Flavor neutrino states and quantum entanglement

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- Flavor states for mixed neutrinos;

- Entanglement in flavor neutrino states;

- Dynamical generation of flavor mixing.

Motivations

• CKM quark mixing, meson mixing, massive neutrino mixing (and oscillations) play a crucial role in phenomenology;

• Theoretical interest: origin of mixing in the Standard Model;

• Bargmann superselection rule*: coherent superposition of states with different masses is not allowed in non-relativistic QM;

• Necessity of a QFT treatment: problems in defining Hilbert space for mixed particles†; oscillation formulas‡;

*V.Bargmann, Ann. Math. (1954); D.M.Greenberger, Phys. Rev. Lett. (2001).

†C.W.Kim and A.Pevsner, *Neutrinos in Physics and Astrophysics*, (Harwood, 1993). C.Giunti, J. Phys. G (2007).

‡M.Beuthe, Phys. Rep. (2003).

Neutrino oscillations in QM *

$$|\nu_e\rangle = \quad \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle$$

$$|\nu_\mu\rangle = -\sin\theta\,|\nu_1\rangle + \cos\theta\,|\nu_2\rangle$$

– Time evolution:

$$|\nu_e(t)\rangle = \cos\theta\, e^{-iE_1 t}\,|\nu_1\rangle + \sin\theta\, e^{-iE_2 t}\,|\nu_2\rangle$$

– Flavor oscillations:

$$P_{\nu_e\to\nu_e}(t) \;=\; |\langle\nu_e|\nu_e(t)\rangle|^2 = 1 - \sin^2 2\theta\, \sin^2\left(\frac{\Delta E}{2}t\right) = 1 - P_{\nu_e\to\nu_\mu}(t)$$

– Flavor conservation:

$$|\langle\nu_e|\nu_e(t)\rangle|^2 + |\langle\nu_\mu|\nu_e(t)\rangle|^2 = 1$$

*S.M.Bilenky and B.Pontecorvo, Phys. Rep. (1978)

Quantum field theory of fermion mixing

– Mixing relations for two Dirac fields

$$\nu_e(x) = \cos\theta\, \nu_1(x) + \sin\theta\, \nu_2(x)$$

$$\nu_\mu(x) = -\sin\theta\, \nu_1(x) + \cos\theta\, \nu_2(x)$$

ν_1, ν_2 are fields with definite masses.

– Mixing transformations connect the two quadratic forms:

$$\mathcal{L} = \bar{\nu}_1\, (i \not\partial - m_1)\, \nu_1 + \bar{\nu}_2\, (i \not\partial - m_2)\, \nu_2$$

and

$$\mathcal{L} = \bar{\nu}_e\, (i \not\partial - m_e)\, \nu_e + \bar{\nu}_\mu\, (i \not\partial - m_\mu)\, \nu_\mu - m_{e\mu}\, (\bar{\nu}_e\, \nu_\mu + \bar{\nu}_\mu\, \nu_e)$$

with $m_e = m_1 \cos^2\theta + m_2 \sin^2\theta$, $m_\mu = m_1 \sin^2\theta + m_2 \cos^2\theta$, $m_{e\mu} = (m_2 - m_1) \sin\theta \cos\theta$.

Currents and charges for mixed fermions *

– Lagrangian in the mass basis:

$$\mathcal{L} = \bar{\nu}_m \, (i \not\partial - M_d) \, \nu_m$$

where $\nu_m^T = (\nu_1, \nu_2)$ and $M_d = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$.

- $\mathcal{L}$ invariant under global $U(1)$ with conserved charge Q= total charge.

– Consider now the $SU(2)$ transformation:

$$\nu'_m = e^{i\alpha_j \tau_j} \, \nu_m \qquad ; \qquad j = 1, 2, 3.$$

with $\tau_j = \sigma_j / 2$ and σ_j being the Pauli matrices.

*M. B., P. Jizba and G. Vitiello, Phys. Lett. B (2001)

The associated currents are:

$$\delta\mathcal{L} \;=\; i\alpha_j\,\bar{\nu}_m\,[\tau_j, M_d]\,\nu_m \;=\; -\alpha_j\,\partial_\mu J^\mu_{m,j}$$

$$J^\mu_{m,j} \;=\; \bar{\nu}_m\,\gamma^\mu\,\tau_j\,\nu_m$$

– The charges $Q_{m,j}(t) \equiv \int d^3\mathbf{x}\, J^0_{m,j}(x)$, satisfy the $su(2)$ algebra:

$$[Q_{m,j}(t), Q_{m,k}(t)] \;=\; i\,\epsilon_{jkl}\,Q_{m,l}(t)\,.$$

– The Casimir operator is proportional to the total charge: $C_m = \frac{1}{2}Q$.

- $Q_{m,3}$ is conserved $\Rightarrow$ charge conserved separately for ν_1 and ν_2:

$$Q_1 \;=\; \frac{1}{2}Q + Q_{m,3} = \int d^3\mathbf{x}\,\nu_1^\dagger(x)\,\nu_1(x)$$

$$Q_2 \;=\; \frac{1}{2}Q - Q_{m,3} = \int d^3\mathbf{x}\,\nu_2^\dagger(x)\,\nu_2(x).$$

These are the flavor charges in the absence of mixing.

The currents in the flavor basis

– Lagrangian in the flavor basis:

$$\mathcal{L} = \bar{\nu}_f \, (i \not\partial - M) \, \nu_f$$

where $\nu_f^T = (\nu_e, \nu_\mu)$ and $M = \begin{pmatrix} m_e & m_{e\mu} \\ m_{e\mu} & m_\mu \end{pmatrix}$.

– Consider the $SU(2)$ transformation:

$$\nu'_f = e^{i\alpha_j \tau_j} \, \nu_f \qquad ; \qquad j = 1, 2, 3.$$

with $\tau_j = \sigma_j/2$ and σ_j being the Pauli matrices.

– The charges $Q_{f,j} \equiv \int d^3\mathbf{x} \, J^0_{f,j}$ satisfy the $su(2)$ algebra:

$$[Q_{f,j}(t), Q_{f,k}(t)] = i \, \epsilon_{jkl} \, Q_{f,l}(t).$$

– The Casimir operator is proportional to the total charge $C_f = C_m = \frac{1}{2} Q$.

- $Q_{f,3}$ is not conserved $\Rightarrow$ exchange of charge between ν_e and ν_μ.

Define the flavor charges as:

$$
\begin{aligned}
Q_e(t) &\equiv \frac{1}{2}Q + Q_{f,3}(t) = \int d^3\mathbf{x}\, \nu_e^\dagger(x)\, \nu_e(x) \\
Q_\mu(t) &\equiv \frac{1}{2}Q - Q_{f,3}(t) = \int d^3\mathbf{x}\, \nu_\mu^\dagger(x)\, \nu_\mu(x)
\end{aligned}
$$

where $Q_e(t) + Q_\mu(t) = Q$.

– We have:

$$
\begin{aligned}
Q_e(t) &= \cos^2\theta\, Q_1 + \sin^2\theta\, Q_2 + \sin\theta\cos\theta \int d^3\mathbf{x} \left[\nu_1^\dagger\nu_2 + \nu_2^\dagger\nu_1\right] \\
Q_\mu(t) &= \sin^2\theta\, Q_1 + \cos^2\theta\, Q_2 - \sin\theta\cos\theta \int d^3\mathbf{x} \left[\nu_1^\dagger\nu_2 + \nu_2^\dagger\nu_1\right]
\end{aligned}
$$

In conclusion:

– In presence of mixing, neutrino flavor charges are defined as

$$Q_e(t) \equiv \int d^3\mathbf{x}\, \nu_e^\dagger(x)\, \nu_e(x) \quad ; \qquad Q_\mu(t) \equiv \int d^3\mathbf{x}\, \nu_\mu^\dagger(x)\, \nu_\mu(x)$$

– They are not conserved charges $\Rightarrow$ flavor oscillations.

– They are still (approximately) conserved in the vertex $\Rightarrow$ define flavor neutrinos as their eigenstates

- Problem: find the eigenstates of the above charges.

Neutrino mixing in QFT

- Mixing relations for two Dirac fields

$$\nu_e(x) \;=\; \cos\theta\, \nu_1(x) + \sin\theta\, \nu_2(x)$$
$$\nu_\mu(x) \;=\; -\sin\theta\, \nu_1(x) + \cos\theta\, \nu_2(x)$$

can be written as*

$$\nu_e^\alpha(x) \;=\; G_\theta^{-1}(t)\, \nu_1^\alpha(x)\, G_\theta(t)$$
$$\nu_\mu^\alpha(x) \;=\; G_\theta^{-1}(t)\, \nu_2^\alpha(x)\, G_\theta(t)$$

– Mixing generator:

$$G_\theta(t) = \exp\left[\theta \int d^3\mathbf{x} \left(\nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x)\right)\right] = \exp[2i\theta\, Q_{m,2}(t)]$$

For ν_e, we get $\frac{d^2}{d\theta^2}\nu_e^\alpha = -\nu_e^\alpha$ with initial conditions $\nu_e^\alpha|_{\theta=0} = \nu_1^\alpha$, $\frac{d}{d\theta}\nu_e^\alpha\big|_{\theta=0} = \nu_2^\alpha$.

*M.B. and G.Vitiello, Annals Phys. (1995)

– ν_i are free Dirac field operators:

$$\nu_i(x) = \sum_{\mathbf{k},r} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{V}} [\, u^r_{\mathbf{k},i}(t)\, \alpha^r_{\mathbf{k},i} \, + \, v^r_{-\mathbf{k},i}(t)\, \beta^{r\dagger}_{-\mathbf{k},i}], \qquad i = 1, 2.$$

– Anticommutation relations:

$$\{\nu^\alpha_i(x), \nu^{\beta\dagger}_j(y)\}_{t=t'} = \delta^3(\mathbf{x}-\mathbf{y})\delta_{\alpha\beta}\delta_{ij} \quad ; \quad \{\alpha^r_{\mathbf{k},i}, \alpha^{s\dagger}_{\mathbf{q},j}\} = \{\beta^r_{\mathbf{k},i}, \beta^{s\dagger}_{\mathbf{q},j}\} = \delta^3(\mathbf{k}-\mathbf{q})\delta_{rs}\delta_{ij}$$

– Orthonormality and completeness relations:

$$u^r_{\mathbf{k},i}(t) \,=\, e^{-i\omega_{k,i}t}\, u^r_{\mathbf{k},i} \quad ; \quad v^r_{\mathbf{k},i}(t) \,=\, e^{i\omega_{k,i}t}\, v^r_{\mathbf{k},i} \quad ; \quad \omega_{k,i} = \sqrt{k^2 + m_i^2}$$

$$u^{r\dagger}_{\mathbf{k},i} u^s_{\mathbf{k},i} = v^{r\dagger}_{\mathbf{k},i} v^s_{\mathbf{k},i} = \delta_{rs} \quad , \quad u^{r\dagger}_{\mathbf{k},i} v^s_{-\mathbf{k},i} = 0 \quad , \quad \sum_r (u^{r\alpha*}_{\mathbf{k},i} u^{r\beta}_{\mathbf{k},i} + v^{r\alpha*}_{-\mathbf{k},i} v^{r\beta}_{-\mathbf{k},i}) = \delta_{\alpha\beta} \,.$$

– Fock space for ν_1, ν_2:

$$\mathcal{H}_{1,2} = \left\{ \alpha^\dagger_{1,2} \,,\, \beta^\dagger_{1,2} \,,\, |0\rangle_{1,2} \right\}.$$

• The vacuum $|0\rangle_{1,2}$ is not invariant under the action of the generator $G_\theta(t)$:

$$|0(t)\rangle_{e,\mu} \equiv G^{-1}_\theta(t)\;|0\rangle_{1,2}$$

• Relation between $|0\rangle_{1,2}$ and $|0(t)\rangle_{e,\mu}$: orthogonality! (for $V \to \infty$)

$$\lim_{V\to\infty}\;{}_{1,2}\langle 0|0(t)\rangle_{e,\mu} = \lim_{V\to\infty}\; e^{V\int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\ln\left(1-\sin^2\theta\,|V_\mathbf{k}|^2\right)^2} = 0$$

with

$$|V_\mathbf{k}|^2 \;\equiv\; \sum_{r,s}|\;v^{r\dagger}_{-\mathbf{k},1}u^{s}_{\mathbf{k},2}\;|^2 \neq 0 \quad for \quad m_1 \neq m_2$$

.

Quantum Field Theory vs. Quantum Mechanics

- Quantum Mechanics:

- finite $\sharp$ of degrees of freedom.

- unitary equivalence of the representations of the canonical commutation relations (von Neumann theorem).

- Quantum Field Theory:

- infinite $\sharp$ of degrees of freedom.

- ∞ many unitarily inequivalent representations of the field algebra $\Leftrightarrow$ many vacua .

- The mapping between interacting and free fields is "weak", i.e. representation dependent (LSZ formalism)*. Example: theories with spontaneous symmetry breaking.

*F.Strocchi, *Elements of Quantum Mechanics of Infinite Systems* (World Scientific, 1985).

- The "flavor vacuum" $|0(t)\rangle_{e,\mu}$ is a $SU(2)$ generalized coherent state*:

$$|0\rangle_{e,\mu} = \prod_{\mathbf{k},r} \Big[(1-\sin^2\theta\,|V_{\mathbf{k}}|^2) - \epsilon^r \sin\theta\,\cos\theta\,|V_{\mathbf{k}}|\,(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1})$$

$$+\,\epsilon^r \sin^2\theta\,|V_{\mathbf{k}}||U_{\mathbf{k}}|\,(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},1} - \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},2}) + \sin^2\theta\,|V_{\mathbf{k}}|^2\,\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\Big]\,|0\rangle_{1,2}$$

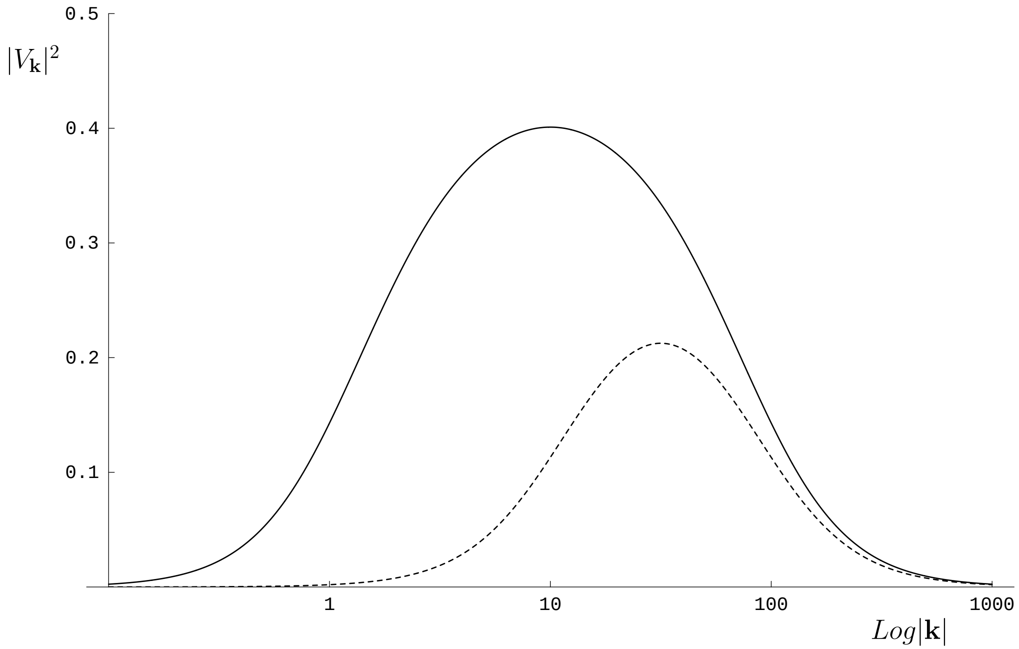
– Condensation density:

$${}_{e,\mu}\langle 0(t)|\alpha^{r\dagger}_{\mathbf{k},i}\alpha^{r}_{\mathbf{k},i}|0(t)\rangle_{e,\mu} = {}_{e,\mu}\langle 0(t)|\beta^{r\dagger}_{\mathbf{k},i}\beta^{r}_{\mathbf{k},i}|0(t)\rangle_{e,\mu} = \sin^2\theta\,|V_{\mathbf{k}}|^2$$

vanishing for $m_1 = m_2$ and/or $\theta = 0$ (in both cases no mixing).

*A. Perelomov, *Generalized Coherent States and Their Applications*, (Springer V., 1986)

Condensation density for mixed fermions



Solid line: $m_1 = 1$, $m_2 = 100$; Dashed line: $m_1 = 10$, $m_2 = 100$.

- $V_{\mathbf{k}} = 0$ when $m_1 = m_2$ and/or $\theta = 0$.

- Max. at $k = \sqrt{m_1 m_2}$ with $V_{max} \to \frac{1}{2}$ for $\frac{(m_2 - m_1)^2}{m_1 m_2} \to \infty$.

- $|V_{\mathbf{k}}|^2 \simeq \frac{(m_2 - m_1)^2}{4k^2}$ for $k \gg \sqrt{m_1 m_2}$.

