}^0 K^+$

$$\mathrm{Im}\,[f_0^{\eta\pi}(s)]_n = \frac{\theta(s-m_n^2)\sqrt{s-4m_K^2}}{32\sqrt{2}\pi\;\; s} \frac{\Delta_{K^0K^+}}{\Delta_{\eta\pi}} f_0^{\bar{K}^0K^+}(s) \times \int_{-1}^{1} dz\; T^*_{\eta\pi^+\to\bar{K}^0K^+}(s,t)$$

[*Also relevant, in principle*]

Contributions involving photons

→ $n = \gamma\pi$

$$\mathrm{Im}\,[f_+^{\eta\pi}(s)] = \frac{i\theta(s-m_\pi^2)(s-m_\pi^2)^2}{256\pi q_{\eta\pi}(s)} \times e F_V^{\pi\gamma}(s) \sum_{\lambda=\pm1} \int_0^\pi d\theta \sin^2\theta\, T^*_{\eta\pi^+\to\gamma(\lambda)\pi^+}(s,t)$$

$\eta\pi \to \gamma\pi$: exotic quantum number, should be small

→ $n = \gamma\pi\pi$

$$\mathrm{Im}\,[f_0^{\eta\pi}(s)] = \frac{-\theta(s-4m_\pi^2)}{2\Delta_{\eta\pi}} \times \sum_{\lambda=\pm1} \int dLips_3\, e F_V^{\pi}(s_{\pi\pi}) e_\gamma(\lambda)(p_{\pi^0} - p_{\pi^+}) T^*_{\eta\pi^+\to\gamma(\lambda)\pi^0\pi^+}$$

Suppressed by phase-space below 1 GeV

ChPT results

- NLO calculations:
  [Scora, Maltman (1995)]
  [Neufeld, Rupertsberger (1995)] (also include $O(e^2 m_q)$)

- What is needed in disp. relations.: $f_+^{\eta\pi}(0) (= f_0^{\eta\pi}(0))$
  and derivatives: $\dot{f}_+^{\eta\pi}(0)$, $\dot{f}_0^{\eta\pi}(0)$

- At LO :

$$f_+^{\eta\pi}(0) = \epsilon \quad = \frac{\sqrt{3}(m_d - m_u)}{4\,(m_s - m_{ud})}$$

$$= \boxed{0.99 \times 10^{-2}} \quad \text{(PS masses +LO ChPT)}$$

and: $\dot{f}_+^{\eta\pi}(0) = \dot{f}_0^{\eta\pi}(0) = 0$

- NLO :

$$\delta f_+^{\eta\pi}(0)|_{NLO} =$$

$$\frac{2\epsilon}{3F_\pi^2(m_\eta^2 - m_\pi^2)}\Big\{64(m_K^2 - m_\pi^2)^2(3L_7 + L_8^r) - m_\eta^2(m_K^2 - m_\pi^2)L_\eta$$

$$-2m_K^2(m_K^2 - 2m_\pi^2)L_K + m_\pi^2(m_K^2 - 3m_\pi^2)L_\pi - \frac{2m_K^2(m_K^2 - m_\pi^2)}{16\pi^2}\Big\}$$

$$-\frac{2\sqrt{3}\,e^2 m_K^2}{27(m_\eta^2 - m_\pi^2)}\Big\{2(2S_2^r + 3S_3^r) - 9Z(L_K - \frac{1}{16\pi^2})\Big\}$$

with: $L_P = \log m_P^2/\mu^2$.

