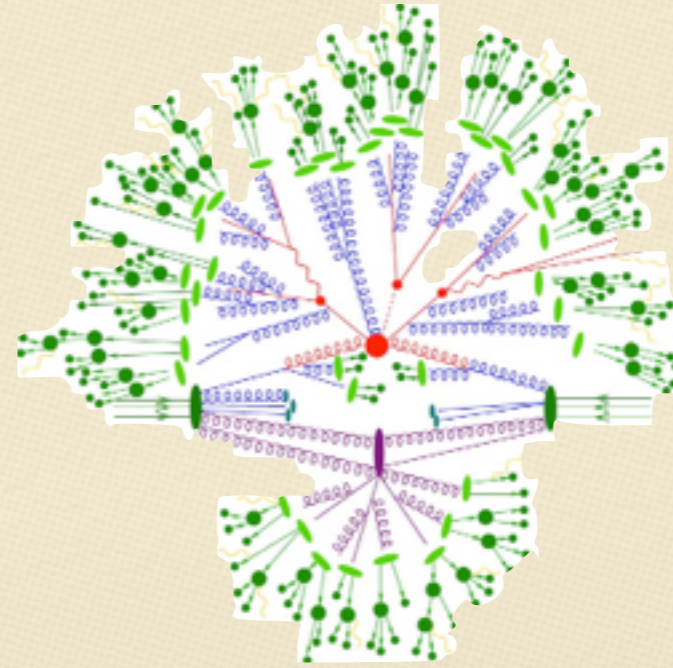


$$\frac{1}{4} \text{Tr}[G^2] - \bar{\psi}(\not{D} - m)\psi$$



QCD

Eric Laenen  
CERN school 2014

# Lecture 1: Partons and Hadrons



---

# Why these lectures?

---

- ◆ In particle physics, QCD is everywhere. The LHC collides protons, which are made up of quarks and gluons (partons). So every collision there involves partons in the initial state.
  - ▶ To see anything new, we must be able to “remove the foreground” (in cosmology-speak)
- ◆ But QCD is not only foreground, it is very interesting how such complicated final states can arise out of beautifully simple Lagrangian, and just two, or even one initial quark or gluon in a collider
  - ▶ In fact, technicalities aside, it is really a beautiful theory. It is the only unbroken, non-abelian gauge theory we have, and we better study it as best as we can.



---

# Real overview of contents

---

- ✦ Lecture 1. QCD, its partons and hadrons
  - ▶ Spectroscopy, colour, flavour, symmetries of QCD. The parton model.
  - ▶ Renormalization, asymptotic freedom, the structure of cross sections.
  - ▶ Parton distribution functions
  - ▶ Event shapes
- ✦ Lecture 2. Higher order aspects
  - ▶ More on  $e^+e^-$  cross sections, IR and collinear divergences, KLN theorem, Serman-Weinberg jets.
  - ▶ Drell-Yan: history, NLO calculation, factorization.
  - ▶ Monte Carlo basics
- ✦ Lecture 3. Modern methods
  - ▶ Spinor helicity methods, recursion relations
  - ▶ Aspects of NNLO methods, IBP's, Mellin-Barnes.
- ✦ Lecture 4. All orders
  - ▶ Basics of resummation. Eikonal approximation, webs.
  - ▶ Heavy quark production, Higgs production, at finite order and resummation

---

# Genesis

---

- ◆ The QCD Lagrangian looks rather beautiful and simple..

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4}\text{Tr}(G_{\mu\nu}G^{\mu\nu}) - \sum_{f=1}^{n_f} \bar{\psi}_f(\not{D} + m_f)\psi_f$$

- ◆ ..but its consequences can be, as we shall see, messy.
- ◆ The strong force at the beginning of the 1960's was not well understood.
- ◆ Lots of mesons found and baryons as well
  - ▶  $\pi, \rho, K, \eta, K^*, \omega, \phi, \dots, p, n, \Lambda, \Sigma, \Xi$  (“cascade”),  $\Omega, \Delta, \dots$ 
    - ✓ and various forms, e.g.  $\Lambda$ , and  $\Delta^-, \Delta^0, \Delta^+, \Delta^{++}$
- ◆ Organized by Gell-Mann and Zweig with SU(3) of flavour (“eightfold way”)
  - ✓  $\Lambda$  (uds),  $\Delta^{++}$  (uuu),  $\Delta^+$ (uud),  $\Delta^0$ (udd),  $\Delta^-$ (ddd), etc
  - ✓ but clearly: the force is strong with these ones..
- ◆ E.g. pion-nucleon coupling is about 13 (from nucleon-nucleon low energy scattering)

# Pauli principle

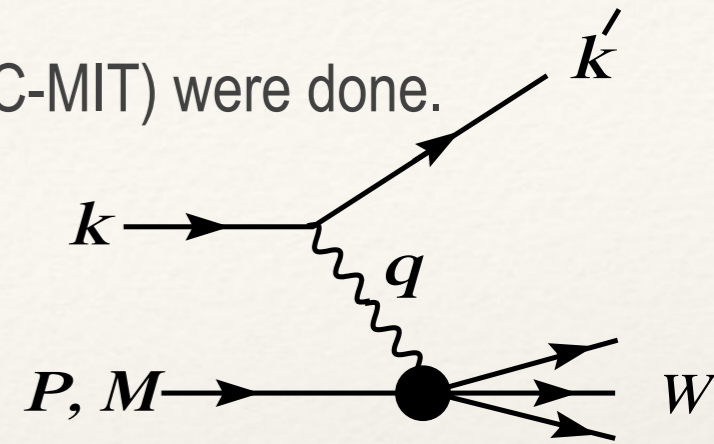
- ◆ So the quarks, if they exist, are strongly bound (no free ones seen). But then the  $\Delta^{++}$  (uuu) and  $\Delta^-$  (ddd), by their existence, pose a quantum puzzle
  - ◆ 3 identical fermions.
  - ◆ In ground state:  $l=0$ .
  - ◆ Spin state:  $3/2$  (inferred via decay patterns)
  - ◆ So *symmetric* total ground state  $|uuu\rangle|\uparrow\uparrow\uparrow\rangle$ . Should we give up Pauli principle?
- ◆ Idea: suppose that quarks come in 3 “colours”. Our theory must be invariant under rotations in colour space.
- ◆ Then we can anti-symmetrize ( $\epsilon_{ijk}$ ) in this variable and therefore maintain the Pauli principle. What does that imply?
  - ◆ **Baryons:**  $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8}_A \oplus \mathbf{8}_S \oplus \mathbf{10}$
  - ◆ **Mesons:**  $\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}_A$
- ◆ The **1** means a “singlet”: there is only one such colour-configuration. Based on the fact that we see only 1 proton (not 27 equal-mass), we can suggest the following statements.

