Uses and methodology of the STRINGVACUA Mathematica package.

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J.G., Y-H. He and A. Lukas: hep-th/0606122
J.G., Y-H. He, A. Ilderton and A. Lukas: hep-th/0703249
J.G., Y-H. He, A. Ilderton and A. Lukas: arXiv:0801.1508 [hep-th]

For download of the STRINGVACUA Mathematica package please see:

http://www-thphys.physics.ox.ac.uk/user/Stringvacua/

This work makes extensive use of the computer algebra systems *Singular* and *Macaulay 2*:

http://www.singular.uni-kl.de/   http://www.math.uiuc.edu/Macaulay2/

Plan of talk:

- Flux vacua systems as a problem in algorithmic algebraic geometry
- Elimination and constraints on flux parameters
- Saturation decomposition and finding vacua

Vacua of flux compactifications as algebraic varieties.

- $K,\ W \to V$ via the usual expression:-

$$V = e^K \left[\mathcal{K}^{A\bar{B}} D_A W D_{\bar{B}} \bar{W} - 3|W|^2 \right]$$

- Let us take $K = -\log(P(M^A, \bar{M}^{\bar{B}}))$ as usual. For example...

$$K = -3\log(T+\bar{T}) - 3\log(Z+\bar{Z}) - \log(S+\bar{S})$$

- Take $W = Q(M^A)$ to be an arbitrary holomorphic polynomial of the fields (simplest case for now).

- $\partial_A V$ is then a rational function $\forall A$.
- So to obtain $\partial V = 0$ we can either set the denominator to infinity or the numerator to zero.
- Denominator $\rightarrow \infty$ corresponds to the runaway extremum at infinity in field space - not physically interesting.
- Therefore we wish to study the solutions given by vanishing numerator polynomials. We denote this by,

$$\langle \partial V \rangle$$

- In terms of complex fields M^A and $\bar{M}^{\bar{B}}$ the polynomials $\langle \partial V \rangle$ are not holomorphic (or purely anti-holomorphic).
- So write each field in terms of its real and imaginary parts...

$$M^A = m^A + i\mu^A$$

- and temporarily complexify these (i.e. pretend m^A and μ^A are complex fields - we will return to this later).
- The locus of extrema of the potential in (complexified) field space, given by $\langle \partial V \rangle = 0$, is now described by the vanishing of a set of holomorphic polynomials.

We have rewritten the extremal locus of the potential as an algebraic variety.

- We can rewrite the F-terms in exactly the same way - also turning those into polynomial expressions (important later).

Elimination orderings: Constraints on flux parameters.

- In what follows $\mathbb{C}[M^A, a_\alpha]$ is the set of all polynomials in the fields and parameters.

- We need an unambiguous *ordering* on the monomials:

$$(M^1)^2 > M^1 a_8 > a_1 a_{94} > \dots$$

- For our purposes we will require a monomial ordering with the *elimination property*:

$$P \in \mathbb{C}[M^A, a_\alpha], \mathrm{LM}(P) \in \mathbb{C}[a_\alpha] \Rightarrow P \in \mathbb{C}[a_\alpha]$$

- Now consider, for example, the equations for a SUSY Minkowski vacuum: $\langle \partial W, W, \mathrm{Constraints} \rangle$

The Buchberger Algorithm:

- Start with our set of polynomials: Call it $G = \{P_I\}$
  - For any pair $P_I, P_J \in G$ multiply by monomials and form difference so as to cancel leading monomials:

    $$S = p_1 P_I - p_2 P_J \text{ s.t. } \mathrm{LM}(P_I), \mathrm{LM}(P_J) \text{ cancel}$$

  - Reduce as much as possible w.r.t. G.

    $$S \xrightarrow{G} h$$

    - If $h = 0$ consider next pair
    - If $h \neq 0$ add h to G and return to beginning
- Algorithm terminates when all pairs reduce to 0. Final set of polynomials is called G_F.

- G_F is a Grobner basis - a form for our equations with lots of nice properties.
- The important property for us today is that $G_F \cap \mathbb{C}\left[a_\alpha\right]$ is a complete set of constraints necessary and sufficient for a solution to exist to our original equations.
- This elimination process has a nice geometrical interpretation - projection onto the space of parameters.
- We can now apply this technology to find the constraints on flux parameters which are necessary and sufficient for the existence of certain types of vacua.