- Remarkable (simple) relation with K^+_{l3}, K^0_{l3} decays

$$f_+^{\eta\pi}(0)|_{LO+NLO} = \frac{1}{\sqrt{3}}\left[\frac{f_+^{K^+\pi^0}(0)}{f_+^{K^0\pi^+}(0)} - 1\right]$$

- Exp. inputs from K-factories [NA48, ISTRA+, KLOE, BNL-E865] quite precise, yielding

$$f_+^{\eta\pi}(0)|_{LO+NLO} = 1.49 \pm 0.23\, 10^{-2}$$

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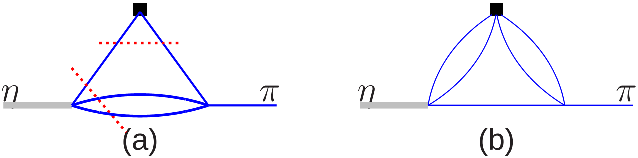
- Derivatives

$$\begin{aligned}
\dot{f}_+^{\eta\pi}(0) &= \frac{\epsilon}{12F_\pi^2}\left[24L_9 - L_K - 2L_\pi - \frac{3}{16\pi^2}\right] \\
\dot{f}_0^{\eta\pi}(0) &= \frac{\epsilon}{12F_\pi^2}\left[48L_5 - 9L_K - \frac{11}{16\pi^2} + 4m_\pi^2\dot{J}_{\eta\pi}(0)\right] \\
&+\frac{\sqrt{3}e^2}{18\Delta_{\eta\pi}}\left[-2(2S_2 + 3S_3) + \frac{11}{16\pi^2}Z\right]
\end{aligned}$$

Dispersion relations

Anomalous thresholds ?

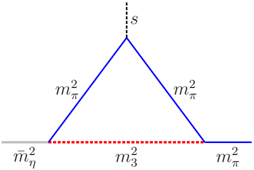
- η-meson is unstable. Impact on analyticity ?
- First investigate in toy model (w. local vertices)



- We consider only (a)-type diagrams

$$f^{\eta\pi}(s) = \frac{1}{16\pi^2}\int_{4m_\pi^2}^{\infty} dm_3^2 \sqrt{1-\frac{4m_\pi^2}{m_3^2}}\, K^{\eta\pi}(m_3^2, s) + \cdots$$

$K^{\eta\pi}(m_3^2, s)$ is a triangle diagram:

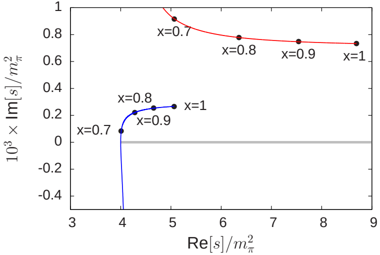


- Disp. representation of $K^{\eta\pi}$ (starting with $m_\eta^2 < 9m_\pi^2$)

$$K^{\eta\pi}(m_3^2, s) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{L^{\eta\pi}(m_3^2, s')}{s' - s}$$

- Watch how singularities of the log. $L^{\eta\pi}(m_3^2, s')$ move when varying m_η^2 ($\equiv m_\eta^2 + i\epsilon$) [Mandelstam (1959)]

Figure:
$m_3^2 = 5m_\pi^2$
$x = m_\eta^2/(m_\eta^2)_{phys.}$



Crossing of real axis: $4m_\pi^2\left(1 - \frac{\epsilon^2}{4m_3^2(m_3^2 - 4m_\pi^2)}\right) < 4m_\pi^2$

$\eta \to 3\pi$ ($\eta\pi \to \pi\pi$) from KT eqs., ChPT matching

- Original [Khuri-Treiman (1958)] (for $K \to 3\pi$), reformulated [Kambor, Wiesendanger, Wyler (1996)], and [Anisovich, Leutwyler (1996)]

- New data:
  $\eta \to 3\pi^0$: [WASA (2009), MAMI-B (2009), MAMI-C (2009), KLOE (2010)]
  $\eta \to \pi^+\pi^-\pi^0$: [KLOE (2008)]

- Recent reconsideration: [S. Lanz, Thesis (2011)], [Colangelo et al. (2011)],[Leutwyler (2013)]: *not a completely settled subject*.