**I. Quarks have 3 possible colours. II Bound states are singlets of this symmetry**

# Probing the proton

- ◆ In the late sixties, early seventies, deep-inelastic scattering experiments (SLAC-MIT) were done.
- ◆ Relation of cross section to “inelastic form factors” of proton  $F_1, F_2, F_3$ :

$$\left(\frac{d^2\sigma}{dx dy}\right)^\gamma = \frac{8\pi\alpha^2 ME}{(Q^2)^2} \left\{ \frac{1 + (1-y)^2}{2} 2xF_1^\gamma(x, Q^2) + (1-y)[F_2^\gamma(x, Q^2) - 2xF_1^\gamma(x, Q^2)] - \frac{M}{2E} xyF_2^\gamma(x, Q^2) \right\}$$

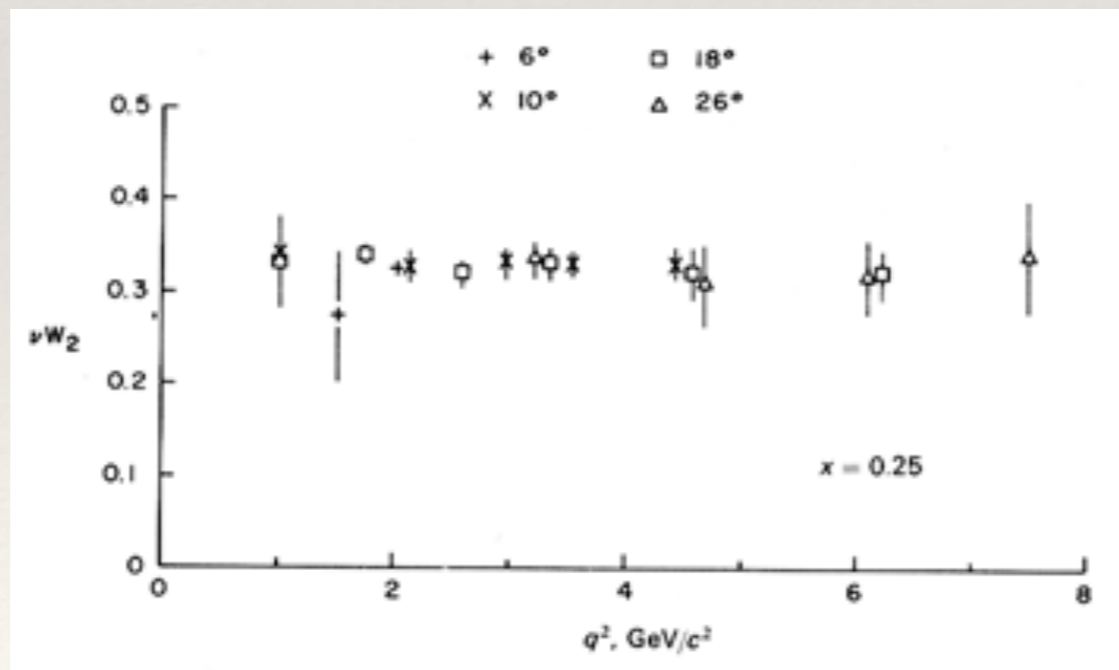


$x = Q^2 / 2M(E - E')$  is momentum fraction of struck quark

$y = (1 - E' / E)$  is fractional energy loss of electron

- ◆ Outcome:  $F_2$  can depend on  $x$  and  $Q^2$ , but seemed to only depend on  $x$

## ▶ “Scaling”

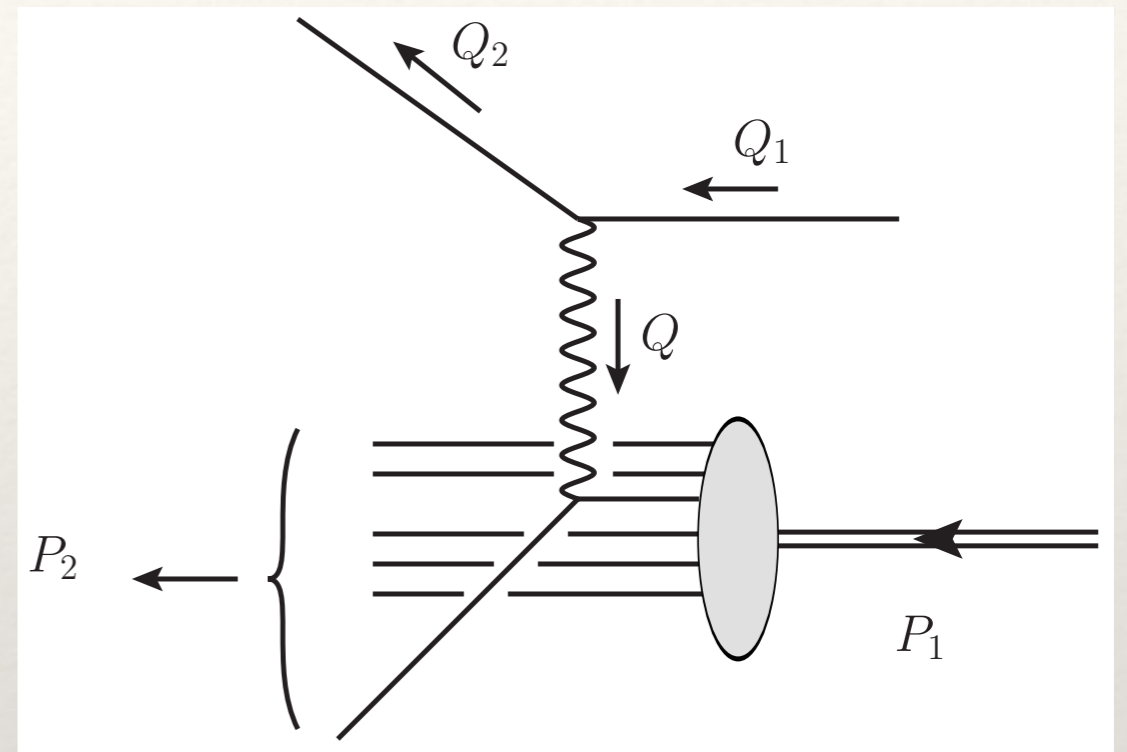




# Parton model

Feynman, Paschos

- ◆ Solution: the Parton model
- ◆ Wonderfully elegant idea, still at the basis of our predictions for the LHC.
- ◆ The scene: an electron at high energy hitting a proton (sitting inside a fixed target, or approaching from colliding beam).
- ◆ From the electron point of view, two relativistic effects occur
  - ▶ The proton is length contracted, looks like a disk
  - ▶ The internal proton dynamics is slowed down, due to time dilation
  - ▶ Assume interactions between constituent “partons” are absent (rather wild assumption at the time)
- ✓ NB Feynman did not call it quarks, perhaps to irritate Gell-Mann



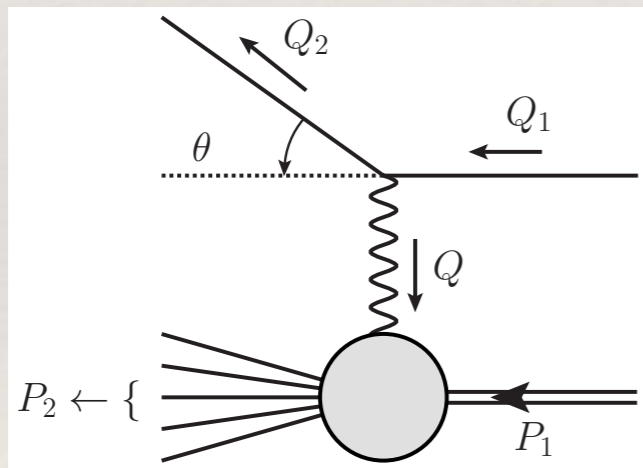
# Parton model

Feynman, Paschos

- ◆ How does this idea translate into a formula?
  - ▶ First, assume we know nothing about the inner workings of the proton. We can then at least isolate our ignorance from the EM part of the process we do know
  - ✓ We assume only photon exchange (can be extended using W or Z exchange)
  - ▶ Definitions of variables Bjorken-x, and y: the fractional energy loss of the electron

$$x \equiv \frac{Q^2}{-2P_1 \cdot Q} \qquad y = (1 - E_1/E_2)$$

- ▶ Then we can derive very generally (we skip that here)



$$\left(\frac{d^2\sigma}{dx dy}\right)^\gamma = \frac{8\pi\alpha^2 ME}{(Q^2)^2} \left\{ \frac{1 + (1-y)^2}{2} 2xF_1^\gamma(x, Q^2) \right. \\ \left. + (1-y)[F_2^\gamma(x, Q^2) - 2xF_1^\gamma(x, Q^2)] - \frac{M}{2E} xyF_2^\gamma(x, Q^2) \right\}$$

- ◆ All our ignorance about the proton structure now resides in the “structure functions” \$F\_i(x, Q^2)\$