• Structure of the annihilation operators for $|0(t)\rangle_{e,\mu}$:

$$\alpha^r_{\mathbf{k},e}(t) = \cos\theta\,\alpha^r_{\mathbf{k},1} + \sin\theta\left(U^*_{\mathbf{k}}(t)\,\alpha^r_{\mathbf{k},2} + \epsilon^r V_{\mathbf{k}}(t)\,\beta^{r\dagger}_{-\mathbf{k},2}\right)$$

$$\alpha^r_{\mathbf{k},\mu}(t) = \cos\theta\,\alpha^r_{\mathbf{k},2} - \sin\theta\left(U_{\mathbf{k}}(t)\,\alpha^r_{\mathbf{k},1} - \epsilon^r V_{\mathbf{k}}(t)\,\beta^{r\dagger}_{-\mathbf{k},1}\right)$$

$$\beta^r_{-\mathbf{k},e}(t) = \cos\theta\,\beta^r_{-\mathbf{k},1} + \sin\theta\left(U^*_{\mathbf{k}}(t)\,\beta^r_{-\mathbf{k},2} - \epsilon^r V_{\mathbf{k}}(t)\,\alpha^{r\dagger}_{\mathbf{k},2}\right)$$

$$\beta^r_{-\mathbf{k},\mu}(t) = \cos\theta\,\beta^r_{-\mathbf{k},2} - \sin\theta\left(U_{\mathbf{k}}(t)\,\beta^r_{-\mathbf{k},1} + \epsilon^r V_{\mathbf{k}}(t)\,\alpha^{r\dagger}_{\mathbf{k},1}\right)$$

• Mixing transformation = Rotation + Bogoliubov transformation .

– Bogoliubov coefficients:

$$U_{\mathbf{k}}(t) = u^{r\dagger}_{\mathbf{k},2} u^r_{\mathbf{k},1}\, e^{i(\omega_{k,2}-\omega_{k,1})t} \quad ; \qquad V_{\mathbf{k}}(t) = \epsilon^r\, u^{r\dagger}_{\mathbf{k},1} v^r_{-\mathbf{k},2}\, e^{i(\omega_{k,2}+\omega_{k,1})t}$$

$$|U_{\mathbf{k}}|^2 + |V_{\mathbf{k}}|^2 = 1$$

- The flavor charge operators are diagonal in the flavor ladder operators:

$$
\begin{aligned}
:: Q_{\nu_\sigma}(t) :: \; &\equiv \int d^3\mathbf{x} \; :: \nu_\sigma^\dagger(x) \, \nu_\sigma(x) :: \\
&= \sum_r \int d^3\mathbf{k} \left(\alpha^{r\dagger}_{\mathbf{k},\sigma}(t) \alpha^r_{\mathbf{k},\sigma}(t) \, - \, \beta^{r\dagger}_{-\mathbf{k},\sigma}(t) \beta^r_{-\mathbf{k},\sigma}(t) \right) , \quad \sigma = e, \mu .
\end{aligned}
$$

Here :: ... :: denotes normal ordering with respect to the flavor vacuum:

$$
:: A :: \equiv A \, - \, {}_{e,\mu}\langle 0 | A | 0 \rangle_{e,\mu}
$$

- Define flavor neutrino states with definite momentum and helicity:

$$
|\nu^r_{\mathbf{k},\sigma}\rangle \, \equiv \, \alpha^{r\dagger}_{\mathbf{k},\sigma}(0) \, |0\rangle_{e,\mu}
$$

– Such states are eigenstates of the flavor charges (at t=0):

$$
:: Q_{\nu_\sigma} :: |\nu^r_{\mathbf{k},\sigma}\rangle \, = \, |\nu^r_{\mathbf{k},\sigma}\rangle
$$

– We have, for an electron neutrino state:

$$\mathcal{Q}_{\mathbf{k},\nu_\sigma}(t) \equiv \langle \nu^r_{\mathbf{k},e}| :: Q_{\nu_\sigma}(t) :: |\nu^r_{\mathbf{k},e}\rangle$$

$$= \left|\left\{\alpha^r_{\mathbf{k},\sigma}(t), \alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2 + \left|\left\{\beta^{r\dagger}_{-\mathbf{k},\sigma}(t), \alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2$$

• Neutrino oscillation formula (exact result)*:

$$\mathcal{Q}_{\mathbf{k},\nu_e}(t) = 1 - |U_{\mathbf{k}}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2} - \omega_{k,1}}{2}t\right) - |V_{\mathbf{k}}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2} + \omega_{k,1}}{2}t\right)$$

$$\mathcal{Q}_{\mathbf{k},\nu_\mu}(t) = |U_{\mathbf{k}}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2} - \omega_{k,1}}{2}t\right) + |V_{\mathbf{k}}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2} + \omega_{k,1}}{2}t\right)$$

- For $k \gg \sqrt{m_1 m_2}$, $|U_{\mathbf{k}}|^2 \to 1$ and $|V_{\mathbf{k}}|^2 \to 0$.

*M.B., P.Henning and G.Vitiello, Phys. Lett. **B** (1999).

Lepton charge violation for Pontecorvo states*

– Pontecorvo states:

$$|\nu^r_{\mathbf{k},e}\rangle_P \;=\; \cos\theta\,|\nu^r_{\mathbf{k},1}\rangle \;+\; \sin\theta\,|\nu^r_{\mathbf{k},2}\rangle$$

$$|\nu^r_{\mathbf{k},\mu}\rangle_P \;=\; -\sin\theta\,|\nu^r_{\mathbf{k},1}\rangle \;+\; \cos\theta\,|\nu^r_{\mathbf{k},2}\rangle\,,$$

are *not* eigenstates of the flavor charges.

⇒ *violation of lepton charge conservation* in the production/detection vertices, at tree level:

$$_P\langle\nu^r_{\mathbf{k},e}|:Q_e(0):|\nu^r_{\mathbf{k},e}\rangle_P \;=\; \cos^4\theta+\sin^4\theta+2|U_{\mathbf{k}}|\sin^2\theta\cos^2\theta \;<\; 1,$$

for any $\theta \neq 0$, $\mathbf{k} \neq 0$ and for $m_1 \neq m_2$.

*M. B., A. Capolupo, F. Terranova and G. Vitiello, Phys. Rev. **D** (2005)
C. C. Nishi, Phys. Rev. **D** (2008).

Other results

- Rigorous mathematical treatment for any number of flavors [*]
- Three flavor fermion mixing: CP violation[†];
- QFT spacetime dependent neutrino oscillation formula[‡];
- Boson mixing[§];
- Majorana neutrinos[¶];

[*] K. C. Hannabuss and D. C. Latimer, J. Phys. A (2000); J. Phys. A (2003);

[†] M.B., A.Capolupo and G.Vitiello, Phys. Rev. **D** (2002)

[‡] M.B., P. Pires Pachêco and H. Wan Chan Tseung, Phys. Rev. **D**, (2003).

[§] M.B., A.Capolupo, O.Romei and G.Vitiello, Phys. Rev. **D**(2001); M.Binger and C.R.Ji. Phys. Rev. **D**(1999); C.R.Ji and Y.Mishchenko, Phys. Rev. **D**(2001); Phys. Rev. **D**(2002).

[¶] M.B. and J.Palmer, Phys. Rev. **D** (2004)

- Flavor vacuum and cosmological constant[*]

- Flavor vacuum induced by condensation of D-particles.[†]

- Geometric phase for mixed particles[‡].

[*]M.B., A.Capolupo, S.Capozziello, S.Carloni and G.Vitiello Phys. Lett. A (2004);

[†]N.E.Mavromatos and S.Sarkar, New J. Phys. (2008); N.E.Mavromatos, S.Sarkar and W.Tarantino, Phys. Rev. D (2008); Phys. Rev. D (2011).

[‡]M.B., P.Henning and G.Vitiello, Phys. Lett. **B** (1999)

The issue of Lorentz invariance

– Canonical energy-momentum tensor for flavor fields:

$$\begin{aligned} T_{\rho\sigma} &= \bar{\nu}_e i\gamma_\rho \partial_\sigma \nu_e - \eta_{\rho\sigma}\bar{\nu}_e(i\gamma^\lambda\partial_\lambda - m_e)\nu_e \\ &+ \bar{\nu}_\mu i\gamma_\rho \partial_\sigma \nu_\mu - \eta_{\rho\sigma}\bar{\nu}_\mu(i\gamma^\lambda\partial_\lambda - m_\mu)\nu_\mu \\ &+ \eta_{\rho\sigma} m_{e\mu}(\bar{\nu}_e\nu_\mu + \bar{\nu}_\mu\nu_e) \end{aligned}$$

– Define momentum and Hamiltonian operators:

$$P^i = \int d^3\mathbf{x}\, T^{0i}; \qquad H = \int d^3\mathbf{x}\, T^{00}.$$

One finds:

$$P^i|\nu_{\mathbf{k},\sigma}\rangle = k^i|\nu_{\mathbf{k},\sigma}\rangle,$$

but

$$H|\nu_{\mathbf{k},\sigma}\rangle \neq \Omega_{\mathbf{k},\sigma}|\nu_{\mathbf{k},\sigma}\rangle.$$

- This happens because: $[H, Q_\sigma] \neq 0$.

Possible scenarios

- ν_e and ν_μ are not fundamental; the fundamental objects are ν_1 and ν_2*;

- ν_e and ν_μ are fundamental but Poincaré invariance is broken (es.nonlinearly realized† as in DSR‡) $\Rightarrow$ modified dispersion relations;

- ν_e and ν_μ are fundamental and Poincaré invariance is recovered in the vertices.

*C. Giunti and C. W. Kim, "Fundamentals of Neutrino Physics and Astrophysics," (2007)
†M. B., J. Magueijo, P. Pires-Pacheco, EPL (2005) ;
‡J. Magueijo, L. Smolin, Phys. Rev. D (2003);

Flavor mixing as a non-abelian gauge theory*

Let us return to the Lagrangian:

$$\mathcal{L} \;=\; \bar{\nu}_e \,(i \not\partial - m_e)\, \nu_e + \bar{\nu}_\mu \,(i \not\partial - m_\mu)\, \nu_\mu \;-\; m_{e\mu}\; (\, \bar{\nu}_e \,\nu_\mu \,+\, \bar{\nu}_\mu \,\nu_e \,)\,.$$

The field equations:

$$\begin{aligned} i\partial_0 \nu_e &= (-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta m_e)\nu_e + \beta m_{e\mu}\nu_\mu \\ i\partial_0 \nu_\mu &= (-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta m_\mu)\nu_\mu + \beta m_{e\mu}\nu_e . \end{aligned}$$

can be written compactly:

$$iD_0 \nu_f = (-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta M_d)\nu_f ,$$

with $\nu_f = (\nu_e, \nu_\mu)^T$, $M_d = \text{diag}(m_e, m_\mu)$.

*M. B., M. Di Mauro, G. Vitiello, Phys. Lett. B (2011)

- Non-abelian covariant derivative:

$$D_0 := \partial_0 + i\, m_{e\mu}\, \beta\, \sigma_1,$$

with $m_{e\mu} = \frac{1}{2} \tan 2\theta\, \delta m$ and $\delta m := m_\mu - m_e$.

- Gauge connection:

$$A_\mu \;:=\; \frac{1}{2} A_\mu^a \sigma_a \;=\; n_\mu \delta m \,\frac{\sigma_1}{2} \in su(2),$$

with $n^\mu := (1,0,0,0)^T$, so that:

$$D_\mu = \partial_\mu + i\, g\, \beta\, A_\mu.$$

We define $g \equiv \tan 2\theta$ as the coupling constant for the mixing interaction.

- The equations of motion and the Lagrangian read:

$$(i\gamma^\mu D_\mu - M_d)\nu_f = 0,$$

$$\mathcal{L} = \bar{\nu}_f (i\gamma^\mu D_\mu - M_d)\nu_f.$$

- Define a new energy-momentum tensor:

$$\widetilde{T}_{\rho\sigma} = \bar{\nu}_f i\gamma_\rho D_\sigma \nu_f - \eta_{\rho\sigma}\bar{\nu}_f(i\gamma^\lambda D_\lambda - M_d)\nu_f.$$

- Momentum and Hamiltonian operators:

$$\begin{aligned}
\widetilde{P}^i &= \int d^3\mathbf{x}\, \widetilde{T}^{0i} \\
&= i\int d^3\mathbf{x}\, \nu_e^\dagger \partial^i \nu_e + i\int d^3\mathbf{x}\, \nu_\mu^\dagger \partial^i \nu_\mu \\
&\equiv \widetilde{P}_e^i(t) + \widetilde{P}_\mu^i(t), \qquad i = 1,2,3;
\end{aligned}$$

$$\begin{aligned}
\widetilde{H}(t) &= \int d^3\mathbf{x}\, \widetilde{T}^{00} \\
&= \int d^3\mathbf{x}\, \nu_e^\dagger \left(-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta m_e\right)\nu_e + \int d^3\mathbf{x}\, \nu_\mu^\dagger \left(-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta m_\mu\right)\nu_\mu \\
&\equiv \widetilde{H}_e(t) + \widetilde{H}_\mu(t).
\end{aligned}$$

This Hamiltonian does *not* generate time evolution.

Flavor fields in a different mass basis

– Flavor fields can be expanded also as*

$$\nu_\sigma(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \sum_r \left[u^r_{\mathbf{k},\sigma}(t) \widetilde{\alpha}^r_{\mathbf{k},\sigma}(t) + v^r_{-\mathbf{k},\sigma}(t) \widetilde{\beta}^{r\dagger}_{-\mathbf{k},\sigma}(t) \right] e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad \sigma = e, \mu,$$

with $u^r_{\mathbf{k},\sigma}(t) = u^r_{\mathbf{k},\sigma} e^{-i\omega_{\mathbf{k},\sigma} t}$ and $v^r_{-\mathbf{k},\sigma}(t) = v^r_{-\mathbf{k},\sigma} e^{i\omega_{\mathbf{k},\sigma} t}$.

The spinor basis is defined by:

$$\begin{aligned} (-\alpha \cdot \mathbf{k} + m_\sigma \beta) u^r_{\mathbf{k},\sigma} &= \omega_{\mathbf{k},\sigma} u^r_{\mathbf{k},\sigma} \\ (-\alpha \cdot \mathbf{k} + m_\sigma \beta) v^r_{-\mathbf{k},\sigma} &= -\omega_{\mathbf{k},\sigma} v^r_{-\mathbf{k},\sigma}, \end{aligned}$$

where $\omega_{\mathbf{k},\sigma} = \sqrt{\mathbf{k}^2 + m_\sigma^2}$.

*K. Fujii, C. Habe, T. Yabuki Phys. Rev. D (1999);

– Operators in different bases are connected by a Bogoliubov transformation:

$$\left(\begin{array}{c}\widetilde{\alpha}^{r}_{\mathbf{k},\sigma}(t)\\ \widetilde{\beta}^{r\dagger}_{-\mathbf{k},\sigma}(t)\end{array}\right)=J^{-1}(t)\left(\begin{array}{c}\alpha^{r}_{\mathbf{k},\sigma}(t)\\ \beta^{r\dagger}_{-\mathbf{k},\sigma}(t)\end{array}\right)J_{\mu}(t),$$

with generator:

$$J(t)=\prod_{\mathbf{k},r}\exp\left\{i\sum_{(\sigma,j)}\xi^{\mathbf{k}}_{\sigma,j}\left[\alpha^{r\dagger}_{\mathbf{k},\sigma}(t)\beta^{r\dagger}_{-\mathbf{k},\sigma}(t)+\beta^{r}_{-\mathbf{k},\sigma}(t)\alpha^{r}_{\mathbf{k},\sigma}(t)\right]\right\},$$

where $(\sigma,j)=(e,1),(\mu,2)$, $\xi^{\mathbf{k}}_{\sigma,j}=(\chi_{\sigma}-\chi_{j})/2$ and $\chi_{\sigma}=\arctan(\mu_{\sigma}/|\mathbf{k}|)$, $\chi_{j}=\arctan(m_{j}/|\mathbf{k}|)$.