- Decomposition theorem [Stern, Sazdjian, Fuchs (1993)] applied to $\eta \to 3\pi$ amplit.

$$T(s,t,u) = \frac{1}{Q^2}\frac{m_K^2(m_K^2-m_\pi^2)}{3\sqrt{3}m_\pi^2 F_\pi^2}\Big[M_0(s) + (s-u)M_1(t) + (s-t)M_1(u) + M_2(t) + M_2(u) - \frac{2}{3}M_2(s)\Big]$$

- Integral equations (non-singular) with 4 parameters:

$$\begin{aligned} M_0(w) &= \Omega_0(w)\left(\alpha_0 + \beta_0 w + \gamma_0 w^2 + w^2 \hat{I}_0(w)\right) \\ M_1(w) &= \Omega_1(w)\left(\beta_1 w + w\hat{I}_1(w)\right) \\ M_2(w) &= \Omega_2(w)\left(w^2 \hat{I}_2(w)\right) \end{aligned}$$

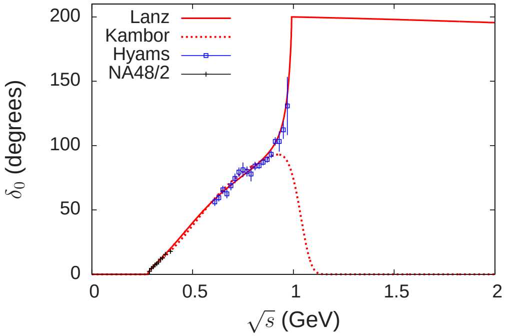
- Determined by 4 matching equations:

$$T(s,t,u) - T^{(4)}_{ChPT}(s,t,u) = O(p^6)$$

- Phase cutoff:

  To include or not to include the $f_0(980)$?

  Value of γ_0 depends strongly on cutoff



- Dalitz plot parameters:

$$\frac{d^2\Gamma_c(X,Y)}{dXdY} = \frac{d^2\Gamma_c(0,0)}{dXdY}\left[1 + aY + bY^2 + dX^2 + fY^3\right]$$

$$\frac{d^2\Gamma_n(X,Y)}{dXdY} = \frac{d^2\Gamma_n(0,0)}{dXdY}\left[1 + \alpha(X^2 + Y^2)\right]$$

Results for $\eta \to 3\pi$

Dalitz	experimental	cutoff(1)	cutoff(2)	Fit
a	$-1.090 \pm 0.005^{+0.008}_{-0.019}$	-1.171	-1.125	-1.062
b	$0.124 \pm 0.006 \pm 0.010$	0.260	0.196	0.163
d	$0.057 \pm 0.006^{+0.007}_{-0.016}$	0.083	0.082	0.067
f	$0.14 \pm 0.01 \pm 0.02$	0.074	0.100	0.102
α	-0.0315 ± 0.0015	-0.0127	-0.0260	-0.0336

(1) $\to$ Lanz
(2) $\to$ KWW

Possibly $a_0 - f_0$ mixing should be included ([M. Albaladejo, work in progress])

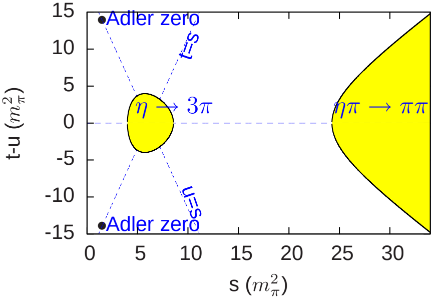
Application to $f_+^{\eta\pi}$ form factor

- DR for vector form factor

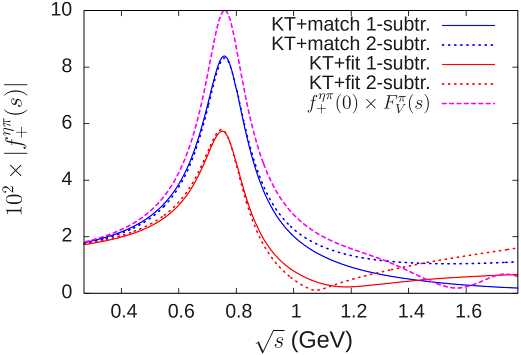
$$f_+^{\eta\pi}(s) = f_+^{\eta\pi}(0) + s\dot{f}_+^{\eta\pi}(0) + \frac{s^2}{\pi}\int_{4m_\pi^2}^{\infty} ds' \, \frac{\text{disc}[f_+^{\eta\pi}(s')]}{(s')^2(s'-s)} \, .$$