# Parton model

Feynman, Paschos

◆ Now for the concrete implementation

▶ First, the collision takes place between the electron and a **constituent** (parton) of the proton

✓ In fact: only one constituent, because the probability of interacting with two constituents is a lot,  $O((1/Q)^2)$ , smaller

▶ The parton has a fraction of the proton energy and momentum  $p^\mu = \xi P_1^\mu$

✓ Assume it is a spin-1/2 fermion

▶ Some kinematics related to electron-parton

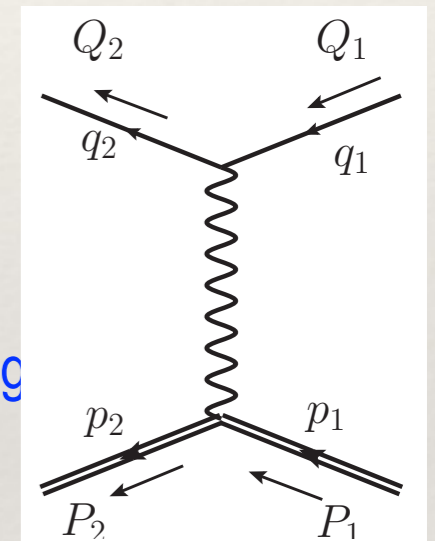
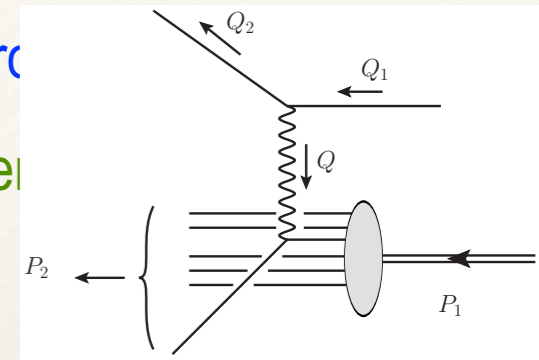
$$Q = Q_1 - Q_2 \quad 0 = p_2^2 = (\xi P_1 + Q)^2 = 2\xi P_1 \cdot Q + Q^2 \quad \xi = \frac{Q^2}{-2P_1 \cdot Q} \equiv x$$

▶ Bjorken-x has therefore the meaning of parton momentum fraction. Electron-parton scattering can now be computed, and gives

$$\left(\frac{d\sigma}{dx dy}\right)^\gamma = \frac{8\pi\alpha^2 ME}{(Q^2)^2} q_i^2 \frac{(1-y)^2 + 1}{2} \delta(x - \xi_i)$$

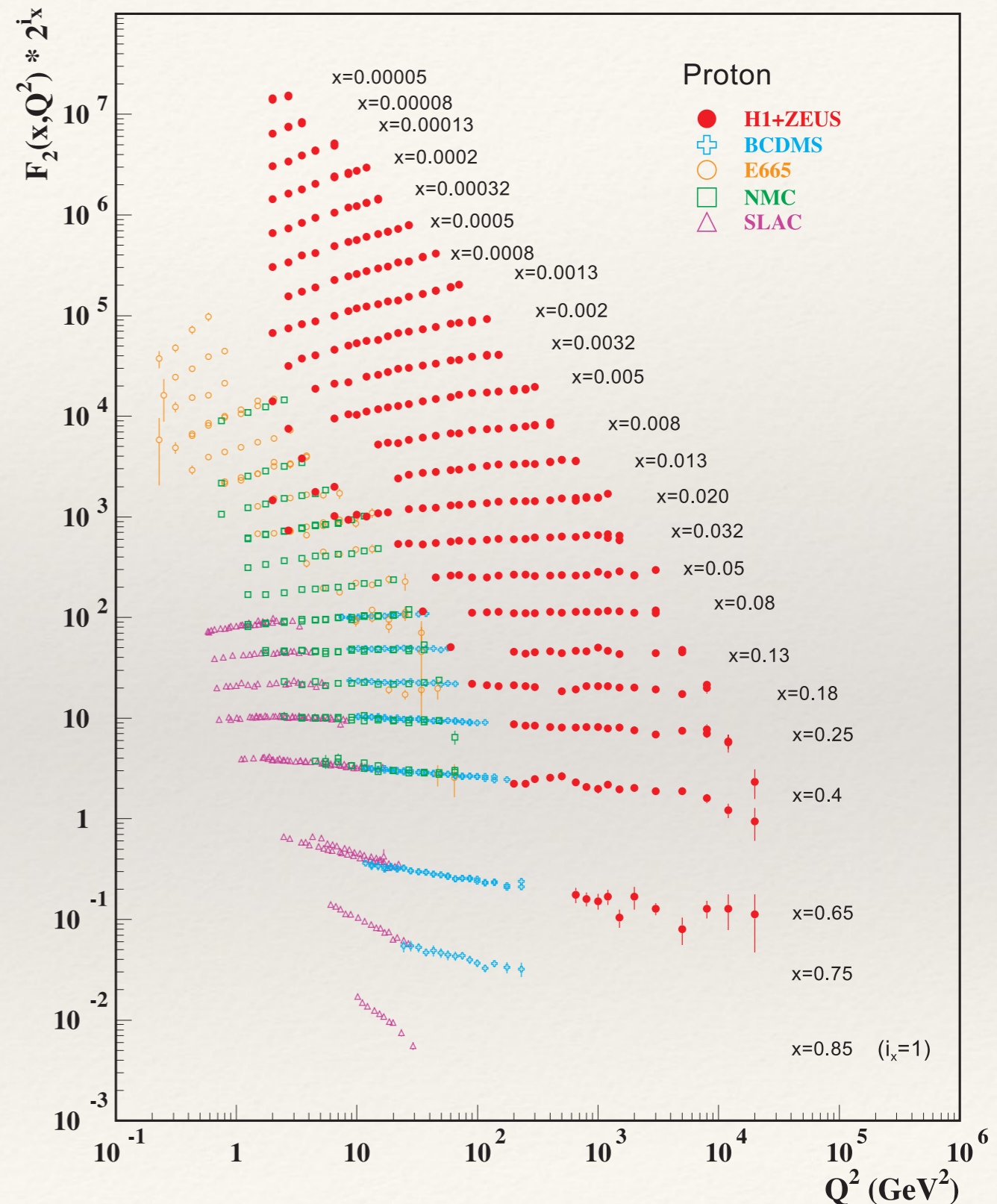
◆ Introduce now the parton distribution function  $\phi_{i/p}(\xi)$ , and integrate over all allowed momentum fractions  $\xi$ . Then we explain scaling!

$$\left(\frac{d\sigma}{dx dy}\right)^\gamma = \frac{8\pi\alpha^2 ME}{(Q^2)^2} \frac{(1-y)^2 + 1}{2} \sum_i q_i^2 x \phi_{i/p}(x)$$