– New flavor vacuum:

$$|\widetilde{0}(t)\rangle_{e\mu}=J^{-1}(t)|0(t)\rangle_{e\mu}.$$

– Momentum and Hamiltonian operators are both diagonalized:

$$\widetilde{\mathbf{P}}_\sigma(t) \;=\; \sum_r \int d^3\mathbf{k}\ \mathbf{k}\left(\widetilde{\alpha}^{r\dagger}_{\mathbf{k},\sigma}(t)\widetilde{\alpha}^{r}_{\mathbf{k},\sigma}(t) + \widetilde{\beta}^{r\dagger}_{\mathbf{k},\sigma}(t)\widetilde{\beta}^{r}_{\mathbf{k},\sigma}(t)\right),$$

$$\widetilde{H}_\sigma(t) \;=\; \sum_r \int d^3\mathbf{k}\ \omega_{\mathbf{k},\sigma}\left(\widetilde{\alpha}^{r\dagger}_{\mathbf{k},\sigma}(t)\,\widetilde{\alpha}^{r}_{\mathbf{k},\sigma}(t) - \widetilde{\beta}^{r}_{\mathbf{k},\sigma}(t)\,\widetilde{\beta}^{r\dagger}_{\mathbf{k},\sigma}(t)\right).$$

– Flavor charges remain diagonal ($[Q_\sigma(t), J(t)] = 0$):

$$\widetilde{Q}_\sigma(t) = \sum_r \int d^3\mathbf{k}\left(\widetilde{\alpha}^{r\dagger}_{\mathbf{k}\sigma}(t)\widetilde{\alpha}^{r}_{\mathbf{k}\sigma}(t) - \widetilde{\beta}^{r\dagger}_{-\mathbf{k}\sigma}(t)\widetilde{\beta}^{r}_{-\mathbf{k}\sigma}(t)\right).$$

• The new flavor states

$$|\widetilde{\nu}^r_{\mathbf{k},\sigma}(t)\rangle = \widetilde{\alpha}^{r\dagger}_{\mathbf{k},\sigma}(t)|\widetilde{0}(t)\rangle_{e\mu}.$$

are locally eigenstates of a four momentum operator:

$$\begin{pmatrix} \widetilde{H}_\sigma(t) \\ \widetilde{\mathbf{P}}_\sigma(t) \end{pmatrix} |\widetilde{\nu}^r_{\mathbf{k},\sigma}(t)\rangle = \begin{pmatrix} \omega_{\mathbf{k},\sigma} \\ \mathbf{k} \end{pmatrix} |\widetilde{\nu}^r_{\mathbf{k},\sigma}(t)\rangle,$$

Poincaré structure

Define the Lorentz generators:

$$\widetilde{M}^{\lambda\rho}(t) = \int d^3\mathbf{x}\left(\widetilde{T}^{0\rho}x^{\lambda} - \widetilde{T}^{0\lambda}x^{\rho}\right) + \frac{1}{2}\int d^3\mathbf{x}\,\nu_f^{\dagger}\sigma^{\lambda\rho}\nu_f = \widetilde{M}_e^{\lambda\rho}(t) + \widetilde{M}_\mu^{\lambda\rho}(t),$$

where $\sigma^{\mu\nu} = -\frac{i}{2}[\gamma^{\mu},\gamma^{\nu}]$.

Algebra of *equal-time* commutators of the generators $(\sigma,\sigma' = e,\mu)$.

$$[\widetilde{P}^{\mu}_{\sigma},\widetilde{P}^{\nu}_{\sigma'}] = 0 \quad ; \qquad [\widetilde{M}^{\mu\nu}_{\sigma},\widetilde{P}^{\lambda}_{\sigma'}] = i\delta_{\sigma\sigma'}\left(\eta^{\mu\lambda}\widetilde{P}^{\nu}_{\sigma} - \eta^{\nu\lambda}\widetilde{P}^{\mu}_{\sigma}\right);$$

$$[\widetilde{M}^{\mu\nu}_{\sigma},\widetilde{M}^{\lambda\rho}_{\sigma'}] = i\delta_{\sigma\sigma'}\left(\eta^{\mu\lambda}\widetilde{M}^{\nu\rho}_{\sigma} - \eta^{\nu\lambda}\widetilde{M}^{\mu\rho}_{\sigma} - \eta^{\mu\rho}\widetilde{M}^{\nu\lambda}_{\sigma} + \eta^{\nu\rho}\widetilde{M}^{\mu\lambda}_{\sigma}\right).$$

- The Poincaré structure is preserved in the interaction vertices.

Physical picture (optical analogy)

• Flavor neutrinos are (locally) on-shell particles, with masses:

$$m_e = m_1 \cos^2\theta + m_2 \sin^2\theta, \qquad m_\mu = m_2 \cos^2\theta + m_1 \sin^2\theta.$$

• Oscillations arise because of interaction with the external gauge field.

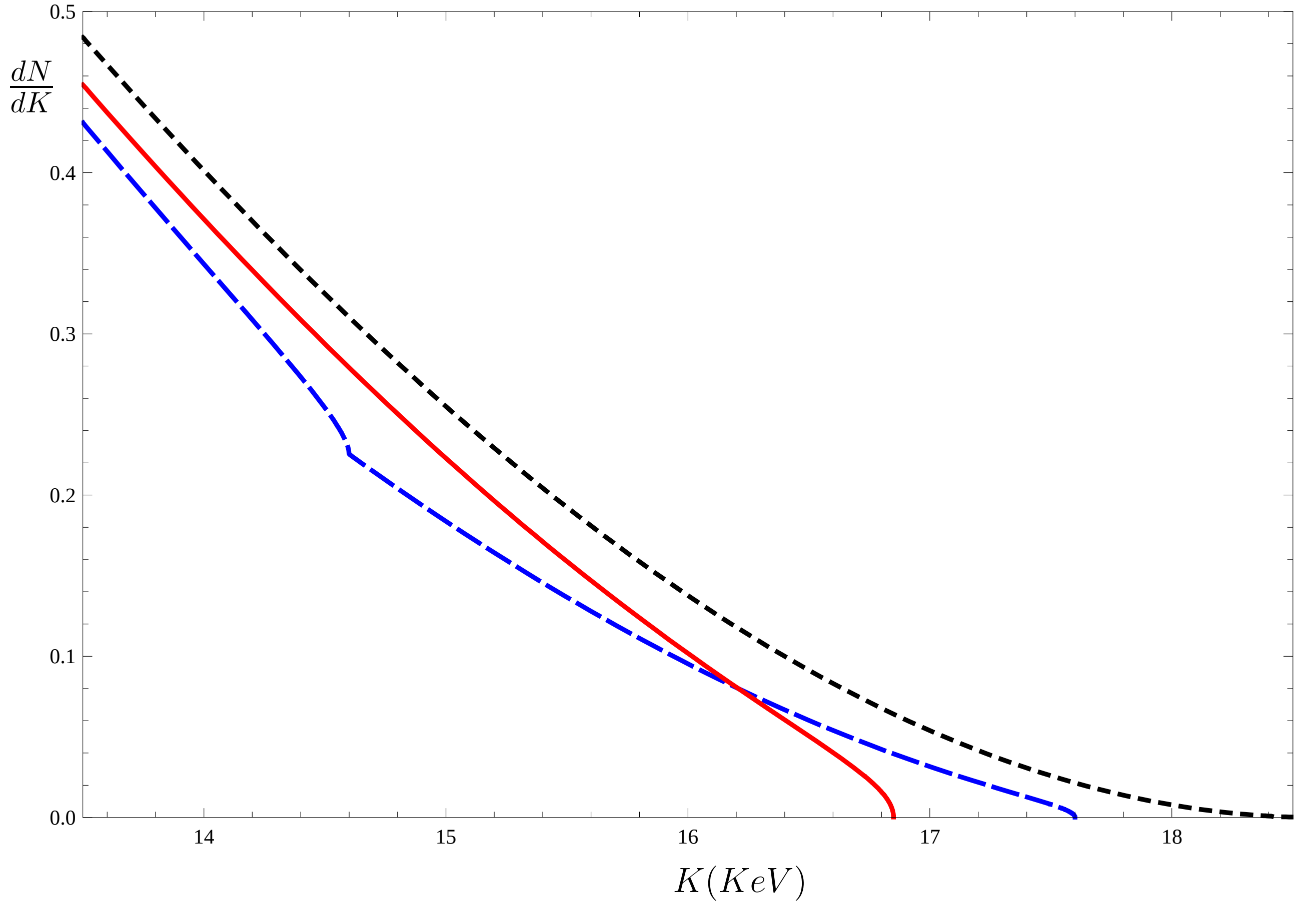
• Lorentz symmetry breaking is due to the external field.

• The vacuum acts as a sort of refractive medium ("*neutrino aether*") with respect to neutrinos.

• Optical analogy: flavor neutrinos as polarizations of the light, oscillations induced by birefringence*.

*C. Weinheimer, Prog. Part. Nucl. Phys., **64** (2010) 205.

Phenomenological consequences



The tail of the tritium β spectrum for:
- a massless neutrino (dotted line);
- fundamental flavor states (continuous line);
- superimposed prediction for 2 mass states (short-dashed line):

We used $m_e = 1.75$ KeV, $m_1 = 1$ KeV, $m_2 = 4$ KeV, $\theta = \pi/6$.

Thermodynamic analogy

Identify

$$F \equiv \widetilde{H}, \qquad T = g = \tan 2\theta$$

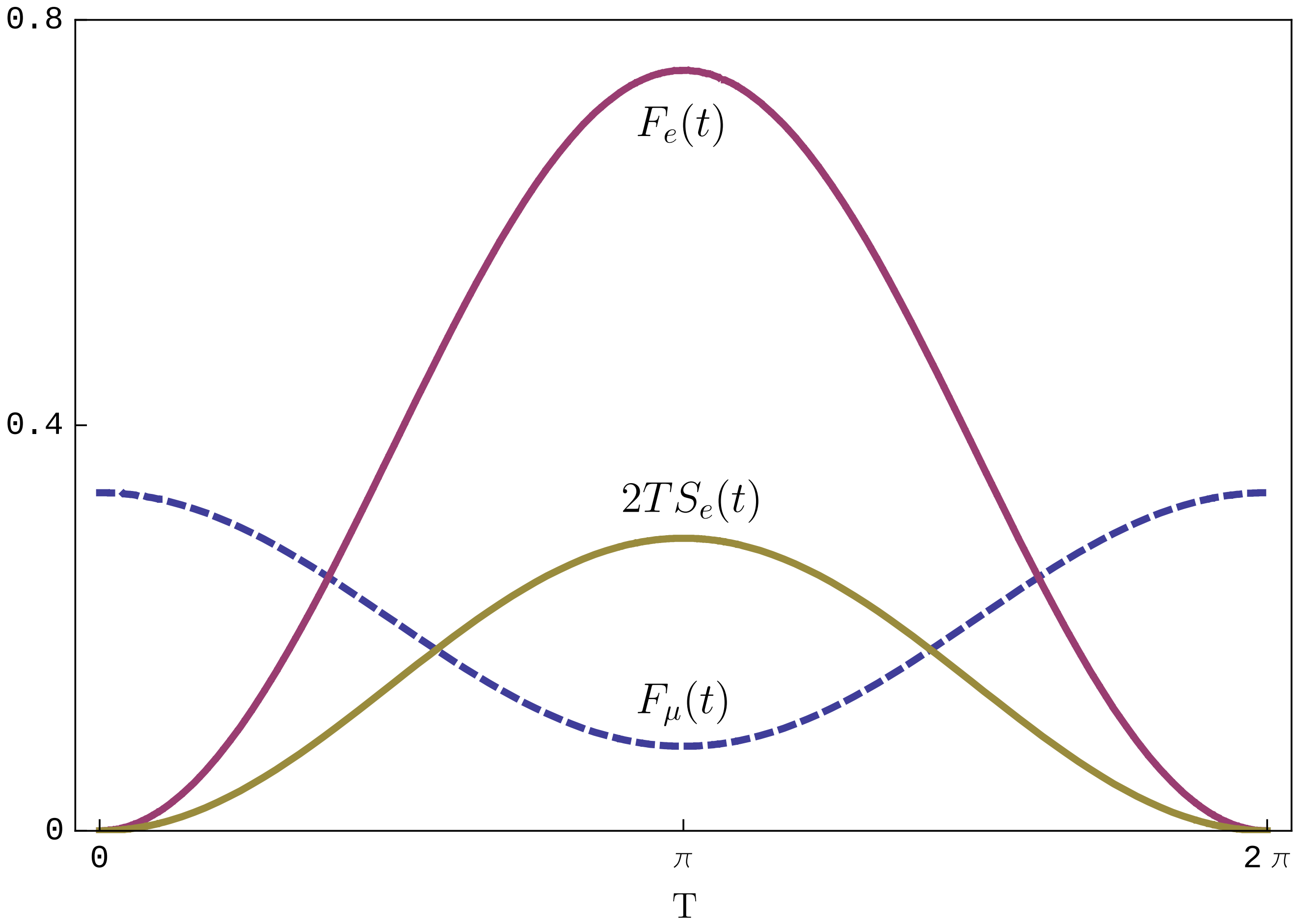
and write

$$H - F = TS,$$

$$S \;=\; \int d^3\mathbf{x}\, \bar{\nu}_f A_0 \nu_f = \frac{1}{2}\delta m \int d^3\mathbf{x}\, (\bar{\nu}_e \nu_\mu + \bar{\nu}_\mu \nu_e).$$

- F is the energy that can be extracted from neutrinos through scattering experiments.

- Each of the two neutrinos can be considered as an open (dissipative) system.



Plot of expectation values on $|\nu_e(0)\rangle$ of $F_e(t)$, $F_\mu(t)$ and $2TS_e(t)$, as functions of dimensionless time $T = (\omega_2 - \omega_1)t$ and $\theta = \pi/6$. Scale on vertical axis is normalized to ω_μ.

Entanglement in neutrino oscillations.

- Flavor mixing and entanglement;
- Entanglement in neutrino oscillations:
  - Flavor entanglement;
  - Decoherence;
- Neutrino oscillations as a resource for quantum information.
- Flavor entanglement in Quantum Field Theory.

Entanglement in particle mixing

– Flavor mixing (neutrinos)

$$|\nu_e\rangle = \quad \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle$$

$$|\nu_\mu\rangle = -\sin\theta\,|\nu_1\rangle + \cos\theta\,|\nu_2\rangle$$

- Correspondence with two-qubit states:

$$|\nu_1\rangle \equiv |1\rangle_1|0\rangle_2 \equiv |10\rangle, \qquad |\nu_2\rangle \equiv |0\rangle_1|1\rangle_2 \equiv |01\rangle,$$

where $|\rangle_i$ denotes states in the Hilbert space for neutrinos with mass m_i.

$\Rightarrow$ flavor states are entangled superpositions of the mass eigenstates:

$$|\nu_e\rangle = \quad \cos\theta\,|10\rangle + \sin\theta\,|01\rangle.$$

Single-particle entanglement*

– A state like $|\psi\rangle_{A,B} = |0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B$ is entangled;

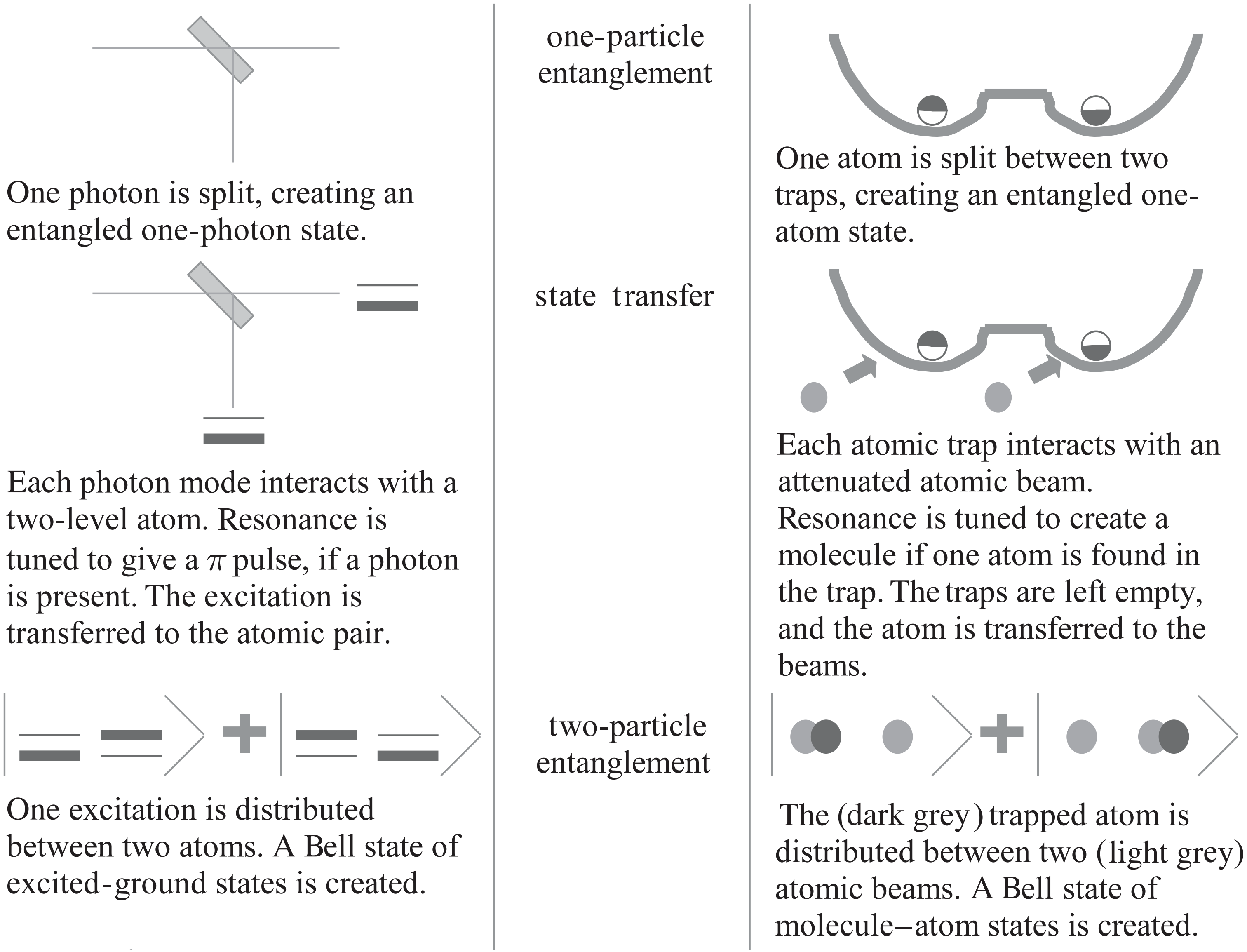
– entanglement among field modes, rather than particles;

– entanglement is a property of composite systems, rather than of many-particle systems;

– entanglement and non-locality are not synonyms;

– single-particle entanglement is as good as two-particle entanglement for applications (quantum cryptography, teleportation, violation of Bell inequalities, etc..).