→ $\eta\pi^+ \to \pi^0\pi^+$ needed in unphysical region

→ But KT formalism applies



- Compare matching cutoff(1), and fitted parameters results



Scalar form factor, τ decay distribution

- Unitarity relation dominated by $\eta\pi$ (below 1 GeV)

- Use phase dispersive representation

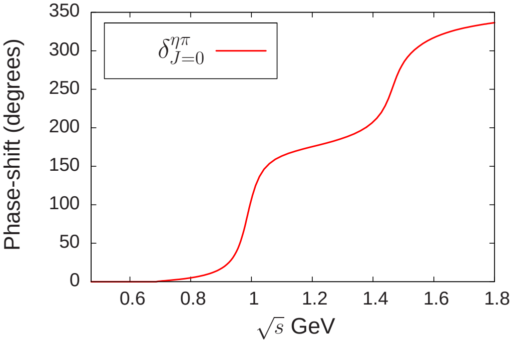
$$f_0^{\eta\pi}(s) = f_0^{\eta\pi}(0) \times \exp\left[\zeta s + \frac{s^2}{\pi}\int_{(m_\eta+m_\pi)^2}^{\infty} \frac{\phi^{\eta\pi}(s')}{(s')^2(s'-s)} ds'\right]$$

→ Watson: below $K\overline{K}$ threshold: $\phi^{\eta\pi}(s) = \delta_0^{\eta\pi}(s)$

→ $\zeta = \dot{f}_0^{\eta\pi}(0)/f_0^{\eta\pi}(0)$ (from ChPT)

→ $s \to \infty$: no exponential behvaviour $\Rightarrow$ Sum rule

- $\eta\pi$ scattering phase: use model [Black, Fariborz, Schechter, (2000)]
  - → LO ChPT+ scalar resonance exchanges, tested for $\pi\pi$, πK, $\eta' \to \eta\pi\pi$
  - → Merits: Simple, crossing symm., approx. unitarity



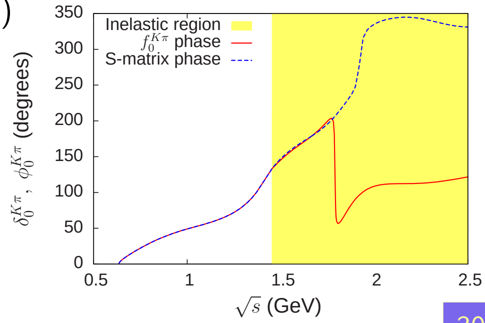
  - → Drawback:
    $a_0 = 3.1 \times 10^{-2}$
    $a_0 = (-0.2 \pm 7.7) \times 10^{-3}$ (NLO LET [Kubis, Schneider (2009)])

- How to deduce form factor phase from scattering phase (in inelastic region)? Use **analogy with $K\pi$**

  → Phase from solving coupled Muskhelishvili-Omnès eqs.[Jamin, Oller, Pich (2001)]

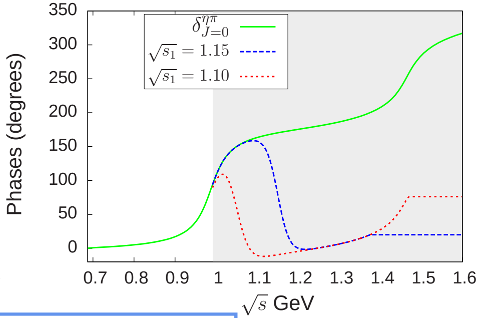
  → Exp. info available on elastic and inelastic scattering (dominated by $K\eta'$)

  → phase displays sharp dip



- Model for form factor phase:

$$\begin{aligned} &\text{Let } s_1,\ s_2: && 4m_K^2 < s_1 < s_2 \\ &s \leqslant s_2: && \phi^{\eta\pi}(s) = \delta_0^{\eta\pi}(s) - \pi\theta(s - s_1) \\ &s > s_2: && \phi^{\eta\pi}(s) = \text{constant}(= \phi^{\eta\pi}(s_2)) \end{aligned}$$