An example:

Shelton et. al. hep-th/0508133

$$
\begin{aligned}
W &= a_0 - 3a_1\tau + 3a_2\tau^2 - a_3\tau^3 \\
&\quad + S(-b_0 + 3b_1\tau - 3b_2\tau^2 + b_3\tau^3) \\
&\quad + 3U(c_0 + (\hat{c}_1 + \check{c}_1 + \tilde{c}_1)\tau - (\hat{c}_2 + \check{c}_2 + \tilde{c}_2)\tau^2 - c_3\tau^3),
\end{aligned}
$$

where

$$a_0 b_3 - 3a_1 b_2 + 3a_2 b_1 - a_3 b_0 = 16$$

$$a_0 c_3 + a_1(\check{c}_2 + \hat{c}_2 - \tilde{c}_2) - a_2(\check{c}_1 + \hat{c}_1 - \tilde{c}_1) - a_3 c_0 = 0$$

$$
\begin{aligned}
c_0 b_2 - \tilde{c}_1 b_1 + \hat{c}_1 b_1 - \check{c}_2 b_0 &= 0 & c_0\tilde{c}_2 - \check{c}_1^2 + \tilde{c}_1\hat{c}_1 - \hat{c}_2 c_0 &= 0 \\
\check{c}_1 b_3 - \hat{c}_2 b_2 + \tilde{c}_2 b_2 - c_3 b_1 &= 0 & c_3\tilde{c}_1 - \check{c}_2^2 + \tilde{c}_2\hat{c}_2 - \hat{c}_1 c_3 &= 0 \\
c_0 b_3 - \tilde{c}_1 b_2 + \hat{c}_1 b_2 - \check{c}_2 b_1 &= 0 & c_3 c_0 - \check{c}_2\hat{c}_1 + \tilde{c}_2\check{c}_1 - \hat{c}_1\tilde{c}_2 &= 0 \\
\check{c}_1 b_2 - \hat{c}_2 b_1 + \tilde{c}_2 b_1 - c_3 b_0 &= 0 & \hat{c}_2\tilde{c}_1 - \tilde{c}_1\check{c}_2 + \check{c}_1\hat{c}_2 - c_0 c_3 &= 0\,.
\end{aligned}
$$

+ additional constraints of same form but with hats and checks switched.

Simplifying the equations for vacua: Saturation Decomposition.

- The equations $\partial V = 0$ are complicated as they contain a lot of information.
- It would be useful to split the equations up into a series of smaller equations - one for each locus of turning points in field space.
- The mathematicians have algorithms (for algebraic varieties) which do precisely this - primary decomposition.
- These algorithms on their own are too slow for our applications. We have to split up the equations a bit first ourselves.

- The main splitting tool used in this work states:

$$L(\langle I\rangle) = L(\langle I, F\rangle) \cup L((I, F^\infty))$$

- Here (I, F^∞) is the set of equations whose roots give the points in $L(\langle I\rangle)$ where $F \neq 0$.

- $\langle I, F\rangle$ is easy to obtain, just add $F = 0$ to eqns!

- How about (I, F^∞)?

  - Consider: $\langle I, tF - 1\rangle \in \mathbb{C}[M^A, a_\alpha, t]$
    These equations have a solution iff $\langle I\rangle$ do and $F \neq 0$.

  - Now eliminate t using the technique we just learned. The result is (I, F^∞) .

$$(I, F^\infty) = \langle I, tf - 1\rangle \cap \mathbb{C}[\phi^i, a_\alpha]$$

- We need some suitable F's : The F-terms! (SUSY theories are the perfect application of these methods).

$$
\begin{aligned}
L(\partial V) \quad = \quad & L(\langle \partial V, f_1, f_2, ..., f_n \rangle) \cup \\
& \bigcup_i L((\langle \partial V, f_1, f_2, \ldots, f_{i-1}, f_{i+1}, \ldots, f_n \rangle : f_i^\infty)) \cup \\
& \bigcup_{i,j} L\big(((\langle \partial V, f_1, f_2, \ldots, f_{i-1}, f_{i+1}, \ldots, f_{j-1}, f_{j+1}, \ldots, f_n \rangle : f_i^\infty) : f_j^\infty\big)\big) \cup \\
& \vdots \\
& L\big(\big(\big(\ldots(\partial V : f_1^\infty) \ldots : f_{n-1}^\infty\big) : f_n^\infty\big)\big) \ .
\end{aligned}
$$

- We can now primary decompose - eqns have become sufficiently simple that this is now fast enough. In addition the vacuum equations are now classified by their supersymmetry breaking.
- The GTZ primary decomposition algorithm works along similar lines to what we have already seen.

Final comments

- All based on polynomials - how do we deal with transcendental functions from non-perturbative effects etc? : Dummy variables.
- **Mathematica package now available**
  - Best on unix based systems (including mac), although there is a windows version.
  - At highest level you don't need to know anything I've been describing - just tell it to look for a given type of vacuum, constraint etc...
  - At lowest level it allows much more freedom. Essentially a Mathematica front end for Singular with nice properties.