*J.van Enk, Phys. Rev. A (2005), (2006);
M.O.Terra Cunha, J.A.Dunningham and V.Vedral, Proc. Royal Soc. A (2007);
J.A.Dunningham and V.Vedral, Phys. Rev. Lett. (2007).
S.B.Papp et al. Science (2009)
D.Salart et al. Phys. Rev. Lett (2010)
G.Björk, A.Laghaout, U.L.Andersen Phys. Rev. A (2012)



Protocols for extraction of single-particle entanglement
(from M.O.Terra Cunha, J.A.Dunningham and V.Vedral, Proc. Royal Soc. A (2007))

Multipartite entanglement in neutrino mixing*

– Neutrino mixing (three flavors):

$$|\underline{\nu}_f\rangle \;=\; U(\tilde{\theta},\delta)\,|\underline{\nu}_m\rangle$$

with $|\underline{\nu}_f\rangle = (|\nu_e\rangle, |\nu_\mu\rangle, |\nu_\tau\rangle)^T$ and $|\underline{\nu}_m\rangle = (|\nu_1\rangle, |\nu_2\rangle, |\nu_3\rangle)^T$.

– Mixing matrix (MNSP)

$$U(\tilde{\theta},\delta) \;=\; \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix},$$

where $(\tilde{\theta},\delta) \equiv (\theta_{12},\theta_{13},\theta_{23};\delta)$, $c_{ij} \equiv \cos\theta_{ij}$ and $s_{ij} \equiv \sin\theta_{ij}$.

- Correspondence with three-qubit states:

$$|\nu_1\rangle \equiv |1\rangle_1|0\rangle_2|0\rangle_3 \equiv |100\rangle, \quad |\nu_2\rangle \equiv |0\rangle_1|1\rangle_2|0\rangle_3 \equiv |010\rangle,$$

$$|\nu_3\rangle \equiv |0\rangle_1|0\rangle_2|1\rangle_3 \equiv |001\rangle$$

*M.B., F.Dell'Anno, S.De Siena, M.Di Mauro and F.Illuminati, Phys. Rev. D (2008).

Multipartite entanglement

– Characterization of entanglement for multipartite systems is a non-trivial task. Several approaches have been developed: global entanglement, tangle, geometric measures*, etc...

In the 3-qubit case, the two fundamental classes† of states are those of the GHZ state and of the W state:

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)\,,$$

$$|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)\,.$$

*T.C.Wei and P.M.Goldbart Phys. Rev. A (2003);
M.B., F.Dell'Anno, S.De Siena and F.Illuminati, Phys. Rev. A (2008).

†W.Dür, G.Vidal, and J.I.Cirac, Phys. Rev. A (2000)

(Flavor) Entanglement in neutrino oscillations*

– Two-flavor neutrino states

$$|\underline{\nu}^{(f)}\rangle = \mathbf{U}(\tilde{\theta}, \delta)\, |\underline{\nu}^{(m)}\rangle$$

where $|\underline{\nu}^{(f)}\rangle = (|\nu_e\rangle, |\nu_\mu\rangle)^T$ and $|\underline{\nu}^{(m)}\rangle = (|\nu_1\rangle, |\nu_2\rangle)^T$ and $\mathbf{U}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

– Flavor states at time t:

$$|\underline{\nu}^{(f)}(t)\rangle = \mathbf{U}(\tilde{\theta}, \delta)\, \mathbf{U}_0(t)\, \mathbf{U}(\tilde{\theta}, \delta)^{-1}\, |\underline{\nu}^{(f)}\rangle \equiv \widetilde{\mathbf{U}}(t)|\underline{\nu}^{(f)}\rangle\,,$$

with $\mathbf{U}_0(t) = \begin{pmatrix} e^{-iE_1 t} & 0 \\ 0 & e^{-iE_2 t} \end{pmatrix}$.

*M.B., F.Dell'Anno, S.De Siena and F.Illuminati, EPL (2009).

– Transition probability for $\nu_\alpha \to \nu_\beta$

$$P_{\nu_\alpha \to \nu_\beta}(t) = |\langle \nu_\beta | \nu_\alpha(t) \rangle|^2 = |\widetilde{\mathbf{U}}_{\alpha\beta}(t)|^2 \, .$$

- We now take the flavor states at initial time as our qubits:

$$|\nu_e\rangle \equiv |1\rangle_e |0\rangle_\mu \equiv |10\rangle_f , \qquad |\nu_\mu\rangle \equiv |0\rangle_e |1\rangle_\mu \equiv |01\rangle_f ,$$

– Starting from $|10\rangle_f$ or $|01\rangle_f$, time evolution generates the (entangled) Bell-like states:

$$|\nu_\alpha(t)\rangle = \widetilde{\mathbf{U}}_{\alpha e}(t)|1\rangle_e|0\rangle_\mu + \widetilde{\mathbf{U}}_{\alpha\mu}(t)|0\rangle_e|1\rangle_\mu , \qquad \alpha = e, \mu.$$

Entanglement measure

– Let $\rho = |\psi\rangle\langle\psi|$ be the density operator for a pure state $|\psi\rangle$

Bipartition of the N-partite system $S = \{S_1, S_2, \ldots, S_N\}$ in two subsystems:

$$S_{A_n} = \{S_{i_1}, S_{i_2}, \ldots, S_{i_n}\}, \quad 1 \leq i_1 < i_2 < \ldots < i_n \leq N; (1 \leq n < N)$$

and

$$S_{B_{N-n}} = \{S_{j_1}, S_{j_2}, \ldots, S_{j_{N-n}}\} \quad , 1 \leq j_1 < j_2 < \ldots < j_{N-n} \leq N; i_q \neq j_p$$

– Reduced density matrix of S_{A_n} after tracing over $S_{B_{N-n}}$:

$$\rho_{A_n} \equiv \rho_{i_1, i_2, \ldots, i_n} = Tr_{B_{N-n}}[\rho] = Tr_{j_1, j_2, \ldots, j_{N-n}}[\rho]$$

• Linear entropy associated to such a bipartition:

$$S_L^{(A_n;B_{N-n})}(\rho) = \frac{d}{d-1}(1 - Tr_{A_n}[\rho_{A_n}^2]) \, ,$$

d is the Hilbert-space dimension: $d = \min\{\dim S_{A_n}, \dim S_{B_{N-n}}\} = \min\{2^n, 2^{N-n}\}$.

• Average linear entropy (global entanglement):

$$\langle S_L^{(n:N-n)}(\rho)\rangle = \begin{pmatrix} N \\ n \end{pmatrix}^{-1} \sum_{A_n} S_L^{(A_n;B_{N-n})}(\rho) \, ,$$

sum over all the possible bi-partitions of the system in two subsystems, respectively with n and $N-n$ elements ($1 \le n < N$).

Entanglement in neutrino oscillations: two-flavors

Consider the density matrix for the electron neutrino state $\rho^{(e)} = |\nu_e(t)\rangle\langle\nu_e(t)|$, and trace over mode $\mu \;\Rightarrow\; \rho_e^{(e)}$.

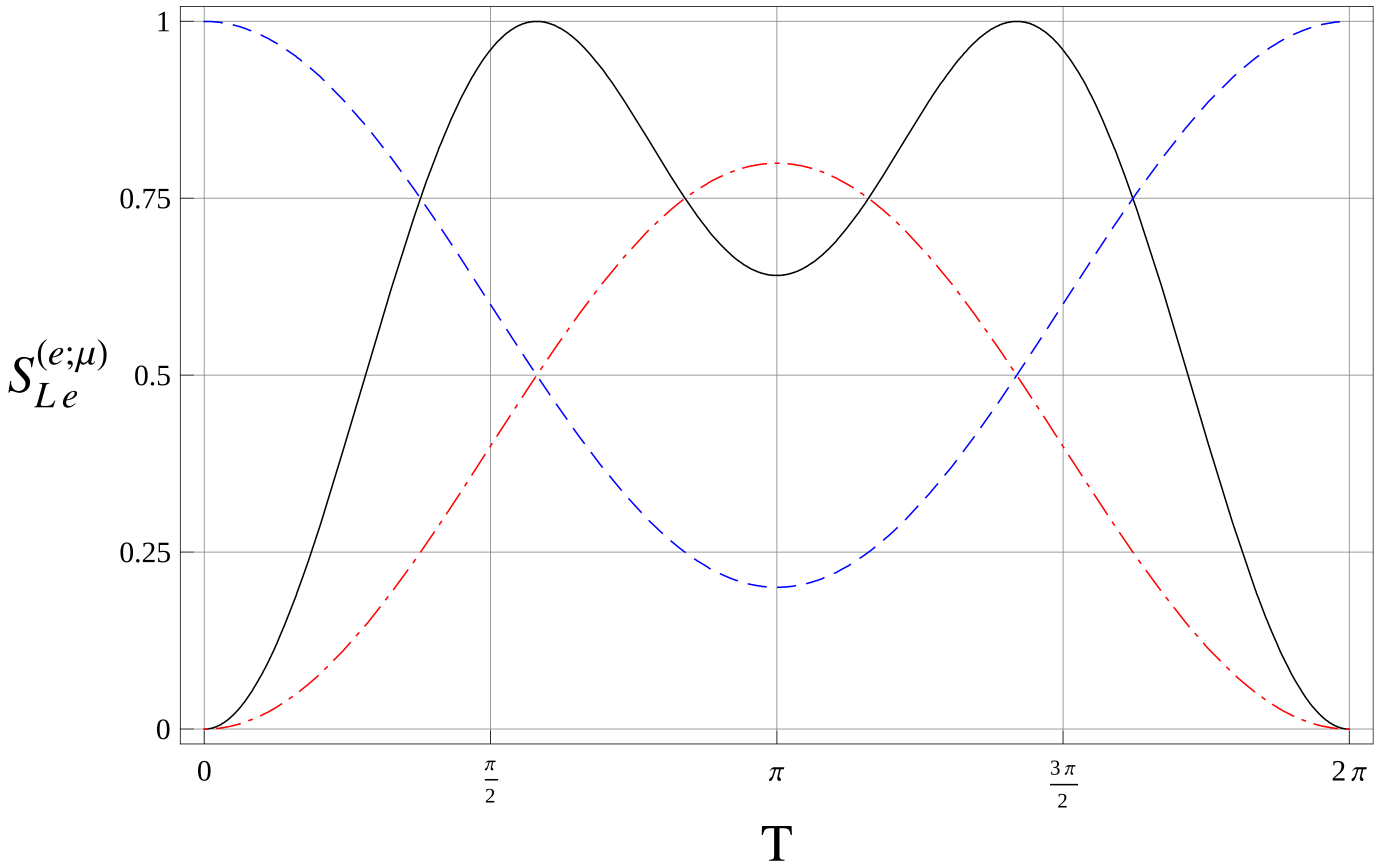
– The associated linear entropy is :

$$S_L^{(e;\mu)}(\rho^{(e)}) \;=\; 4\,|\widetilde{\mathbf{U}}_{e\mu}(t)|^2\,|\widetilde{\mathbf{U}}_{ee}(t)|^2 \;=\; 4\,P_{\nu_e\to\nu_e}(t)\,P_{\nu_e\to\nu_\mu}(t)$$

– The linear entropy for the state $\rho^{(\alpha)}$ is:

$$\begin{aligned} S_{L\alpha}^{(e;\mu)} = S_{L\alpha}^{(\mu;e)} = \langle S_{L\alpha}^{(1:1)}\rangle \;&=\; 4\,|\widetilde{\mathbf{U}}_{\alpha\mu}(t)|^2\,|\widetilde{\mathbf{U}}_{\alpha e}(t)|^2 \\ &=\; 4|\widetilde{\mathbf{U}}_{\alpha e}(t)|^2\,(1-|\widetilde{\mathbf{U}}_{\alpha e}(t)|^2) \\ &=\; 4|\widetilde{\mathbf{U}}_{\alpha\mu}(t)|^2\,(1-|\widetilde{\mathbf{U}}_{\alpha\mu}(t)|^2)\,. \end{aligned}$$

- Linear entropy given by product of transition probabilities !



Linear entropy $S_{Le}^{(e;\mu)}$ (full) as a function of the scaled time $T = \frac{2Et}{\Delta m_{12}^2}$, with $\sin^2\theta = 0.314$. Transition probabilities $P_{\nu_e \to \nu_e}$ (dashed) and $P_{\nu_e \to \nu_\mu}$ (dot-dashed) are reported for comparison.

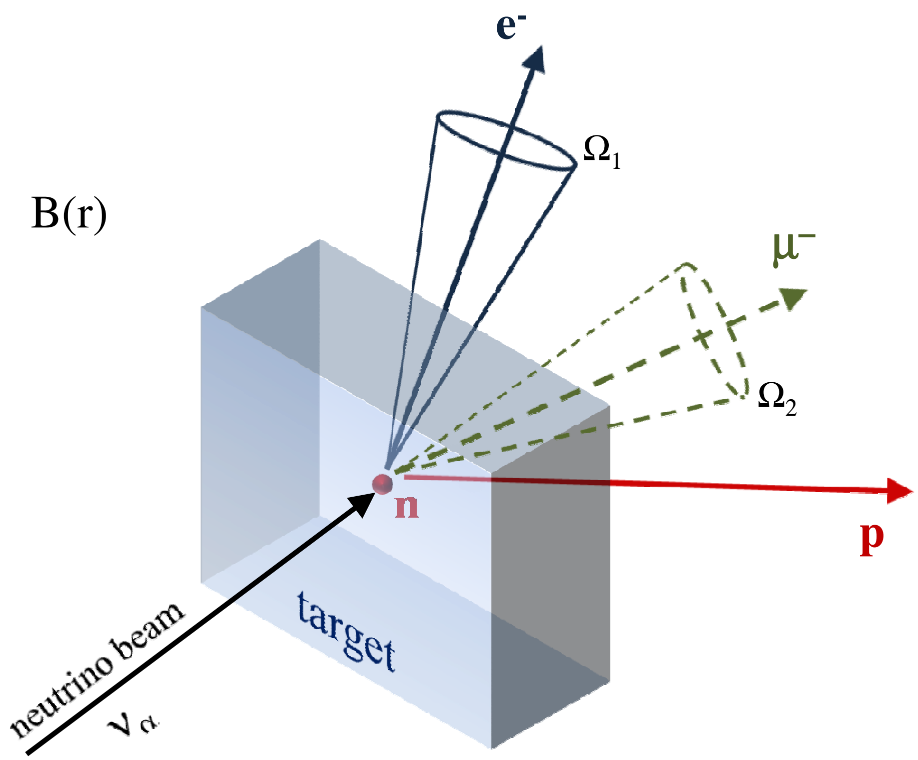
Neutrino oscillations as a resource for quantum information

- Single-particle entanglement encoded in flavor states $|\underline{\nu}^{(f)}(t)\rangle$ is a real physical resource that can be used, at least in principle, for protocols of quantum information.

– Experimental scheme for the transfer of the flavor entanglement of a neutrino beam into a single-particle system with *spatially separated modes*.

Charged-current interaction between a neutrino ν_α with flavor α and a nucleon N gives a lepton α^- and a baryon X:

$$\nu_\alpha + N \longrightarrow \alpha^- + X \,.$$



Generation of a single-particle entangled lepton state (two flavors):

In the target the charged-current interaction occurs: $\nu_\alpha + n \longrightarrow \alpha^- + p$ with $\alpha = e\,,\mu$.

A spatially nonuniform magnetic field $\mathbf{B}(\mathbf{r})$ constraints the momentum of the outgoing lepton within a solid angle Ω_i, and ensures spatial separation between lepton paths.

The reaction produces a superposition of electronic and muonic spatially separated states.

• Given the initial Bell-like superposition $|\nu_\alpha(t)\rangle$ the unitary process associated with the weak interaction leads to the superposition

$$|\alpha(t)\rangle = \Lambda_e|1\rangle_e|0\rangle_\mu + \Lambda_\mu|0\rangle_e|1\rangle_\mu\,,$$

where $|\Lambda_e|^2 + |\Lambda_\mu|^2 = 1$, and $|k\rangle_\alpha$, with $k = 0, 1$, represents the lepton qubit.

The coefficients Λ_α are proportional to $\widetilde{\mathbf{U}}_{\alpha\beta}(t)$ and to the cross sections associated with the creation of an electron or a muon.

• Analogy with single-photon system: quantum uncertainty on the so-called "*which path*" of the photon at the output of an unbalanced beam splitter $\Leftrightarrow$ uncertainty on the "*which flavor*" of the produced lepton.