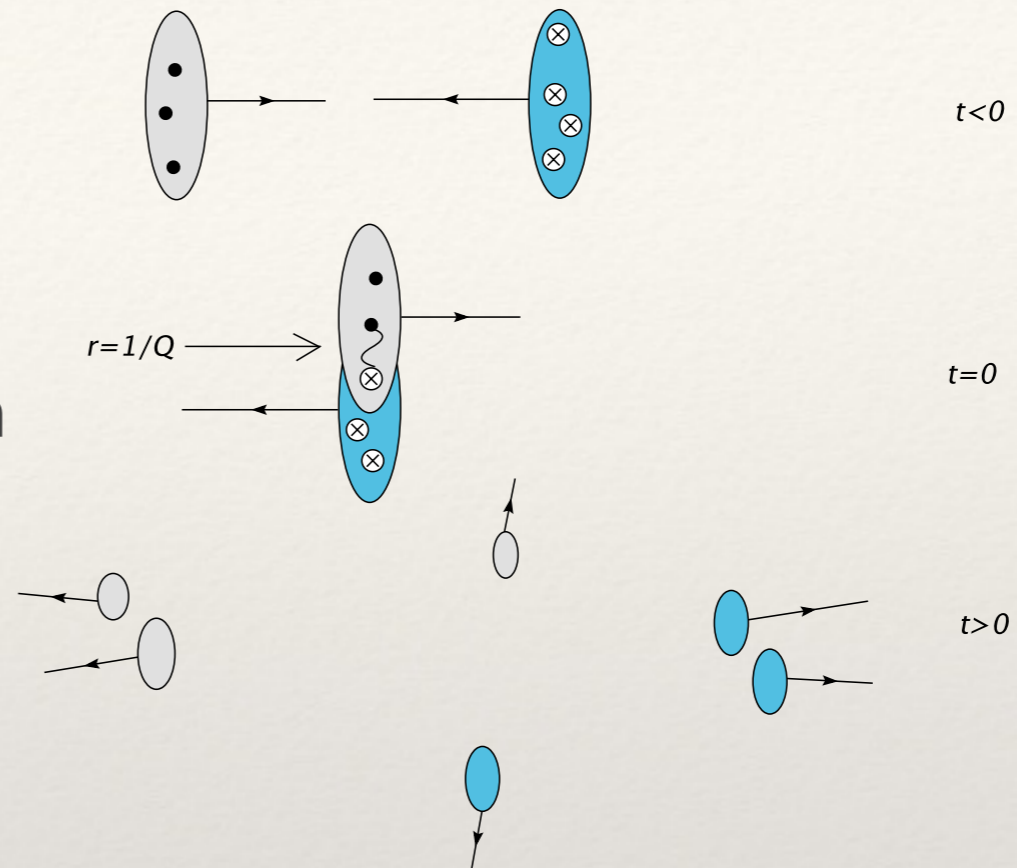
# Deep-inelastic scattering

- ◆ But in better measurements: no scaling anymore
  - “violation” very mild. Logarithmic, in fact
- ◆ Notice: steeper slopes for smaller  $x$
- ◆ Data are large  $x$  earlier cannot reach high  $Q^2$ 
  - Can all be explained, qualitatively
- ◆ Data take in 90's and early naughties at fixed target experiments, and at the HERA ep collider in DESY, Hamburg



# Universality of parton model, and paradox

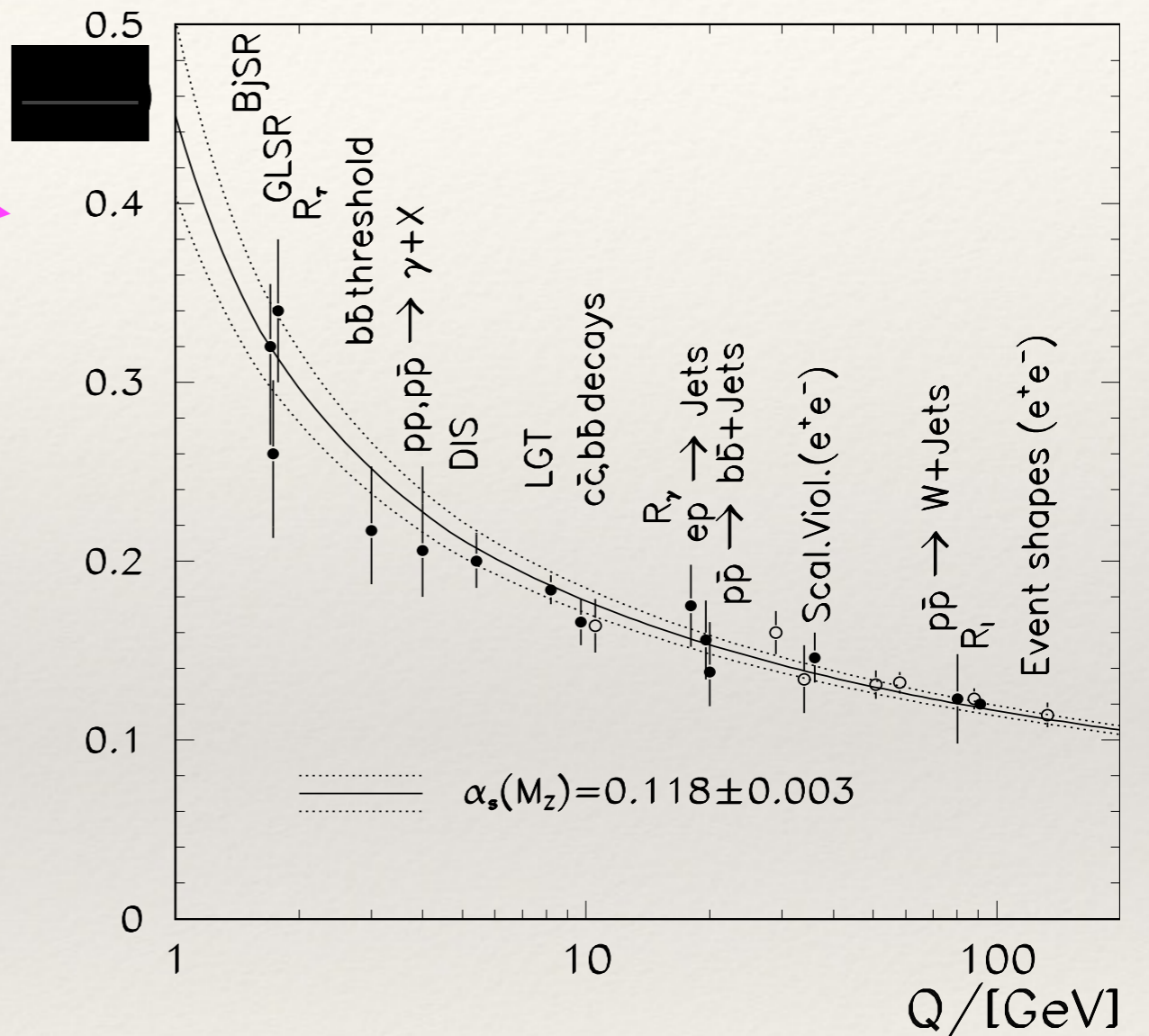
- ◆ Ideas generalizes to hadron-hadron scattering.
  - ▶ Brings predictive power, if the PDF's are the same all processes
    - ✓ This is an assumption in the parton model
- ◆ In QCD this can be proven. Such proofs, though formal, are important.
  - ▶ They don't always work..
- ◆ But the succes of the parton model presents a great paradox



How can quark be both strongly bound into hadrons, and act as free “partons” in deep-inelastic scattering??

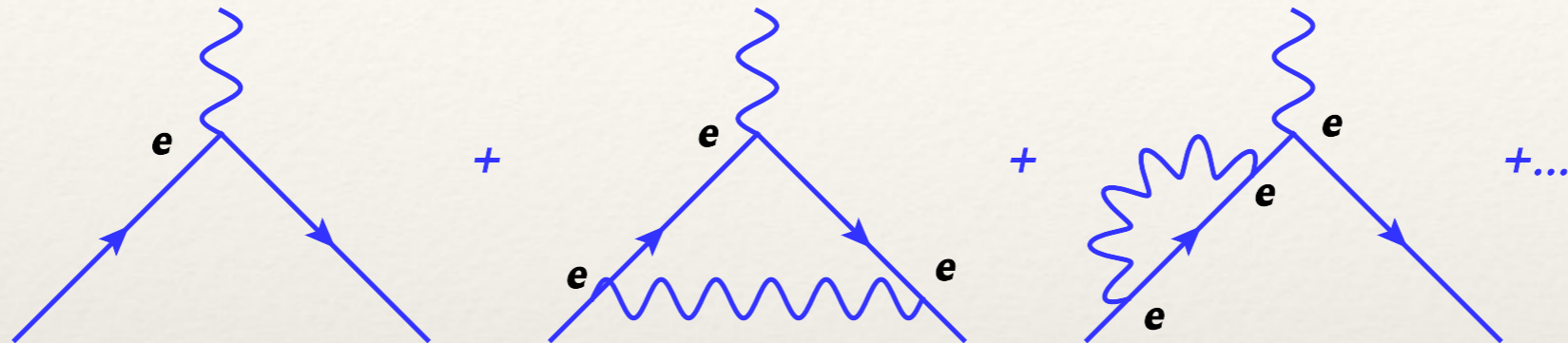
# Towards a solution of the paradox

- ◆ To solve this paradox, the coupling would have to behave like this
  - ▶ At low  $Q$  coupling is strong
  - ▶ For increasing  $Q$ , the coupling decreases
- ◆ But: how does a coupling become  $Q$  dependent in the first place. In the Lagrangian it is just a number: “ $g$ ”?
- ◆ For this we need to consider the effect of renormalization.