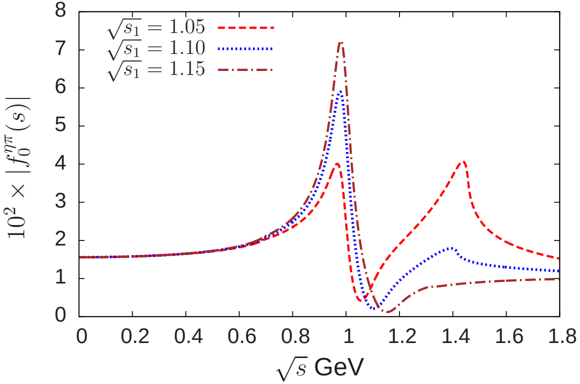


- s_1 given, s_2 determined from sum rule constraint

$$\zeta \equiv \frac{\dot{f}_0^{\eta\pi}(0)}{f_0^{\eta\pi}(0)} = \frac{1}{\pi}\int_{(m_\eta+m_\pi)^2}^{\infty} \frac{\phi^{\eta\pi}(s')}{(s')^2}\, ds' \ .$$

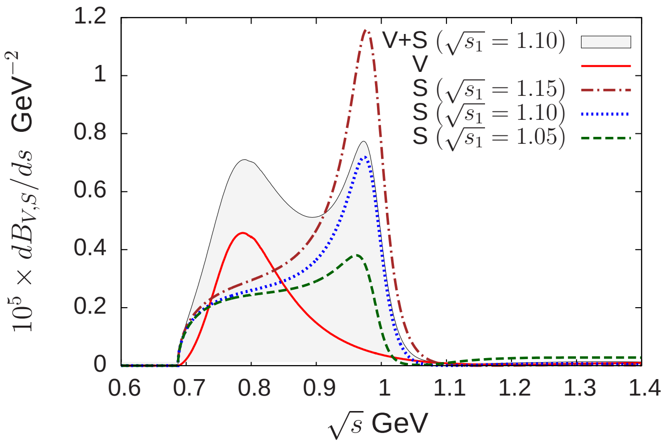
Ensures no exponential divergence of form factor.

Results for scalar form factor



- Exotic $a_0(980)$ (or $a_0(1450)$): dip is close to resonance mass

Results on τ decay energy distribution:



- Significant scalar contribution *below* $a_0(980)$ peak

Results on τ decay branching fractions:

- Our estimate:

$$BF_{vect} \simeq 0.11 \times 10^{-5}$$
$$BF_{scal} \simeq 0.37^{+0.30}_{-0.20} \times 10^{-5} \quad \text{(var. of } s_{dip}\text{)}$$

Results on τ decay branching fractions:

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  is on lower side of previous ones:

V	S	total (x10^5)	ref.
0.25	1.60	1.85	Tisserant, Truong (1982)
0.12	1.38	1.50	Pich (1987)
0.15	1.06	1.21	Neufeld, Rupertsberger(1995)
0.36	1.00	1.36	Nussinov, Soffer (2008)
[0.2-0.6]	[0.2-2.3]	[0.4-2.9]	Paver, Riazuddin (2010)
0.44	0.04	0.48	Volkov,Kostunin (2012) (NJL)

- Experimental: $BF \leqslant 9.9 \times 10^{-5}$ [Babar (2011)]

Summary

- New estimates for isospin violation $\eta\pi$ form factors: use informations from $\eta \to 3\pi$ and K^+_{l3}/K^0_{l3}
- Shape of ρ resonance distorted (compared to naive VMD) when using KT $\eta \to 3\pi$ solution close to experiment
- Estimate of $f_0^{\eta\pi}(s)$ assumes cusp in the phase: reduced size compared to previous work
- Experimentally, large number of τ's produced at B factories:

$$N_\tau = 5 \times 10^8 \text{ (Babar)}$$
$$N_\tau = 9 \times 10^8 \text{ (Belle)}$$

but not optimized for τ physics (large backgrounds). Better situation at future facilities ? Super B, Super charm-tau ...