The coefficients Λ_α plays the role of the transmissivity and of the reflectivity of the beam splitter.

Entanglement for neutrino oscillations in QFT*

– Extension of the above analysis to QFT

– Non-trivial nature of mixing transformations in QFT

– Dynamical symmetry approach to entanglement

– Entropic measures in QFT

*M.B., F.Dell'Anno, F.Illuminati and S.De Siena, EPL (2014).

Entanglement in relativistic systems

- Necessity for a treatment of entanglement in the context of Quantum Field Theory.*

- Lorentz invariance of entanglement: two particles entangled in spin and momentum†.

*M.O.Terra Cunha, J.A.Dunningham and V.Vedral, Proc. Royal Soc. A (2007); Y.Shi, Phys. Rev. D (2004);

†N.Friis, R.A.Bertlmann, M.Huber and B.C.Hiesmayr Phys. Rev. A (2010); M.Huber, N.Friis, A.Gabriel, C.Spengler and B.C.Hiesmayr, EPL (2011)

Dynamical symmetry approach to entanglement

• Entanglement can be characterized by total variance of the operators generating the dynamical algebra*.

– Consider the observables X_i elements of the basis of a Lie algebra $\mathcal{L}$ such that the Lie group $G = \exp(i\mathcal{L})$ defines the dynamic symmetry of the system.

• Entanglement of a state $|\psi\rangle$ is given by the total amount of uncertainty:

$$\Delta(\psi) \;=\; \sum_i \left(\langle\psi|X_i^2|\psi\rangle - \langle\psi|X_i|\psi\rangle^2 \right)$$

*A. A. Klyachko, B. Öztop, and A. S. Shumovsky, *Phys. Rev.* **A** (2007);

Entanglement for flavor neutrino states in QFT I

– Entanglement for flavor neutrino states in QFT can be expressed by means of the variances of the neutrino charges: Q_i, $Q_\sigma(t)$

– Variance of $Q_i \Rightarrow$ static entanglement:

$$
\begin{aligned}
\Delta Q_i(\nu_e) &= \langle \nu^r_{\mathbf{k},e}|Q_i^2(t)|\nu^r_{\mathbf{k},e}\rangle - \langle \nu^r_{\mathbf{k},e}|Q_i|\nu^r_{\mathbf{k},e}\rangle^2 \\
&= \cos^2\theta \sin^2\theta
\end{aligned}
$$

– Variance of $Q_\sigma \Rightarrow$ flavor entanglement:

$$
\begin{aligned}
\Delta Q_\sigma(\nu_e)(t) &= \langle \nu^r_{\mathbf{k},e}|Q_\sigma^2(t)|\nu^r_{\mathbf{k},e}\rangle - \langle \nu^r_{\mathbf{k},e}|Q_\sigma(t)|\nu^r_{\mathbf{k},e}\rangle^2 \\
&= \mathcal{Q}^{\mathbf{k}}_{\nu_e\to\nu_e}(t)\, \mathcal{Q}^{\mathbf{k}}_{\nu_e\to\nu_\mu}(t)
\end{aligned}
$$

in formal agreement with results obtained in QM.

Entanglement for flavor states in QFT II

Rewrite QFT flavor neutrino state as:

$$|\nu_e(t)\rangle = [\mathbf{U}_{ee}(t)\,\alpha^\dagger_e + \mathbf{U}_{e\mu}(t)\,\alpha^\dagger_\mu + \mathbf{U}^{e\bar{e}}_{e\mu}(t)\,\alpha^\dagger_e\alpha^\dagger_\mu\beta^\dagger_e + \mathbf{U}^{\mu\bar{\mu}}_{ee}(t)\,\alpha^\dagger_e\alpha^\dagger_\mu\beta^\dagger_\mu]\,|0\rangle_{e,\mu}$$

with

$$\mathbf{U}_{ee}(t) = e^{-i\omega_1 t}[\cos^2\theta + \sin^2\theta(e^{-i(\omega_2-\omega_1)t}|U|^2 + e^{-i(\omega_2+\omega_1)t}|V|^2)]$$

$$\mathbf{U}_{e\mu}(t) = e^{-i\omega_1 t}U\cos\theta\sin\theta(e^{-i(\omega_2-\omega_1)t} - 1)$$

$$\mathbf{U}^{e\bar{e}}_{e\mu}(t) = e^{-i\omega_1 t}V\cos\theta\sin\theta(1 - e^{-i(\omega_2+\omega_1)t})$$

$$\mathbf{U}^{\mu\bar{\mu}}_{ee}(t) = e^{-i\omega_1 t}UV\sin^2\theta(e^{-i(\omega_2+\omega_1)t} - e^{-i(\omega_2-\omega_1)t}),$$

$$|\mathbf{U}_{ee}(t)|^2 + |\mathbf{U}_{e\mu}(t)|^2 + |\mathbf{U}^{e\bar{e}}_{e\mu}(t)|^2 + |\mathbf{U}^{\mu\bar{\mu}}_{ee}(t)|^2 = 1.$$

- In QFT, flavor neutrino states exhibit multiparticle components.

Linear entropies for $|\nu_e\rangle$ associated to unbalanced bipartitions $S_L^{(\alpha;\beta,\gamma,\delta)}$:

$$S_L^{(\nu_e;\nu_\mu,\bar{\nu}_e,\bar{\nu}_\mu)} = 4|\mathbf{U}_{e\mu}|^2(1-|\mathbf{U}_{e\mu}|^2)\,,$$

$$S_L^{(\nu_\mu;\nu_e,\bar{\nu}_e,\bar{\nu}_\mu)} = 4|\mathbf{U}_{ee}|^2(1-|\mathbf{U}_{ee}|^2)\,,$$

$$S_L^{(\bar{\nu}_e;\nu_e,\nu_\mu,\bar{\nu}_\mu)} = 4|\mathbf{U}_{e\mu}^{e\bar{e}}|^2(1-|\mathbf{U}_{e\mu}^{e\bar{e}}|^2)\,,$$

$$S_L^{(\bar{\nu}_\mu;\nu_e,\nu_\mu,\bar{\nu}_e)} = 4|\mathbf{U}_{ee}^{\mu\bar{\mu}}|^2(1-|\mathbf{U}_{ee}^{\mu\bar{\mu}}|^2)\,.$$

In the quantum mechanical limit, first two expressions reduce to the Pontecorvo analogs, while the other two go to zero.

The linear entropies associated with balanced bipartitions $S_{L,e}^{(a,b;c,d)}$ are:

$$S_{L,e}^{(\nu_e,\nu_\mu;\bar{\nu}_e,\bar{\nu}_\mu)} = \frac{4}{3}\left[1-(|\mathbf{U}_{ee}|^2+|\mathbf{U}_{e\mu}|^2)^2-(|\mathbf{U}_{e\mu}^{e\bar{e}}|^2+|\mathbf{U}_{ee}^{\mu\bar{\mu}}|^2)^2\right],$$

$$S_{L,e}^{(\nu_e,\bar{\nu}_e;\nu_\mu,\bar{\nu}_\mu)} = \frac{4}{3}\left[1-(|\mathbf{U}_{ee}|^2+|\mathbf{U}_{ee}^{\mu\bar{\mu}}|^2)^2-(|\mathbf{U}_{e\mu}|^2+|\mathbf{U}_{e\mu}^{e\bar{e}}|^2)^2\right],$$

$$S_{L,e}^{(\nu_e,\bar{\nu}_\mu;\nu_\mu,\bar{\nu}_e)} = \frac{4}{3}\left[1-(|\mathbf{U}_{ee}|^2+|\mathbf{U}_{e\mu}^{e\bar{e}}|^2)^2-(|\mathbf{U}_{e\mu}|^2+|\mathbf{U}_{ee}^{\mu\bar{\mu}}|^2)^2\right].$$

The linear entropies $S_L^{(a;b,c,d)}$, coincide, apart from a constant factor, with the variances associated with the particle number:

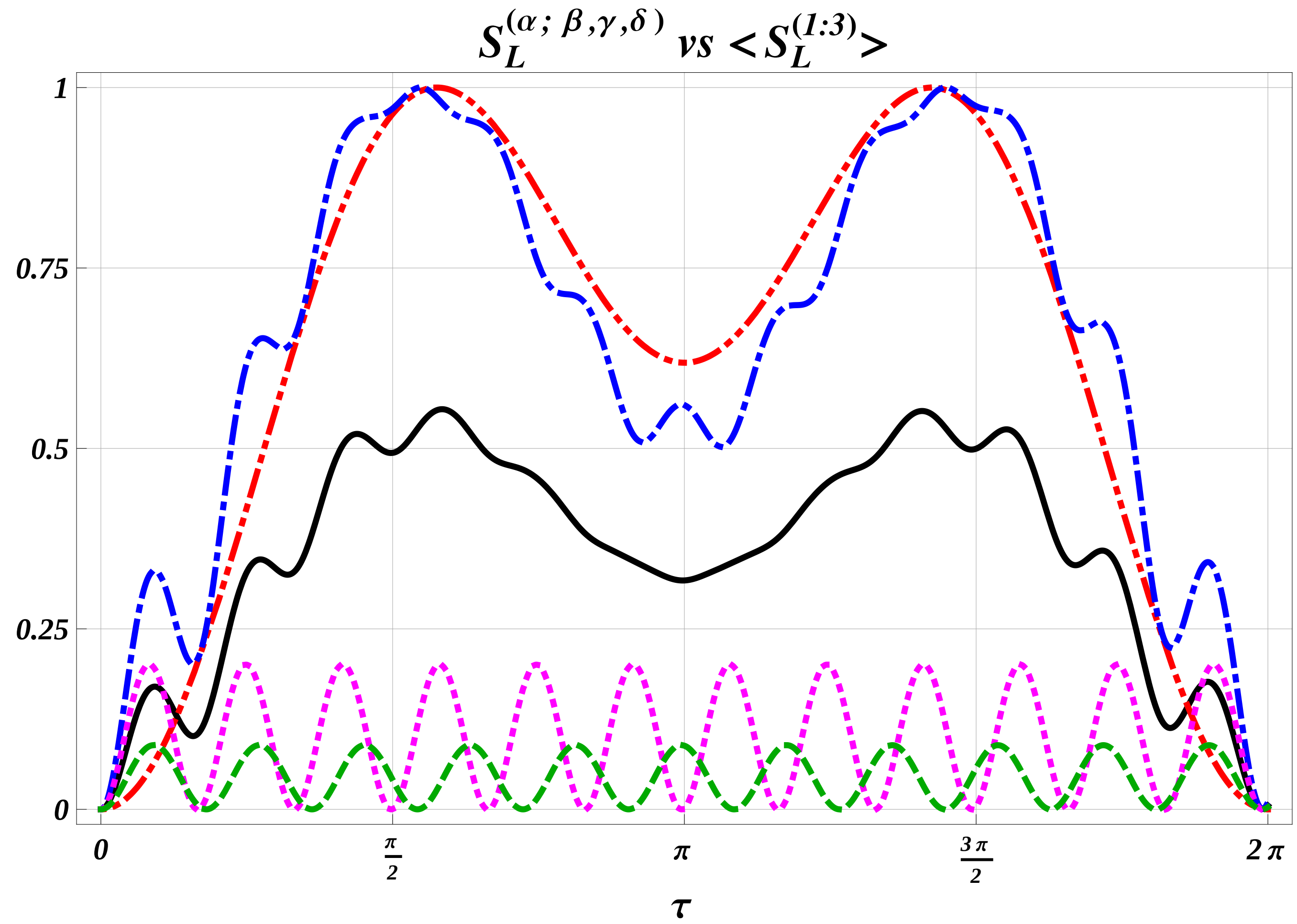
$$\langle(\Delta N_a)^2\rangle \equiv \langle N_a\rangle\,(1-\langle N_a\rangle) = \frac{1}{4}\,S_L^{(a;b,c,d)}\,.$$

On the other hand, the linear entropies $S_L^{(a,b;c,d)}$ are proportional to variances of linear combination of number operators

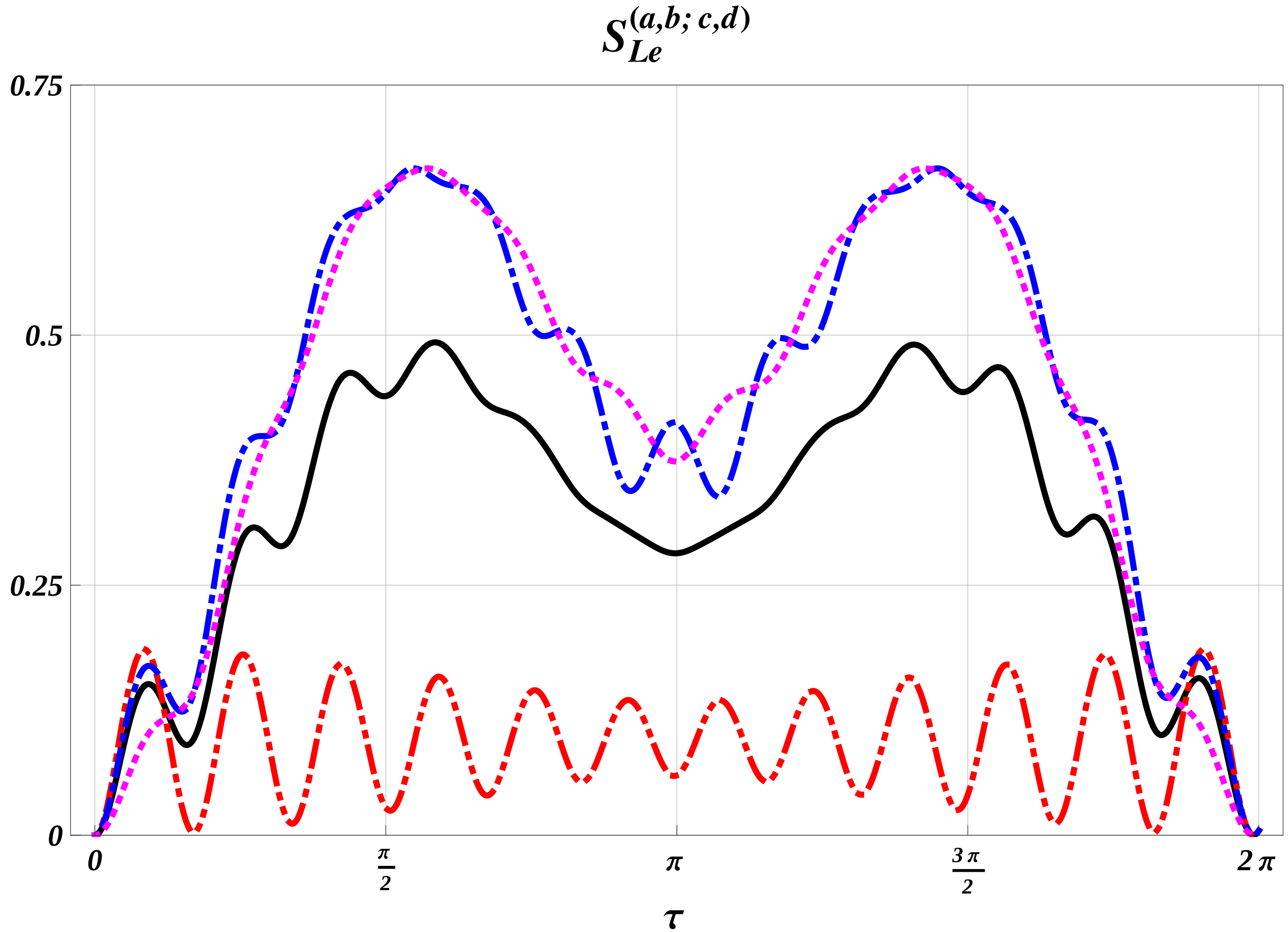
$$\langle[\Delta(N_e+N_\mu)]^2\rangle = \frac{3}{4}S_{L,e}^{(\nu_e,\nu_\mu;\bar{\nu}_e,\bar{\nu}_\mu)}\,,$$

$$\langle[\Delta(N_e-N_{\bar{e}})]^2\rangle \equiv \langle[\Delta Q_e]^2\rangle = \frac{3}{4}S_{L,e}^{(\nu_e,\bar{\nu}_e;\nu_\mu,\bar{\nu}_\mu)}\,,$$

$$\langle[\Delta(N_e-N_{\bar{\mu}})]^2\rangle = \frac{3}{4}S_{L,e}^{(\nu_e,\bar{\nu}_\mu;\nu_\mu,\bar{\nu}_e)}\,.$$



Linear entropies: $S_L^{(\nu_e;\nu_\mu,\bar{\nu}_e,\bar{\nu}_\mu)}$ (double-dot-dashed line), $S_L^{(\nu_\mu;\nu_e,\bar{\nu}_e,\bar{\nu}_\mu)}$ (dot-dashed line), $S_L^{(\bar{\nu}_e;\nu_e,\nu_\mu,\bar{\nu}_\mu)}$ (dotted line), $S_L^{(\bar{\nu}_\mu;\nu_e,\nu_\mu,\bar{\nu}_e)}$ (dashed line), and the average linear entropy $\langle S_L^{(1:3)}\rangle$ (full line) as functions of the scaled time $\tau = (\omega_2 - \omega_1)t$.