# Loops and regularization

- ✦ In fact, quantum effects do lead to a scale-dependent coupling, through **renormalization**.
- ✦ Computing any Green function at higher orders in a coupling leads to loops.



- ✦ Some loop integrals are divergent, and need to be regularized before being able to “handle” them

$$\int d^4l \frac{1}{[l^2 - m^2][(l + p)^2 - m^2]} \sim \int \frac{d^4l}{(l^2)^2} \sim i \int \frac{d\Omega l^3 dl}{l^4} \sim 2\pi^2 i \int \frac{dl}{l}$$

- ✦ One can put a cut-off on the  $l$  integral, but everyone uses dimensional regularization:  $4 \rightarrow 4 - 2\varepsilon$

$$\int \frac{dl}{l} \rightarrow \int \frac{dl}{l^{1+2\varepsilon}} = \frac{-1}{2\varepsilon}$$

- ✦ Very elegant. So loop integral results are would-be divergent. How to get rid of this? Renormalize

---

# Renormalization

---

- ◆ We focus on the key point (see also the QFT lectures). Write in this case

$$e = Z_e \left( \frac{1}{\varepsilon}, e_R(\mu) \right) e_R(\mu)$$

$$Z_e = 1 + e_R^2(\mu) \left( z^{1,1} \frac{1}{\varepsilon} + z^{1,0} \right) + \mathcal{O}(e_R^4)$$

- ◆ So beside the loop integrals, there is now a second source of  $1/\varepsilon$ : the renormalization of the coupling  $e$  in the tree-level graph.
  - ▶ Choose now the number  $z^{1,1}$  such that the  $1/\varepsilon$  from the loops is cancelled.
  - ▶ Is this not ridiculous? I could cancel any  $1/\varepsilon$  divergence in that way...!
  - ▶ BUT: the magic of renormalizable theories is that fixing  $z^{1,1}$  in this way, will fix this type of  $1/\varepsilon$  divergence in any other one-loop diagram in this theory.
  - ▶ I can renormalize a finite number of quantities: couplings, fields and masses. I can fix the Z-factors in a few calculations, but they must then work also in all other situations.
- ◆ Observe that on the right hand side a scale  $\mu$  appears, in both Z-factor and renormalized coupling  $e_R$ . The product does not depend on it. This is the renormalization scale. Sketchwise:

$$\left( 1 + e_R^2 \ln\left(\frac{\Lambda}{\mu}\right) + \mathcal{O}(e_R^4) \right) \times \left( 1 + e_R^2 \ln\left(\frac{\mu}{Q}\right) + \mathcal{O}(e_R^4) \right) = 1 + e_R^2 \ln\left(\frac{\Lambda}{Q}\right) + \mathcal{O}(e_R^4)$$



# Beta-function

- ✦ A very useful reminder when using dimensional regularization:  $\frac{1}{\varepsilon} \left(\frac{\mu}{Q}\right)^\varepsilon = \frac{1}{\varepsilon} \exp \left[ \varepsilon \ln \left(\frac{\mu}{Q}\right) \right]$
- ✦ In analogy to  $e_R$ , now for  $\alpha_s = g^2/4\pi$   $\simeq \frac{1}{\varepsilon} (1 + \varepsilon \ln \left(\frac{\mu}{Q}\right)) = \frac{1}{\varepsilon} + \ln \left(\frac{\mu}{Q}\right)$

$$\alpha_s = Z_\alpha \left( \frac{1}{\varepsilon}, \alpha_{s,R}(\mu) \right) \alpha_{s,R}(\mu)$$

- ✦ We derive from this  $Z_\alpha = 1 + \frac{\alpha_{s,R}(\mu)}{4\pi} \left( \frac{11C_A - 2n_f}{3} \frac{1}{\varepsilon} + c_\alpha \right) + \mathcal{O}(\alpha_{s,R}^2)$

$$\mu \frac{d}{d\mu} \ln \alpha_{s,R}(\mu) = -\mu \frac{d}{d\mu} \ln Z_\alpha \left( \frac{1}{\varepsilon}, \alpha_{s,R}(\mu) \right) = -\frac{\beta(\alpha_{s,R}(\mu))}{\alpha_{s,R}(\mu)}$$

- ✦ The QCD beta function is known to 4th order, with the 5th order being computed. Keep only the first term gives the differential equation

$$\mu \frac{d}{d\mu} \alpha_s(\mu) = -\frac{\alpha_s^2(\mu)}{2\pi} \left( \frac{11C_A - 2n_f}{3} \right) = -\frac{\beta_0}{2\pi} \alpha_s^2$$

- ▶ Observe already that an increase in  $\mu$  leads to decrease in  $\alpha$ . But for higher  $\mu$  the decrease decreases..

- ▶ Solution

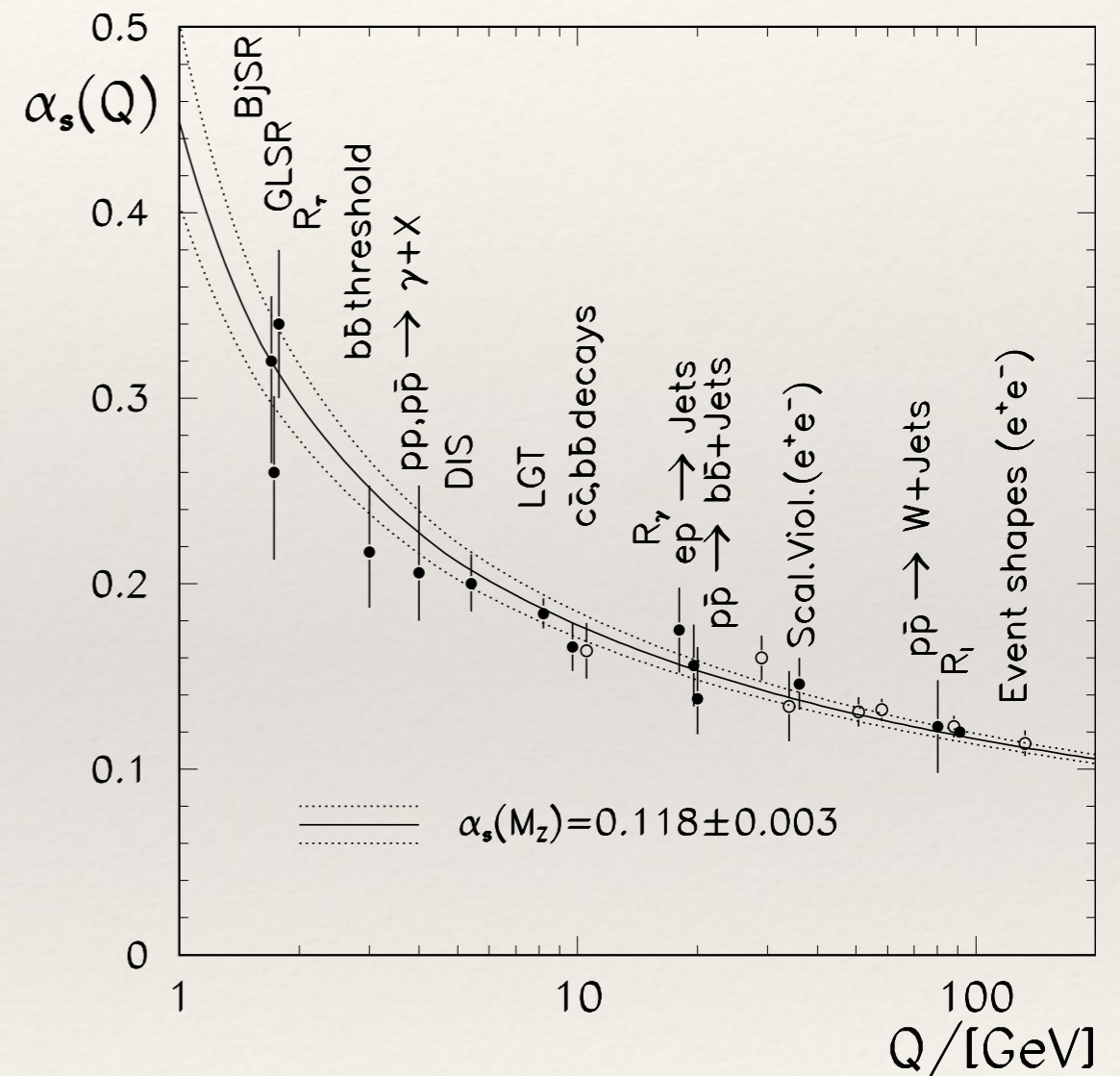
$$\alpha_s(\mu) = \frac{4\pi/\beta_0}{\ln \left( \frac{\mu^2}{\Lambda_{QCD}^2} \right)}$$