The linear entropies $S_{L,e}^{(\nu_e,\nu_\mu;\bar{\nu}_e,\bar{\nu}_\mu)}$ (double-dot-dashed line), $S_{L,e}^{(\nu_e,\bar{\nu}_e;\nu_\mu,\bar{\nu}_\mu)}$ (dot-dashed line), $S_{L,e}^{(\nu_e,\bar{\nu}_\mu;\nu_\mu,\bar{\nu}_e)}$ (dotted line), and the average linear entropy $\langle S_{L,e}^{(2:2)} \rangle$ (full line) as functions of the scaled time $\tau = (\omega_2 - \omega_1)t$. The mixing angle θ is fixed at the experimental value $\sin^2\theta = 0.314$; the parameters x and p are fixed as $x = 10$ and $p = 5$.

Conclusions

• Mixing transformations are not trivial in Q.F.T. $\Leftrightarrow$ they are associated to inequivalent representations.

• The vacuum for mixed fields has the structure of a $SU(N)$ generalized coherent state (condensate of particle-antiparticle pairs).

• Flavor neutrinos are produced as entangled states in the SM;

• Neutrino oscillations as a resource for quantum information;

• Extension to QFT: Entanglement vs. inequivalent representations.

Dynamical generation of flavor mixing*

• The non trivial nature of flavor vacuum should result from the SSB process and the Higgs mechanism in the Standard Model;

• We consider dynamical symmetry breaking in a toy model with two flavors and quartic interaction term, as a generalization of Nambu and Jona-Lasinio model†;

• The approach of Umezawa, Takahashi and Kamefuchi for describing mass generation using inequivalent representations‡ is suitable for our purposes.

*M.B., P. Jizba, G. Lambiase and N. Mavromatos, arXiv:1312.4924 [hep-ph]

†Y. Nambu and G. Jona-Lasinio, Phys. Rev. (1961);

‡H. Umezawa, Y. Takahashi and S. Kamefuchi, Ann. Phys. (1964)

Dynamical mass generation and inequivalent representations

Consider a free Dirac field (at finite volume V):

$$\psi = \frac{1}{\sqrt{V}} \sum_{\mathbf{k},r} \left[u_{\mathbf{k}}\, a^r_{\mathbf{k}} e^{-ik\cdot x} + v_{\mathbf{k}}\, b^{r\dagger}_{\mathbf{k}} e^{ik\cdot x} \right] , \qquad a^r_{\mathbf{k}}|0\rangle = b^r_{\mathbf{k}}|0\rangle = 0$$

The *same* field operator can be expanded as

$$\psi(x) \;=\; \frac{1}{\sqrt{V}} \sum_{\mathbf{k},r} \left[u^r_{\mathbf{k}}(\vartheta,\phi) \alpha^r_{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{x}} + v^r_{\mathbf{k}}(\vartheta,\phi) \beta^{r\dagger}_{\mathbf{k}}\, e^{-i\mathbf{k}\cdot\mathbf{x}} \right] ,$$

with $\alpha^r_{\mathbf{k}}\,|0(\vartheta,\varphi)\rangle \;=\; \beta^r_{\mathbf{k}}\,|0(\vartheta,\varphi)\rangle \;=\; 0\,.$ Bogoliubov transformation:

$$\begin{aligned} \alpha^r_{\mathbf{k}} \;&=\; \cos\vartheta^r_{\mathbf{k}}\; a^r_{\mathbf{k}} \;+\; \sin\vartheta^r_{\mathbf{k}}\; e^{i\varphi(\mathbf{k},r)}\; b^{r\dagger}_{-\mathbf{k}} \\ \beta^r_{-\mathbf{k}} \;&=\; \cos\vartheta^r_{\mathbf{k}}\; b^r_{-\mathbf{k}} \;-\; \sin\vartheta^r_{\mathbf{k}}\; e^{i\varphi(\mathbf{k},r)}\; a^{r\dagger}_{\mathbf{k}} \end{aligned}$$

and

$$\begin{aligned} u^r_{\mathbf{k}}(\vartheta,\phi) \;&=\; u^r_{\mathbf{k}}\, \cos\vartheta_k \;+\; v^r_{-\mathbf{k}}\, e^{-i\varphi^r_k} \sin\vartheta_k\,, \\ v^r_{\mathbf{k}}(\vartheta,\phi) \;&=\; v^r_{\mathbf{k}}\, \cos\vartheta_k \;-\; u^r_{-\mathbf{k}}\, e^{i\varphi^r_k} \sin\vartheta_k\,. \end{aligned}$$

V-limit for operator products

In the infinite volume limit, one has the following relations:

$$V\text{-lim}\left[\int d^3\boldsymbol{x}\,\bar{\psi}_\alpha(x)\psi_\beta(x)\right] = \int d^3\boldsymbol{x}\ :\bar{\psi}_\alpha(x)\psi_\beta(x):\ + \int d^3\boldsymbol{x}\, iS^-_{\alpha\beta}(\vartheta,\varphi)\,,$$

$$V\text{-lim}\left[\int d^3\boldsymbol{x}\,\bar{\psi}_\alpha(x)\psi_\beta(x)\bar{\psi}_\gamma(x)\psi_\delta(x)\right] =$$

$$= iS^-_{\alpha\beta}(\vartheta,\varphi)\int d^3\boldsymbol{x}\ :\bar{\psi}_\gamma(x)\psi_\delta(x):\ + iS^+_{\gamma\delta}(\vartheta,\varphi)\int d^3\boldsymbol{x}\ :\bar{\psi}_\alpha(x)\psi_\beta(x):$$

$$+\ iS^-_{\alpha\delta}(\vartheta,\varphi)\int d^3\boldsymbol{x}\ :\bar{\psi}_\gamma(x)\psi_\beta(x):\ + iS^+_{\gamma\beta}(\vartheta,\varphi)\int d^3\boldsymbol{x}\ :\bar{\psi}_\alpha(x)\psi_\delta(x):$$

$$+ \int d^3\boldsymbol{x} \sum_{\text{contractions}} S^+(\vartheta,\varphi)S^+(\vartheta,\varphi)\,.$$

$S^\pm_{\alpha\beta}(\theta,\varphi)$ are free two-point Wightman functions evaluated in $|0(\theta,\varphi)\rangle$:

$$iS^+_{\alpha\beta}(\vartheta,\varphi) = \langle 0(\vartheta,\varphi)|\bar{\psi}_\alpha(x)\psi_\beta(x)|0(\vartheta,\varphi)\rangle,$$

$$iS^-_{\alpha\beta}(\vartheta,\varphi) = \langle 0(\theta,\varphi)|\bar{\psi}_\alpha(x)\psi_\beta(x)|0(\vartheta,\varphi)\rangle$$

We consider the following hamiltonian:

$$\begin{aligned} H &= H_0 + H_{\text{int}}\,, \\ H_0 &= \int d^3\mathbf{x}\, \bar{\psi}\, (-i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} \; + \; m)\, \psi\,, \\ H_{\text{int}} &= \lambda \int d^3\mathbf{x}\, \left[\left(\bar{\psi}\psi\right)^2 - \left(\bar{\psi}\gamma^5\psi\right)^2 \right]. \end{aligned}$$

In the lowest order in the Yang-Feldman eq. the V-limit of H gives:

$$V\text{-lim}\,[H] = \bar{H}_0 + c - number.$$

with

$$\begin{aligned} \bar{H}_0 &= H_0 + \delta H_0 \\ \delta H_0 &= \int d^3x \left(f\bar{\psi}\psi \, + \, ig\bar{\psi}\gamma_5\psi \right) \end{aligned}$$

where f, g depend on the set of parameters (ϑ, φ):

$$f = \lambda\, C_s\,, \qquad g = \lambda\, C_p\,.$$

$$
\begin{aligned}
C_p &\equiv i \lim_{V\to\infty} \langle 0(\vartheta,\varphi)|\bar{\psi}(x)\gamma_5\psi(x)|0(\vartheta,\varphi)\rangle \\
&= \frac{2}{(2\pi)^3}\int d^3\mathbf{k}\, \sin 2\vartheta_k \, \sin\varphi_k
\end{aligned}
$$

$$
\begin{aligned}
C_s &\equiv \lim_{V\to\infty} \langle 0(\vartheta,\varphi)|\bar{\psi}(x)\psi(x)|0(\vartheta,\varphi)\rangle \\
&= -\frac{2}{(2\pi)^3}\int d^3\mathbf{k}\, \left[\frac{m}{\omega_k}\cos 2\vartheta_k - \frac{k}{\omega_k}\sin 2\vartheta_k \cos\varphi_k\right].
\end{aligned}
$$

We then require that $\bar{H}_0$ has the form of a free Hamiltonian:

$$\bar{H}_0 \;=\; \sum_r \int d^3\mathbf{k}\; E_\mathbf{k} \left(\alpha_\mathbf{k}^{r\dagger}\,\alpha_\mathbf{k}^r + \beta_\mathbf{k}^{r\dagger}\,\beta_\mathbf{k}^r\right) + W_0.$$

with

$$E_\mathbf{k} = \sqrt{\mathbf{k}^2 + M^2} \quad ; \qquad W_0 = -2\int d^3\mathbf{k}\; E_\mathbf{k}.$$

by fixing the Bogoliubov transformation parameters. One obtains:

$$\cos 2\vartheta_\mathbf{k}^r \;=\; \frac{1}{E_k}\left[\omega_k + f\frac{m}{\omega_k}\right]$$

$$\cos\varphi(\mathbf{k},\mathbf{r}) \;=\; -f\frac{k}{\omega_k}\frac{1}{\sqrt{g^2 + f^2(k^2/\omega_k^2)}}$$

$$M^2 \;=\; (m+f)^2 + g^2\,.$$

Two possibilities:

$$C_p = 0, \qquad M = m - \frac{2\lambda}{(2\pi)^3} M \int \frac{d^3\mathbf{k}}{E_k},$$

$$m = 0, \qquad 1 + \frac{2\lambda}{(2\pi)^3} \int \frac{d^3\mathbf{k}}{E_k} = 0.$$

The second case is only allowed for $\lambda < 0$.

Dynamical generation of flavor mixing

We consider the following hamiltonian:

$$\begin{aligned} H &= H_0 + H_{int} \\ H_0 &= \int d^3x \bar{\Psi} \left(-i\gamma^i \partial_i + M_0 \right) \Psi \end{aligned}$$

with $\Psi^T = (\psi_I, \psi_{II})$ and $M_0 = diag(m_I, m_{II})$.

The interaction Hamiltonian H_{int} can be assumed in the generic form

$$\mathcal{H}_{\text{int}} = \left(\bar{\psi}\, \Gamma\, \psi \right) \left(\bar{\psi}\, \Gamma'\, \psi \right),$$

where Γ and Γ' are some doublet spinor matrices.

In this case the V-limit renormalization term $\delta\mathcal{H}_0$ has the following structure

$$\begin{aligned} \delta\mathcal{H}_0 &= \delta\mathcal{H}_0^I + \delta\mathcal{H}_0^{II} + \delta\mathcal{H}_{\text{mix}} \\ &= f_I\, \bar{\psi}_I \psi_I \;+\; f_{II}\, \bar{\psi}_{II} \psi_{II} \;+\; h \left(\bar{\psi}_I \psi_{II} \;+\; \bar{\psi}_{II} \psi_I \right). \end{aligned}$$

Generalized Bogoliubov transformation

We consider the 4×4 canonical transformation

$$\begin{pmatrix} \alpha_A \\ \alpha_B \\ \beta_A^\dagger \\ \beta_B^\dagger \end{pmatrix} = \begin{pmatrix} c_\theta\, \rho_{AI} & s_\theta\, \rho_{AII} & c_\theta\, \lambda_{AI} & s_\theta\, \lambda_{AII} \\ -s_\theta\, \rho_{BI} & c_\theta\, \rho_{BII} & -s_\theta\, \lambda_{BI} & c_\theta\, \lambda_{BII} \\ c_\theta\, \lambda_{AI} & s_\theta\, \lambda_{AII} & c_\theta\, \rho_{AI} & s_\theta\, \rho_{AII} \\ -s_\theta\, \lambda_{BI} & c_\theta\, \lambda_{BII} & -s_\theta\, \rho_{BI} & c_\theta\, \rho_{BI} \end{pmatrix} \begin{pmatrix} a_I \\ a_{II} \\ b_I^\dagger \\ b_{II}^\dagger \end{pmatrix}$$

where $c_\theta \equiv \cos\theta$, $s_\theta \equiv \sin\theta$ and

$$\rho_{ab} \equiv \cos\frac{\chi_a - \chi_b}{2}, \quad \lambda^{\mathbf{k}}_{ab} \equiv \sin\frac{\chi_a - \chi_b}{2}, \quad \chi_a \equiv \cot^{-1}\left[\frac{k}{m_a}\right], \qquad a, b = I, II, A, B.$$

Thus we have three parameters (θ, m_A, m_B) to fix in terms of (f_I, f_{II}, h) in order to diagonalize the Hamiltonian.

Partial diagonalization

A possible representation is obtained by a partial diagonalization of $\bar{H}_0$, leaving untouched $\delta\mathcal{H}_{\text{mix}}$:

$$\bar{\mathcal{H}}_0 \;=\; \sum_{\sigma=e,\mu} \bar{\psi}_\sigma \left(-i\gamma\cdot\boldsymbol{\nabla} \;+\; m_\sigma\right)\psi_\sigma + h\left(\bar{\psi}_e\psi_\mu \;+\; \bar{\psi}_\mu\psi_e\right).$$

Such a representation is obtained by setting

$$\begin{aligned} \theta &\;\to\; 0\,, \\ m_A &\;\to\; m_e \;\equiv\; m_I + f_I\,, \\ m_B &\;\to\; m_\mu \;\equiv\; m_{II} + f_{II}\,. \end{aligned}$$

The vacuum in this representation is denoted as

$$|0(\theta=0, m_e, m_\mu)\rangle \;\equiv\; |0\rangle_{e\mu}\,,$$

In this representation we have

$${}_{e,\mu}\langle 0|\bar{H}_0|0\rangle_{e,\mu} \;=\; -2\int d^3\mathbf{k}\,\left(\sqrt{k^2 \;+\; m_e^2} \;+\; \sqrt{k^2 \;+\; m_\mu^2}\right),$$

since ${}_{e,\mu}\langle 0|\delta\mathcal{H}_{\text{mix}}|0\rangle_{e,\mu} \;=\; 0$.

Complete diagonalization

Another possibility is to require that the Hamiltonian $\bar{\mathcal{H}}_0$ becomes fully diagonal in two fermion fields, ψ_1 and ψ_2, with masses m_1 and m_2:

$$\bar{\mathcal{H}}_0 \;=\; \sum_{j=1,2} \bar{\psi}_j \left(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} \;+\; m_j\right)\psi_j\,.$$

The condition for the complete diagonalization is found to be:

$$\begin{aligned}
\theta \;&\to\; \bar{\theta} \;\equiv\; \frac{1}{2}\tan^{-1}\left[\frac{2h}{m_\mu - m_e}\right],\\
m_{_A} \;&\to\; m_1 \;\equiv\; \frac{1}{2}\left(m_e + m_\mu - \sqrt{(m_\mu - m_e)^2 + 4h^2}\right),\\
m_{_B} \;&\to\; m_2 \;\equiv\; \frac{1}{2}\left(m_e + m_\mu + \sqrt{(m_\mu - m_e)^2 + 4h^2}\right).
\end{aligned}$$

where we introduced the notation $m_e = m_{_I} + f_{_I}$, $m_\mu = m_{_{II}} + f_{_{II}}$.