- Exercise: find this solution. How does  $\Lambda_{QCD}$  come in?

This solves the paradox!

# QCD and asymptotic freedom

- ◆ The QCD couplings is asymptotically free in the UV, and very strong in the IR
- ◆ Crucial was the minus sign in front of  $\beta_0$ .
  - higher order terms in  $\beta$  do not spoil this
- ◆ In the 70's lots of theories were examined, but only this strange non-abelian gauge theory yielded a negative beta-function
- ◆ Nobelprize 2004: Gross, Wilczek, Politzer



---

# QCD gauge/local symmetry

---

- ✦ Before diving into the heart of the perturbative QCD treatment, let us devote a few slides to formal aspects of this non-abelian gauge theory.
- ✦ Let us start from symmetry. We would like to build a theory that is invariant under local SU(3) transformations. SU(3) is a non-abelian group (elements don't commute).
- ✦ We take a fermion field that has 3 components (each in turn a 4-component spinor), that transforms as

$$\psi(x)' = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \end{pmatrix}' = U(x)\psi(x) = \begin{pmatrix} & & \\ & U(x) & \\ & & \end{pmatrix} \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \end{pmatrix}$$

- ▶ The three components are also called R,B,G sometimes.
- ▶ This is a “covariant transformation”. We also have  $\bar{\psi}' = \bar{\psi}U^\dagger(x) = \bar{\psi}'U^{-1}(x)$
- ✦ Problem: derivative of fermion field (part of Dirac Lagrangian), does not transform covariantly.

$$\partial_\mu\psi(x)' = U(x)\partial_\mu\psi(x) + (\partial_\mu U(x))\psi$$

- ✦ Solution: introduce better, covariant derivative  $D_\mu$ , that does transform nicely, so that...

# QCD gauge/local symmetry

- ◆ ..we have

$$\bar{\psi}' \gamma^\mu D'_\mu \psi' = \bar{\psi} \gamma^\mu \underbrace{U^{-1}(x)U(x)}_{=1} D_\mu \psi$$

- ◆ To construct the covariant derivative, introduce the gauge field (any many as there are SU(3) generators)

$$D_\mu \psi(x) = \left( \left( \begin{array}{ccc} \partial_\mu & & \\ & \partial_\mu & \\ & & \partial_\mu \end{array} \right) - g A_\mu^a \underbrace{(T_a)}_{3 \times 3 \text{ matrices}} \right) \psi(x)$$

$$\equiv (\partial_\mu - g A_\mu) \psi(x)$$

- ▶ Notice there are now quark-gluon interactions!

$$\bar{\psi} \gamma^\mu D_\mu \psi = \bar{\psi} \gamma^\mu \partial_\mu \psi - g \bar{\psi}_i \gamma^\mu [\mathbf{T}_a]_{ij} \psi_j A_\mu^a$$

- ▶ Exercise: how must A transform to have  $D_\mu \psi$  transform covariantly?

- ◆ The field strength is also derived from covariant derivative

$$[D_\mu, D_\nu] \psi = -g G_{\mu\nu} \psi = -g G_{\mu\nu}^a \mathbf{T}_a \psi$$

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f_{bc}^a A_\mu^b A_\nu^c$$

- ▶ Exercise: check this

# QCD gauge symmetry

- Transformation of field strength (as a 3x3 matrix [=“Lie algebra valued”]) goes as

$$G'_{\mu\nu}(x) \equiv G'^a_{\mu\nu}(x) \mathbf{T}_a = U(x) G_{\mu\nu}(x) U^{-1}(x)$$

- Now construct the QCD Lagrangian according to the principle: everything not forbidden is mandatory.
  - and also all terms must have dimension maximum 4

- Unique solution (for one quark flavour)

$$\mathcal{L} = \frac{1}{4} \text{Tr} [G_{\mu\nu} G^{\mu\nu}] - \bar{\psi} \not{D} \psi - m \bar{\psi} \psi$$

- There is one more possible term:  $\mathcal{L} = \theta \text{Tr} [G_{\mu\nu} G_{\rho\sigma}] \epsilon^{\mu\nu\rho\sigma}$  but that is a total derivative, and a separate story (strong CP problem)

- can add fermion copies/flavours, each with its own mass, and arrive at the QCD Lagrangian

$$\mathcal{L}_{QCD} = \frac{1}{4} \text{Tr} [G_{\mu\nu} G^{\mu\nu}] - \sum_f \bar{\psi}_f \not{D} \psi_f - m_f \bar{\psi}_f \psi_f$$

- with multiple flavours, if masses of some are identical, they can mix: extra flavour symmetry
- if masses are zero, left and right handed parts of fermion act as independent fields/particles

$$\sum_f \bar{\psi}_f \not{D} \psi_f = \sum_f \overline{(\psi_L)_f} \not{D} (\psi_L)_f + \sum_f \overline{(\psi_R)_f} \not{D} (\psi_R)_f$$

# QCD other symmetries

- ◆ We could then mix them independently (with global transformations, otherwise we would need more gauge fields): flavour symmetry matrices ( $N_l \times N_l$ ) of QCD ( $N_l$ : # of light flavours [u,d, sometimes s])

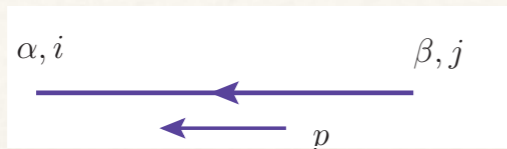
$$U_L \otimes U_R$$

- ▶ This is a fascinating field within QCD itself. It explains why pions are light (there are almost Goldstone bosons), with the whole field of strange (Kaon) physics, CP violation etc
- ▶ Because we cannot compute a meson or hadron ab initio (but we do know an enormous amount from the lattice QCD community), we cannot use the QCD Lagrangian directly to describe low-energy meson dynamics
- ▶ Very succesful *effective* theory: chiral perturbative theory. Won't discuss that in these lectures
- ◆ The QCD Lagrangian in components

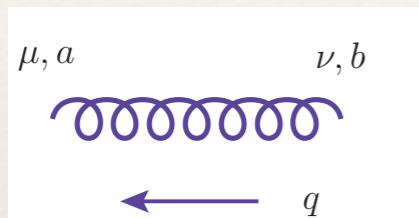
$$\begin{aligned} \mathcal{L}_{QCD} = & - \sum_f \bar{\psi}_{f,i} (\not{\partial} - m) \psi_{f,i} + g \sum_f \bar{\psi}_{f,i} \gamma^\mu [\mathbf{T}_a]_{ij} \psi_{f,j} \\ & + \text{Tr} [t_a t_b] \left[ \frac{1}{4} (\partial_\mu W_\nu^a - \partial_\nu W_\mu^a) (\partial^\mu W^{\nu b} - \partial^\nu W^{\mu b}) \right. \\ & \left. - g f_{cd}^b \partial_\mu W_\nu^a W^{\mu c} W^{\nu d} + \frac{1}{4} g^2 f_{cd}^a f_{ef}^b W_\mu^c W^{\mu e} W_\nu^d W^{\nu f} \right] \end{aligned}$$

- ▶ from which one can read off the Feynman rules

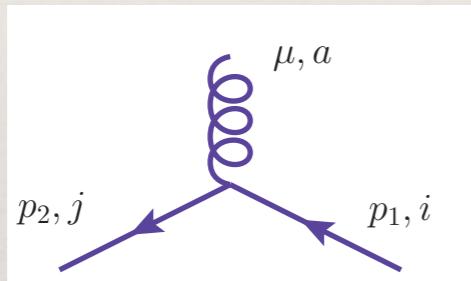
# QCD Feynman Rules



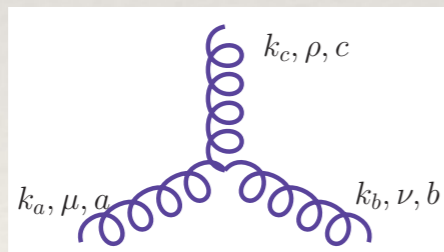
$$\frac{1}{i(2\pi)^4} \frac{\delta_{ij}(-i\not{p} + m)_{\alpha\beta}}{p^2 + m^2}$$



$$\frac{1}{i(2\pi)^4} \frac{\delta_{ab}}{q^2} \left( \eta_{\mu\nu} - (1 - \lambda^{-2}) \frac{q_\mu q_\nu}{q^2} \right)$$



$$i(2\pi)^4 \delta^4(p_1 - p_2 + q)(-g)[\mathbf{T}_a]_{ij}(\gamma_\mu)_{\alpha\beta}$$



$$i(2\pi)^4 (-ig) f^{abc} [\eta_{\mu\nu}(k^a - k^b)_\rho + \eta_{\nu\rho}(k^b - k^c)_\mu + \eta_{\rho\mu}(k^c - k^a)_\nu]$$

- ◆ For completeness.
- ◆ Rules involves Lorentz (vector, spinor) and SU(3) (fundamental, adjoint) parts
  - ▶ Not directly linked

# QCD and UV divergences

- When computing loop integrals, and UV divergences result from them, not all of them can be cancelled by renormalization of just the QCD coupling

$$\alpha_s = Z_\alpha \left( \frac{1}{\varepsilon}, \alpha_{s,R}(\mu) \right) \alpha_{s,R}(\mu)$$

$$Z_\alpha = 1 + \frac{\alpha_{s,R}(\mu)}{4\pi} \left( \frac{11C_A - 2n_f}{3} \frac{1}{\varepsilon} + c_\alpha \right) + \mathcal{O}(\alpha_{s,R}^2)$$

- In fact, in general, all the fields, couplings and parameters get their Z-factors.

$$W_\mu^a \rightarrow \sqrt{Z_W} W_\mu^a, \quad \psi \rightarrow \sqrt{Z_\psi} \psi, \quad c^a \rightarrow \sqrt{Z_c} c^a, \quad b^a \rightarrow \sqrt{Z_b} b^a, \\ g \rightarrow Z_g g, \quad m \rightarrow Z_m m, \quad \lambda \rightarrow Z_\lambda \lambda$$

- As a consequence

$$\mathcal{L} \longrightarrow \mathcal{L} + \Delta\mathcal{L} \quad \Delta\mathcal{L} = - (Z_\psi - 1) \bar{\psi} \not{\partial} \psi - m (Z_m Z_\psi - 1) \bar{\psi} \psi \\ + g (Z_g Z_\psi Z_W^{1/2} - 1) W_\mu^a \bar{\psi} \gamma^\mu t_a \psi + \dots$$

- From one-loop calculations

$$Z_W = 1 + \frac{g^2 \mu^\varepsilon}{6\pi^2} \frac{1}{\varepsilon} \left[ N_f C_2(\text{R}) \frac{\dim \text{R}}{\dim \text{G}} - \frac{1}{8} C_2(\text{G}) (13 - 3\lambda^{-2}) \right] + \mathcal{O}(g^4),$$

$$Z_\lambda = 1 - \frac{g^2 \mu^\varepsilon}{12\pi^2} \frac{1}{\varepsilon} \left[ N_f C_2(\text{R}) \frac{\dim \text{R}}{\dim \text{G}} - \frac{1}{8} C_2(\text{G}) (13 - 3\lambda^{-2}) \right] + \mathcal{O}(g^4),$$

$$Z_\psi = 1 + \frac{g^2 \mu^\varepsilon}{8\pi^2} \frac{1}{\varepsilon} C_2(\text{R}) \lambda^{-2} + \mathcal{O}(g^4),$$

$$Z_m = 1 + \frac{3g^2 \mu^\varepsilon}{8\pi^2} \frac{1}{\varepsilon} C_2(\text{R}) + \mathcal{O}(g^4),$$

$$Z_g = 1 - \frac{g^2 \mu^\varepsilon}{12\pi^2} \frac{1}{\varepsilon} \left[ N_f C_2(\text{R}) \frac{\dim \text{R}}{\dim \text{G}} - \frac{11}{4} C_2(\text{G}) \right] + \mathcal{O}(g^4),$$

$$\sqrt{Z_b Z_c} = 1 + \frac{g^2 \mu^\varepsilon}{12\pi^2} \frac{1}{\varepsilon} \left[ N_f C_2(\text{R}) \frac{\dim \text{R}}{\dim \text{G}} - \frac{1}{4} C_2(\text{G}) (11 - 3\lambda^{-2}) \right] + \mathcal{O}(g^4).$$



---

# Renormalizability of QCD

---

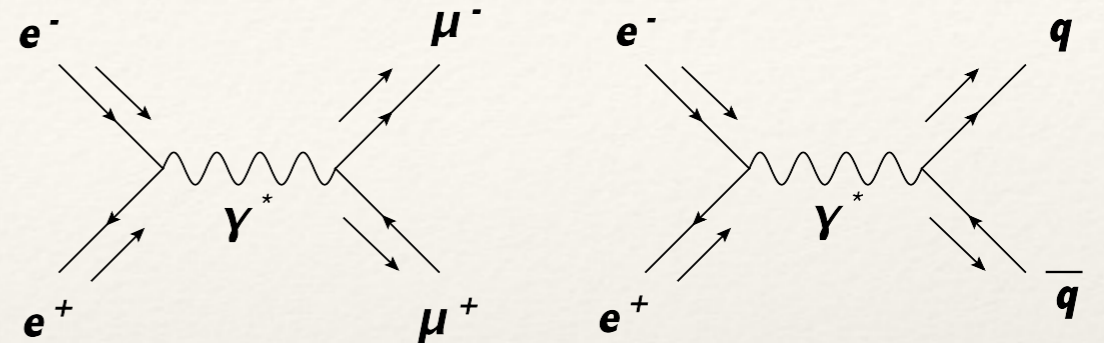
- ✦ In fact, with these Z-factors, *every* UV divergence in *any* one-loop QCD amplitude is cancelled.
- ✦ But if it goes wrong at higher orders, all is for naught..
- ✦ This was a key worry in the early 70's. Renormalizability of QED was known, and of numerous scalar, Yukawa and other field theories. Non-abelian gauge seemed too hard.
- ✦ This was the problem that Gerard 't Hooft tackled as a PhD student, together with his advisor Martinus Veltman [\[after a summer school!\]](#)
- ✦ The solution was presented by 't Hooft at a EPS meeting in Amsterdam in 1971, leaving most participants stunned. He and Veltman proved that no new Z-factors are needed to any order. One just needs to determine them to higher order.
- ✦ They used lots of diagrammatic clever techniques. More modern is the use of “BRST symmetry”, but the proof was there.
  - ▶ [Only then was QCD, and in fact the Standard Model, taken more seriously, now that it was a legitimate theory.](#)
- ✦ We shall take it otherwise for granted. Our problem will be mostly other types of divergences.