We set

$$|0(\bar{\theta}, m_1, m_2)\rangle \;\equiv\; |0\rangle_{1,2}\,,$$

The vev of the Hamiltonian in this representation has the form:

$${}_{1,2}\langle 0|\bar{H}_0|0\rangle_{1,2} \;=\; -2\int d^3\mathbf{k}\;\left(\sqrt{k^2 \;+\; m_1^2} \;+\; \sqrt{k^2 \;+\; m_2^2}\right).$$

Notice that the difference in energy among $|0\rangle_{12}$ and $|0\rangle_{e\mu}$ is given by

$$\begin{aligned}\Delta E &\equiv {}_{e\mu}\langle 0|\bar{H}_0|0\rangle_{e\mu} - {}_{12}\langle 0|\bar{H}_0|0\rangle_{12} \\ \Delta E &= -2\int d^3p \left[\sqrt{p^2+m_e^2}+\sqrt{p^2+m_\mu^2}-\sqrt{p^2+m_1^2}-\sqrt{p^2+m_2^2}\right].\end{aligned}$$

For large p, i.e. $p \gg m_{1,2}$, we have $\sqrt{p^2+m^2} \simeq p + \frac{m^2}{2p}$ and

$$\Delta E \simeq \int d^3p \, \frac{(m_2-m_1)^2}{2p} \sin^2 2\theta \,.$$

We now compare this result with the flavor vacuum condensation density, namely

$${}_{e\mu}\langle 0|H|0\rangle_{e\mu} = \int d^3p \, (\omega_2+\omega_1)|V_p|^2 \sin^2\theta \,.$$

For large p, this has the same form of ΔE since $|V_{\mathbf{p}}|^2 \simeq \frac{(m_2-m_1)^2}{4p^2}$ for $p \gg \sqrt{m_1 m_2}$

Three-flavor fermion mixing*

Mixing relations:

$$\Psi_f(x) \,=\, \mathbf{M}\,\Psi_m(x)$$

where $\Psi_f^T = (\nu_e, \nu_\mu, \nu_\tau)$, $\Psi_m^T = (\nu_1, \nu_2, \nu_3)$ and

$$\mathbf{M} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

with $c_{ij} = \cos\theta_{ij}, s_{ij} = \sin\theta_{ij}$

*M.B., A.Capolupo and G.Vitiello, Phys. Rev. **D** (2002)

We have:

$$\nu_\sigma^\alpha(x) = G_\theta^{-1}(t)\, \nu_i^\alpha(x)\, G_\theta(t),$$

where $(\sigma, i) = (e, 1), (\mu, 2), (\tau, 3)$, and

$$G_\theta(t) \;=\; G_{23}(t) G_{13}(t) G_{12}(t)$$

$$\begin{aligned}
G_{12}(t) &= exp\left[\theta_{12}\int d^3\mathbf{x}(\nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x))\right], \\
G_{13}(t) &= exp\left[\theta_{13}\int d^3\mathbf{x}(\nu_1^\dagger(x)\nu_3(x)e^{-i\delta} - \nu_3^\dagger(x)\nu_1(x)e^{i\delta})\right], \\
G_{23}(t) &= exp\left[\theta_{23}\int d^3\mathbf{x}(\nu_2^\dagger(x)\nu_3(x) - \nu_3^\dagger(x)\nu_2(x))\right],
\end{aligned}$$

Flavor vacuum:

$$|0\rangle_f \;=\; G_\theta^{-1}(t)\; |0\rangle_m$$

Flavor annihilation operators:

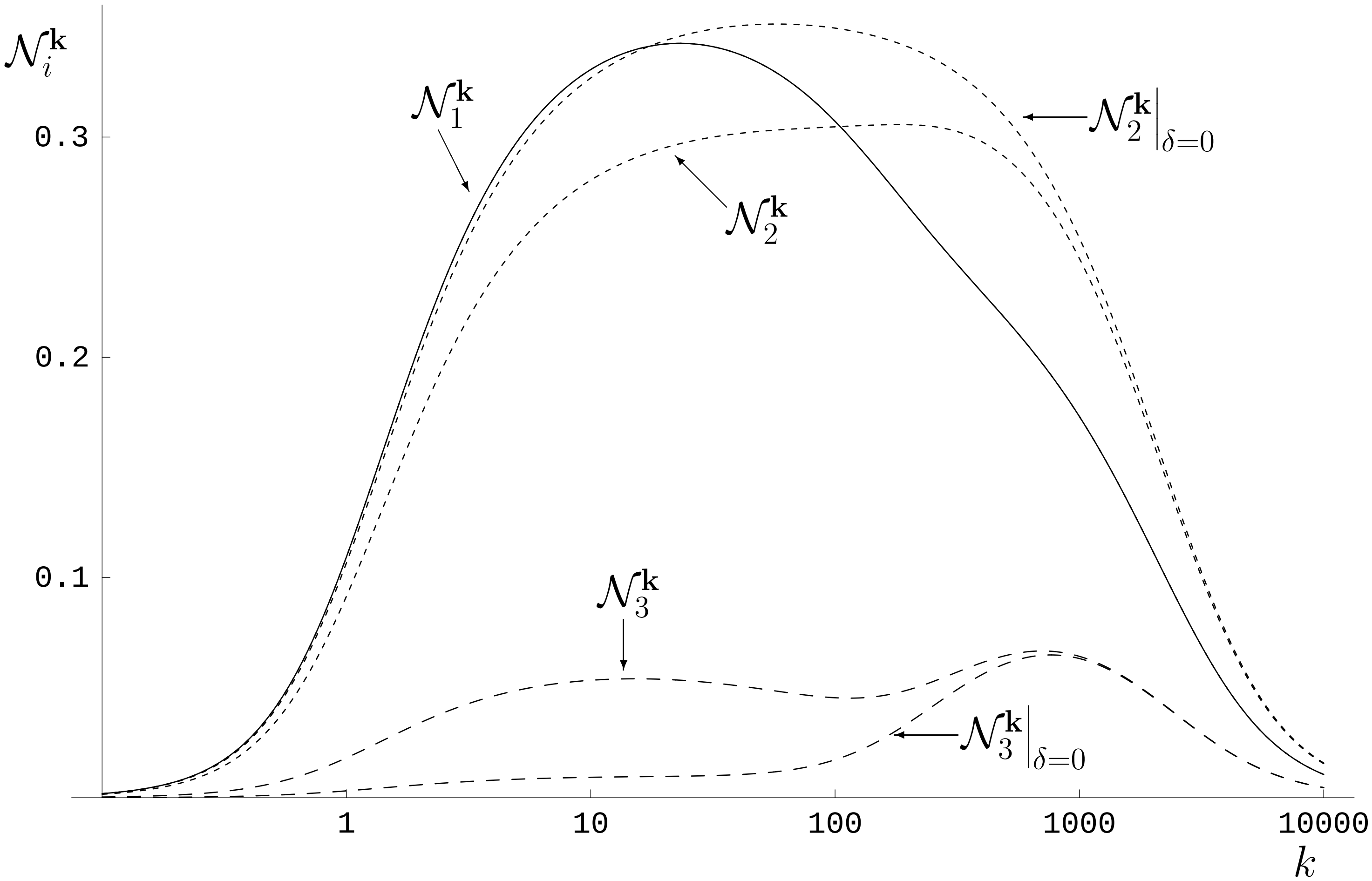
$$\alpha^r_{\mathbf{k},e} \;=\; c_{12}c_{13}\,\alpha^r_{\mathbf{k},1} + s_{12}c_{13}\left(U^{\mathbf{k}*}_{12}\,\alpha^r_{\mathbf{k},2} + \epsilon^r V^{\mathbf{k}}_{12}\,\beta^{r\dagger}_{-\mathbf{k},2}\right) + e^{-i\delta}\,s_{13}\left(U^{\mathbf{k}*}_{13}\,\alpha^r_{\mathbf{k},3} + \epsilon^r V^{\mathbf{k}}_{13}\,\beta^{r\dagger}_{-\mathbf{k},3}\right)\,,$$

$$\begin{aligned}\alpha^r_{\mathbf{k},\mu} \;=\;& \left(c_{12}c_{23} - e^{i\delta}\,s_{12}s_{23}s_{13}\right)\,\alpha^r_{\mathbf{k},2} - \left(s_{12}c_{23} + e^{i\delta}\,c_{12}s_{23}s_{13}\right)\left(U^{\mathbf{k}}_{12}\,\alpha^r_{\mathbf{k},1} - \epsilon^r V^{\mathbf{k}}_{12}\,\beta^{r\dagger}_{-\mathbf{k},1}\right)\\ &+ s_{23}c_{13}\left(U^{\mathbf{k}*}_{23}\,\alpha^r_{\mathbf{k},3} + \epsilon^r V^{\mathbf{k}}_{23}\,\beta^{r\dagger}_{-\mathbf{k},3}\right)\,,\end{aligned}$$

$$\begin{aligned}\alpha^r_{\mathbf{k},\tau} \;=\;& c_{23}c_{13}\,\alpha^r_{\mathbf{k},3} - \left(c_{12}s_{23} + e^{i\delta}\,s_{12}c_{23}s_{13}\right)\left(U^{\mathbf{k}}_{23}\,\alpha^r_{\mathbf{k},2} - \epsilon^r V^{\mathbf{k}}_{23}\,\beta^{r\dagger}_{-\mathbf{k},2}\right)\\ &+ \left(s_{12}s_{23} - e^{i\delta}\,c_{12}c_{23}s_{13}\right)\left(U^{\mathbf{k}}_{13}\,\alpha^r_{\mathbf{k},1} - \epsilon^r V^{\mathbf{k}}_{13}\,\beta^{r\dagger}_{-\mathbf{k},1}\right)\end{aligned}$$

and similar ones for antiparticles ($\delta \to -\delta$).

Condensation densities



Condensation densities $\mathcal{N}_i^{\mathbf{k}}$ for sample values of masses and mixings

Parameterizations of mixing matrix

$$\nu_\sigma^\alpha(x) = G_\theta^{-1}(t)\, \nu_i^\alpha(x)\, G_\theta(t),$$

Define the more general generators:

$$G_{12} \equiv \exp\left[\theta_{12} \int d^3x \left(\nu_1^\dagger \nu_2 e^{-i\delta_2} - \nu_2^\dagger \nu_1 e^{i\delta_2}\right)\right]$$

$$G_{13} \equiv \exp\left[\theta_{13} \int d^3x \left(\nu_1^\dagger \nu_3 e^{-i\delta_5} - \nu_3^\dagger \nu_1 e^{i\delta_5}\right)\right]$$

$$G_{23} \equiv \exp\left[\theta_{23} \int d^3x \left(\nu_2^\dagger \nu_3 e^{-i\delta_7} - \nu_3^\dagger \nu_2 e^{i\delta_7}\right)\right]$$

There are six different matrices obtained by permutations of the above generators.

We can obtain all possible parameterizations of the matrix by setting to zero two of the phases and permuting rows/columns.

Currents and charges for 3-flavor fermion mixing

Lagrangian for three free Dirac fields with different masses

$$\mathcal{L}(x) \,=\, \bar{\Psi}_m(x)\,(i\,\not\partial - M_d)\,\Psi_m(x)$$

where $\Psi_m^T = (\nu_1, \nu_2, \nu_3)$ and $M_d = diag(m_1, m_2, m_3)$.

The $SU(3)$ transformations:

$$\Psi'_m(x) \,=\, e^{i\alpha_j \lambda_j/2}\,\Psi_m(x) \quad ; \qquad j = 1,..,8$$

with α_j real constants, and λ_j the Gell-Mann matrices, give the currents:

$$J^\mu_{m,j}(x) \,=\, \frac{1}{2}\bar{\Psi}_m(x)\,\gamma^\mu\,\lambda_j\,\Psi_m(x)$$

The combinations:

$$Q_1 \equiv \frac{1}{3}Q + Q_{m,3} + \frac{1}{\sqrt{3}}Q_{m,8},$$

$$Q_2 \equiv \frac{1}{3}Q - Q_{m,3} + \frac{1}{\sqrt{3}}Q_{m,8}$$

$$Q_3 \equiv \frac{1}{3}Q - \frac{2}{\sqrt{3}}Q_{m,8}$$

$$Q_i = \sum_r \int d^3\mathbf{k} \left(\alpha^{r\dagger}_{\mathbf{k},i}\alpha^{r}_{\mathbf{k},i} - \beta^{r\dagger}_{-\mathbf{k},i}\beta^{r}_{-\mathbf{k},i}\right), \quad i = 1, 2, 3.$$

are the Noether charges for the fields ν_i with $\sum_i Q_i = Q$.

Flavor charges:

$$:: Q_\sigma(t) ::= G_\theta^{-1}(t) \;:Q_i:\; G_\theta(t) = \sum_r \int d^3\mathbf{k} \left(\alpha^{r\dagger}_{\mathbf{k},\sigma}(t)\alpha^{r}_{\mathbf{k},\sigma}(t) - \beta^{r\dagger}_{-\mathbf{k},\sigma}(t)\beta^{r}_{-\mathbf{k},\sigma}(t)\right)$$

CP violation and $SU(3)$

Modified Gell-Mann matrices:

$$\tilde{\lambda}_1 = \begin{pmatrix} 0 & e^{i\delta_2} & 0 \\ e^{-i\delta_2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad \tilde{\lambda}_2 = \begin{pmatrix} 0 & -ie^{i\delta_2} & 0 \\ ie^{-i\delta_2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad \tilde{\lambda}_4 = \begin{pmatrix} 0 & 0 & e^{-i\delta_5} \\ 0 & 0 & 0 \\ e^{i\delta_5} & 0 & 0 \end{pmatrix}$$

$$\tilde{\lambda}_5 = \begin{pmatrix} 0 & 0 & -ie^{-i\delta_5} \\ 0 & 0 & 0 \\ ie^{i\delta_5} & 0 & 0 \end{pmatrix} \quad , \quad \tilde{\lambda}_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e^{i\delta_7} \\ 0 & e^{-i\delta_7} & 0 \end{pmatrix} \quad , \quad \tilde{\lambda}_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -ie^{i\delta_7} \\ 0 & ie^{-i\delta_7} & 0 \end{pmatrix} ,$$

$$\tilde{\lambda}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad \tilde{\lambda}_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} .$$

Neutrino oscillations (wave packets)*

– Consider, in one dimension, a neutrino with definite flavor, propagating along the x direction:

$$|\nu_\alpha(x,t)\rangle = \sum_j U_{\alpha,j}\, \psi_j(x,t)\, |\nu_j\rangle\,,$$

where $U_{\alpha,j}$ is an element of the mixing matrix, $|\nu_j\rangle$ the mass eigenstate with mass m_j, and $\psi_j(x,t)$ its wave function.

– Assume Gaussian distribution $\psi_j(p)$ for the momentum of the massive neutrino $|\nu_j\rangle$:

$$\psi_j(x,t) = \frac{1}{\sqrt{2\pi}} \int dp\, \psi_j(p)\, e^{ipx - iE_j(p)t}\,, \qquad \psi_j(p) = \frac{1}{(2\pi\sigma_p^2)^{1/4}}\, e^{-\frac{1}{4\sigma_p^2}(p-p_j)^2}\,,$$

where $E_j(p) = \sqrt{p^2 + m_j^2}$.

– The associated density matrix writes:

$$\rho_\alpha(x,t) = |\nu_\alpha(x,t)\rangle\langle\nu_\alpha(x,t)|\,.$$

If $\sigma_p \ll E_j^2(p_j)/m_j$, one can write $E_j(p) \simeq E_j + v_j(p - p_j)$, with $E_j \equiv \sqrt{p_j^2 + m_j^2}$, and $v_j \equiv \frac{\partial E_j(p)}{\partial p}\big|_{p=p_j} = \frac{p_j}{E_j}$ is the group velocity of the wave packet for ν_j.

*C. Giunti, C. W. Kim, Phys. Rev. **D** (1998); *Fundamentals of Neutrino Physics and Astrophysics*, Oxford Univ. Pr. (2007)

– In this case, a Gaussian integration yields:

$$\rho_\alpha(x,t) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \sum_{j,k} U_{\alpha j} U_{\alpha k}^* e^{-i(E_j-E_k)t + i(p_j-p_k)x - \frac{1}{4\sigma_x^2}[(x-v_j t)^2 + (x-v_k t)^2]} |\nu_j\rangle\langle\nu_k| \,,$$

where $\sigma_x = (2\sigma_p)^{-1}$. For extremely relativistic neutrinos, one has

$$E_j \simeq E + \xi \frac{m_j^2}{2E} \,, \qquad p_j \simeq E - (1-\xi)\frac{m_j^2}{2E} \,, \qquad v_j \simeq 1 - \frac{m_j^2}{2E_j^2}$$

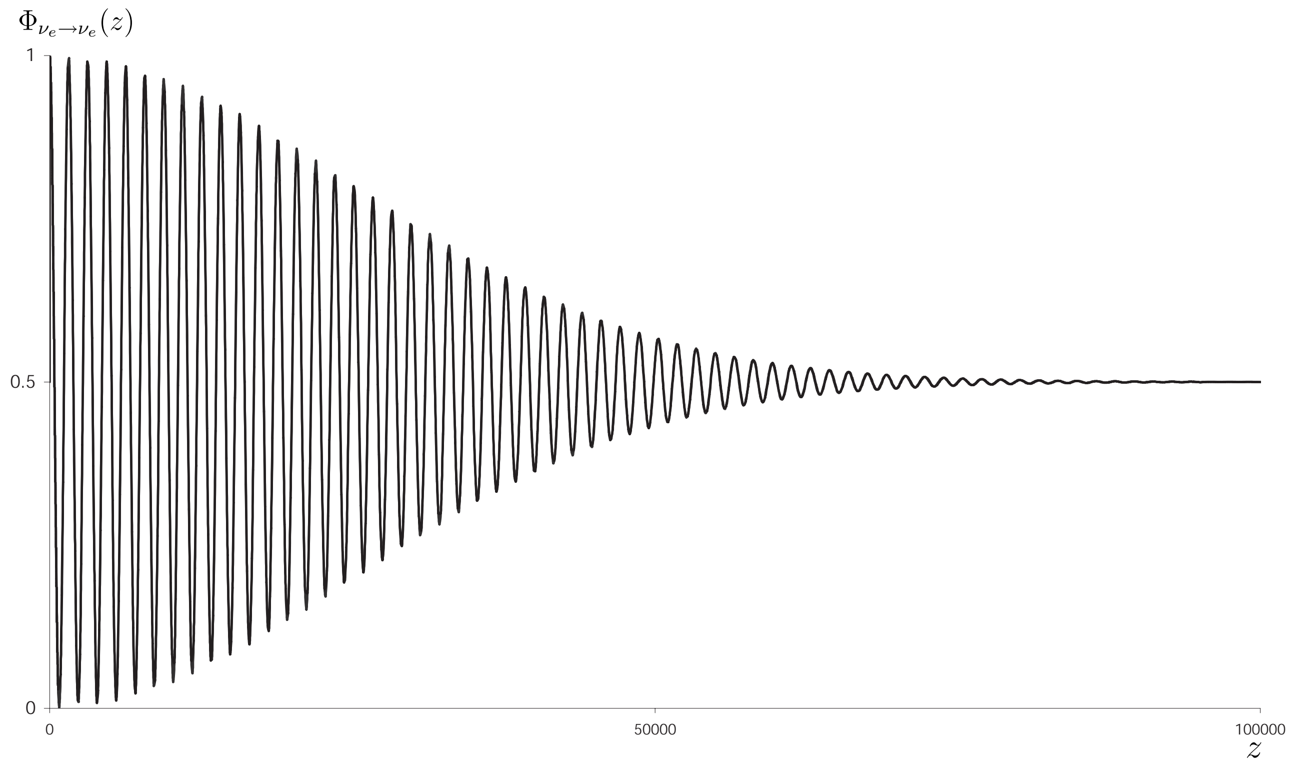
where E is the neutrino energy in the limit of zero mass, and ξ a dimensionless constant depending on the characteristic of the production process.

– The density matrix $\rho_\alpha(x,t)$ provides a space-time description of neutrino dynamics.

– In realistic situations, it is convenient to consider the time-independent density matrix $\rho_\alpha(x)$ obtained by the time average of $\rho_\alpha(x,t)$:

$$\rho_\alpha(x) = \sum_{j,k} U_{\alpha j} U_{\alpha k}^* \exp\left[-i \frac{\Delta m_{jk}^2 x}{2E} - \left(\frac{\Delta m_{jk}^2 x}{4\sqrt{2} E^2 \sigma_x} \right)^2 - \left(\xi \frac{\Delta m_{jk}^2}{4\sqrt{2} E \sigma_p} \right)^2 \right] |\nu_j\rangle\langle\nu_k| \,,$$

with $\Delta m_{jk}^2 = m_j^2 - m_k^2$.



Flavor oscillations in space.

$$P_{\nu_e\to\nu_e}(x) \;\simeq\; 1-\frac{1}{2}\sin^2(2\theta)\left\{1-\cos\left(2\pi\frac{x}{L^{osc}}\right)\exp\left[-\left(\frac{x}{L^{coh}}\right)^2-2\pi^2\left(\frac{\sigma_x}{L^{osc}}\right)^2\right]\right\}$$

- Oscillation length: $L^{osc}=4\pi p/\triangle m^2$

- Coherence length: $L^{coh}=(L^{osc}p)/(\sqrt{2}\pi\sigma_p)$.

Decoherence in neutrino oscillations

• We analyze the coherence of the quantum superposition of the neutrino mass eigenstates, by looking at the spatial behavior of the multipartite entanglement of the above state*.

By means of the identification $|\nu_i\rangle = |\delta_{i,1}\rangle_1 |\delta_{i,2}\rangle_2 |\delta_{i,3}\rangle_3 \equiv |\delta_{i,1}\delta_{i,2}\delta_{i,3}\rangle$, with $i = 1, 2, 3$, we construct the matrix with elements

$$\langle lmn|\rho_\alpha(x)|ijk\rangle, \qquad \textit{where} \quad i, j, k, l, m, n = 0, 1$$

– We analytically compute logarithmic negativities $E_{\mathcal{N}\,\alpha}^{(i,j;k)}$,for $i, j, k = 1, 2, 3$ and $i \neq j \neq k$, and average logarithmic negativity $\langle E_{\mathcal{N}\,\alpha}^{(2:1)}\rangle$, for the neutrino states with flavor $\alpha = e, \mu, \tau$.

*M.B., F.Dell'Anno, S.De Siena, M.Di Mauro and F.Illuminati, Phys. Rev. D (2008).

We assume for the mixing angles the experimental values

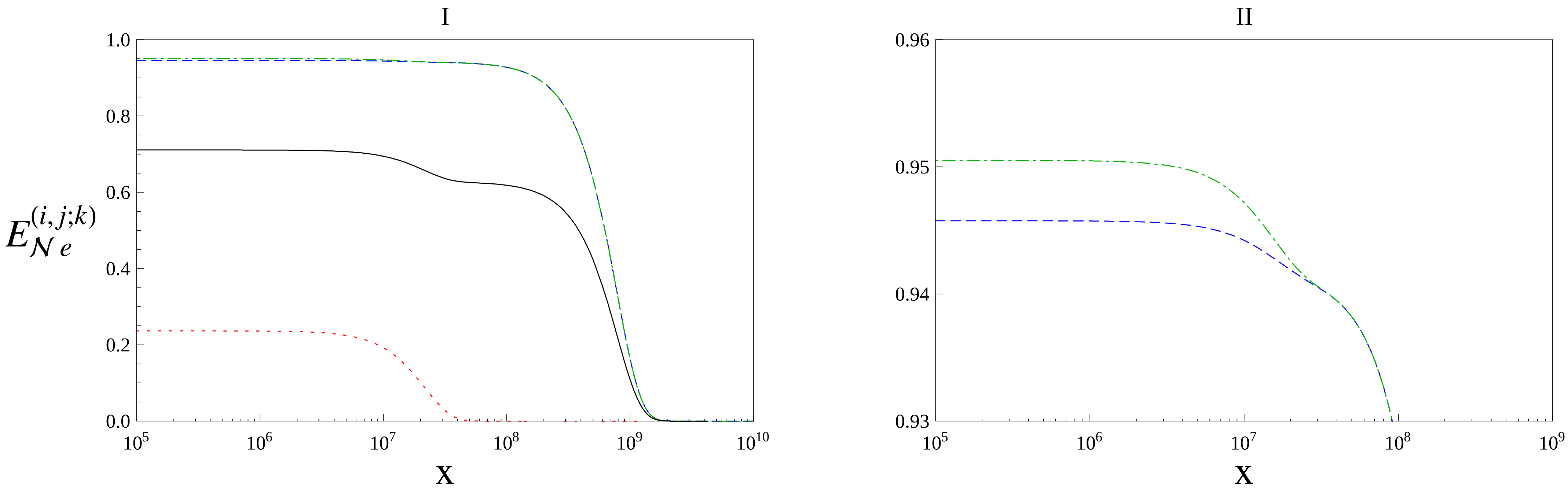
$$\sin^2\theta_{12}^{MNSP} = 0.314(1\ {}^{+0.18}_{-0.15}\ ), \quad \sin^2\theta_{13}^{MNSP} = (0.8\ {}^{+2.3}_{-0.8}\ )\times 10^{-2}, \quad \sin^2\theta_{23}^{MNSP} = 0.45(1\ {}^{+0.35}_{-0.20}\ )$$

The squared mass differences are fixed at the experimental values*

$$\Delta m_{21}^2 = \delta m^2, \qquad \Delta m_{31}^2 = \Delta m^2 + \frac{\delta m^2}{2}, \qquad \Delta m_{32}^2 = \Delta m^2 - \frac{\delta m^2}{2},$$
$$\delta m^2 = 7.92\times 10^{-5}\, eV^2, \qquad \delta m^2 = 2.6\times 10^{-3}\, eV^2.$$

We take $E = 10\, GeV$ and $\sigma_p = 1\, GeV$. The parameter ξ is put to zero for simplicity.

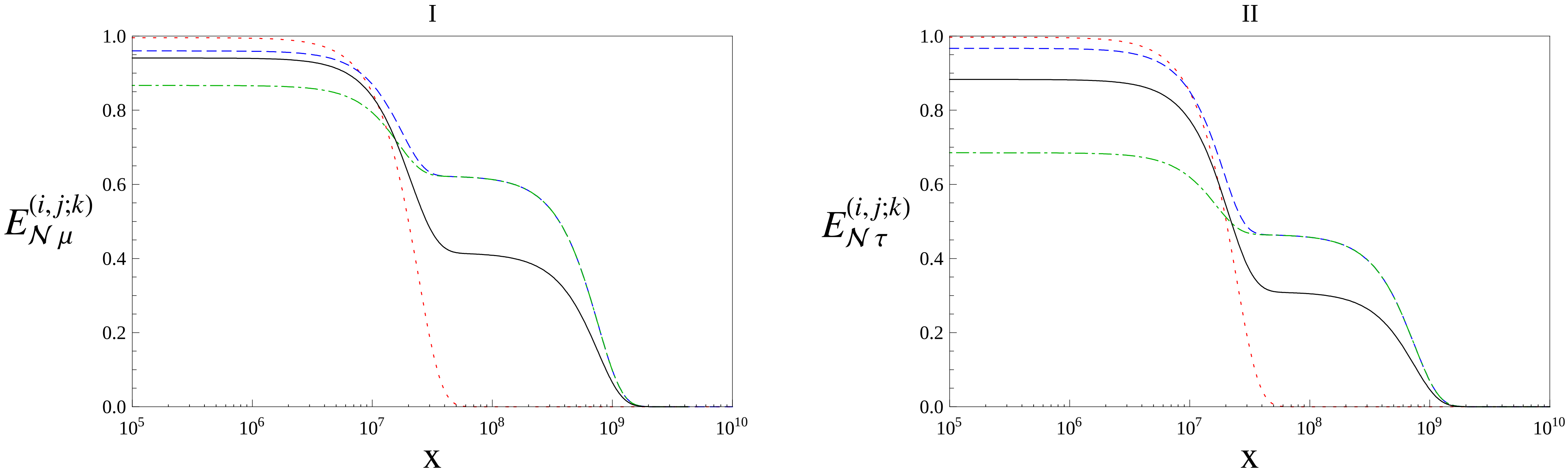
*G. L. Fogli, E. Lisi, A. Marrone, A. Melchiorri, A. Palazzo, P. Serra, J. Silk, and A. Slosar, Phys. Rev. D (2007).



Logarithmic negativities $E_{\mathcal{N}\,e}^{(i,j;k)}$ for all possible bipartitions and average logarithmic negativity $\langle E_{\mathcal{N}\,e}^{(2:1)} \rangle$ (solid line) as functions of the distance x (meters).

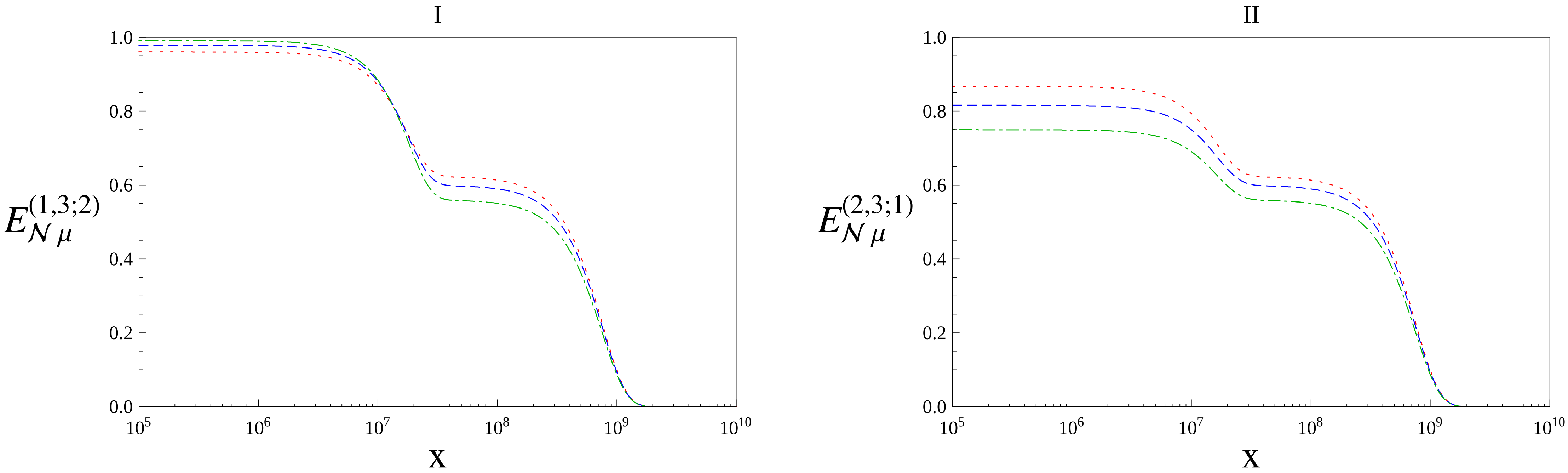
In panel II we plot a zoom of $E_{\mathcal{N}\,e}^{(1,3;2)}$ and $E_{\mathcal{N}\,e}^{(2,3;1)}$

All plotted quantities are independent of the CP-violating phase δ.



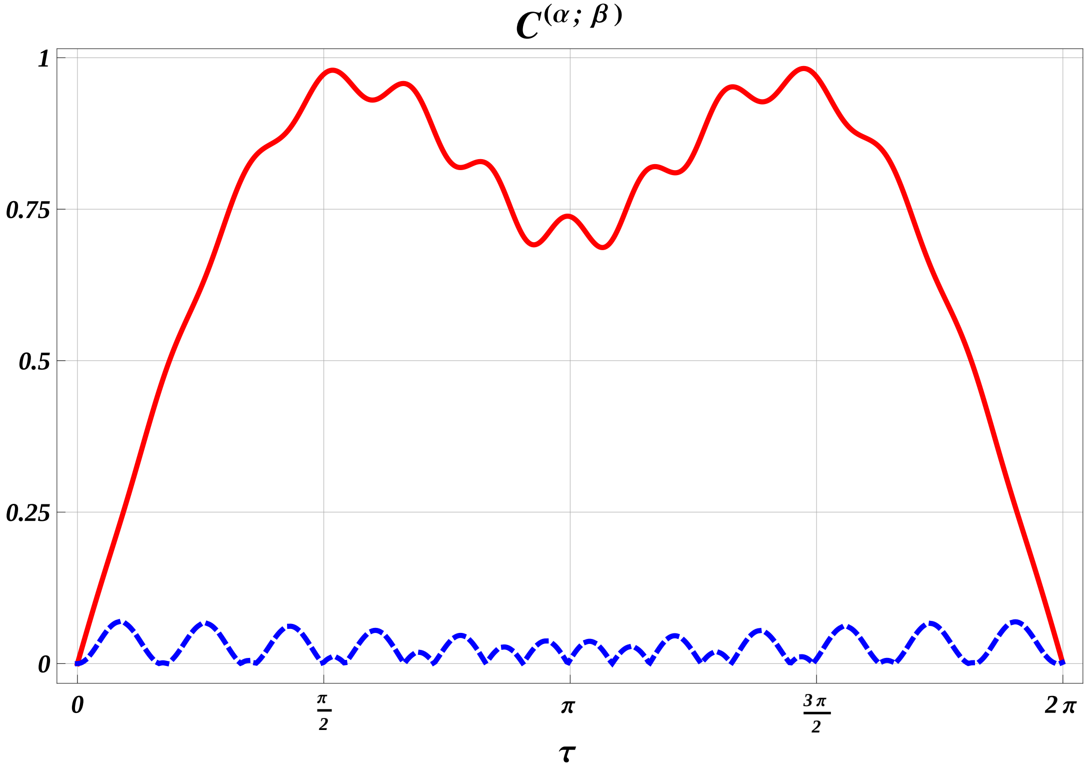
Logarithmic negativities $E_{\mathcal{N}\,\alpha}^{(i,j;k)}$ for all possible bipartitions and average logarithmic negativity $\langle E_{\mathcal{N}\,\alpha}^{(2:1)} \rangle$ (solid line), with $\alpha = \mu, \tau$, as functions of the distance x (meters).

The CP-violating phase δ is put to zero. The x axis is in logarithmic scale, and the dimensions are meters.



Logarithmic negativities $E_{\mathcal{N}\,\mu}^{(1,3;2)}$ (panel I) and $E_{\mathcal{N}\,\mu}^{(2,3;1)}$ (panel II) as functions of the distance x (meters) for different choices of the CP-violating phase δ: (a) $\delta = 0$ (dotted line); (b) $\delta = \frac{\pi}{2}$ (dashed line); (b) $\delta = \pi$ (dot-dashed line). $E_{\mathcal{N}\,\mu}^{(1,2;3)}$ is independent of δ.

• Alternatively, we can quantify entanglement between two single parties, by tracing over other degrees of freedom. The resulting state is a mixed state for which we calculate the concurrence.



The concurrences $C^{(\alpha;\beta)}$ associated with the reduced mixed states ρ_{ν_e,ν_μ}, ($C^{(\nu_e;\nu_\mu)}$, full line), and $\rho_{\bar{\nu}_e,\bar{\nu}_\mu}$ ($C^{(\bar{\nu}_e;\bar{\nu}_\mu)}$, dashed line), as functions of the scaled time $\tau = (\omega_2 - \omega_1)t$.
The concurrences $C^{(\nu_e;\bar{\nu}_e)}$, $C^{(\nu_e;\bar{\nu}_\mu)}$, $C^{(\nu_\mu;\bar{\nu}_e)}$, and $C^{(\nu_\mu;\bar{\nu}_\mu)}$ vanish for every τ.

– Connection among the above QFT results...