# Evidence for colors

Since the colour quantum number is confined, how can we tell there are 3 colors? Indirectly!

R-ratio

$$R(s) = \frac{\sigma[e^+e^- \rightarrow \text{hadrons}](s)}{\sigma[e^+e^- \rightarrow \mu^+\mu^-](s)}$$

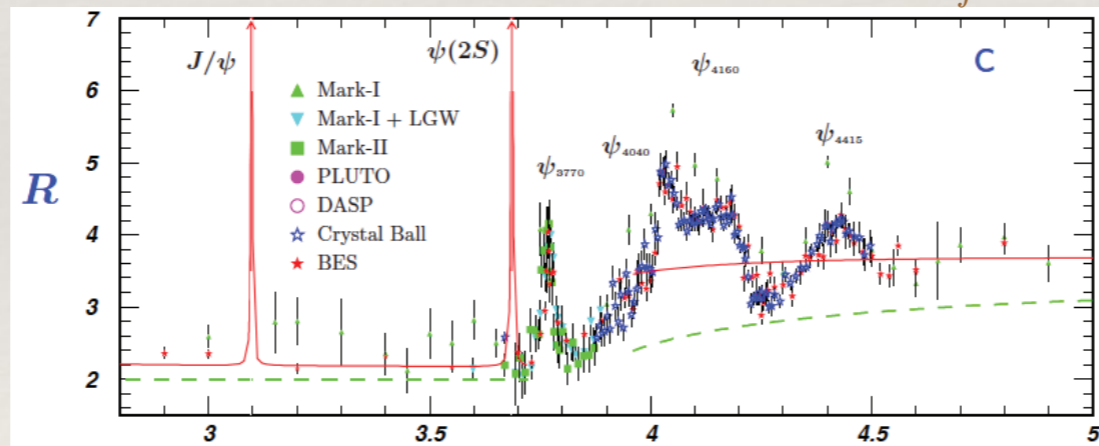


Leading order

$$R(s) = N_c \sum_f Q_f^2 \theta(s - 4m_f^2) = 3 \times \left( \frac{4}{9} + \frac{1}{9} + \frac{1}{9} + \frac{4}{9} \right) = \frac{10}{3} \simeq 3.3$$

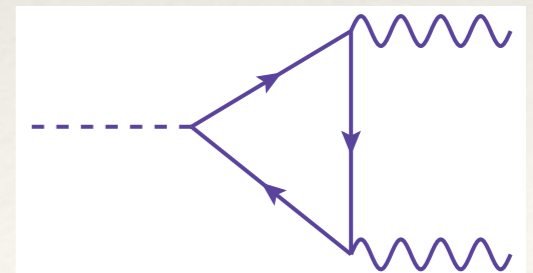
Next-to-leading order not a huge correction (10%)

$$R(s) = N_c \sum_f Q_f^2 \theta(s - 4m_f^2) \left( 1 + \frac{\alpha_s}{\pi} \right) \simeq 3.6$$

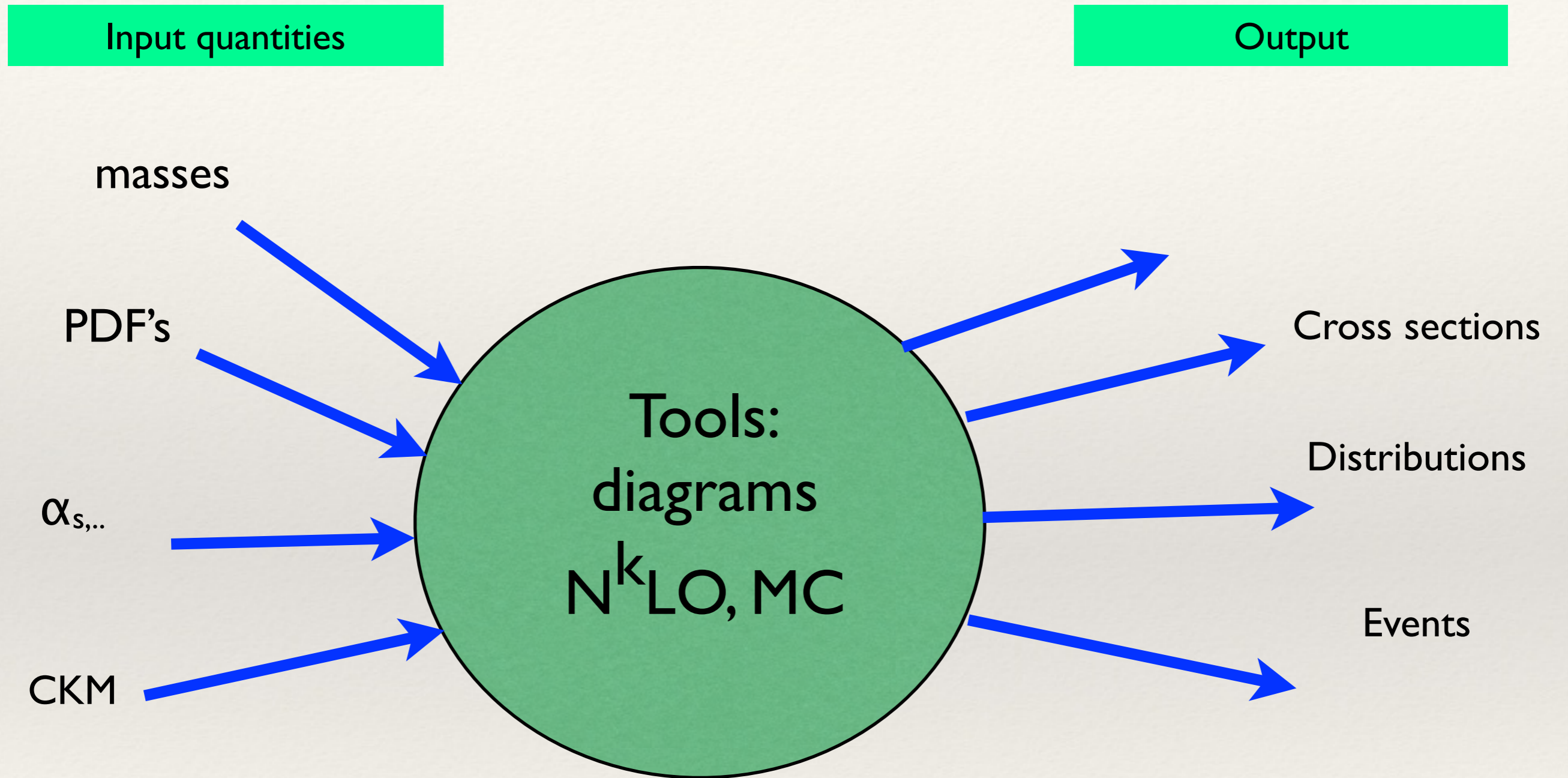


Other evidence: pion decay rate to two photons only correct with factor 9

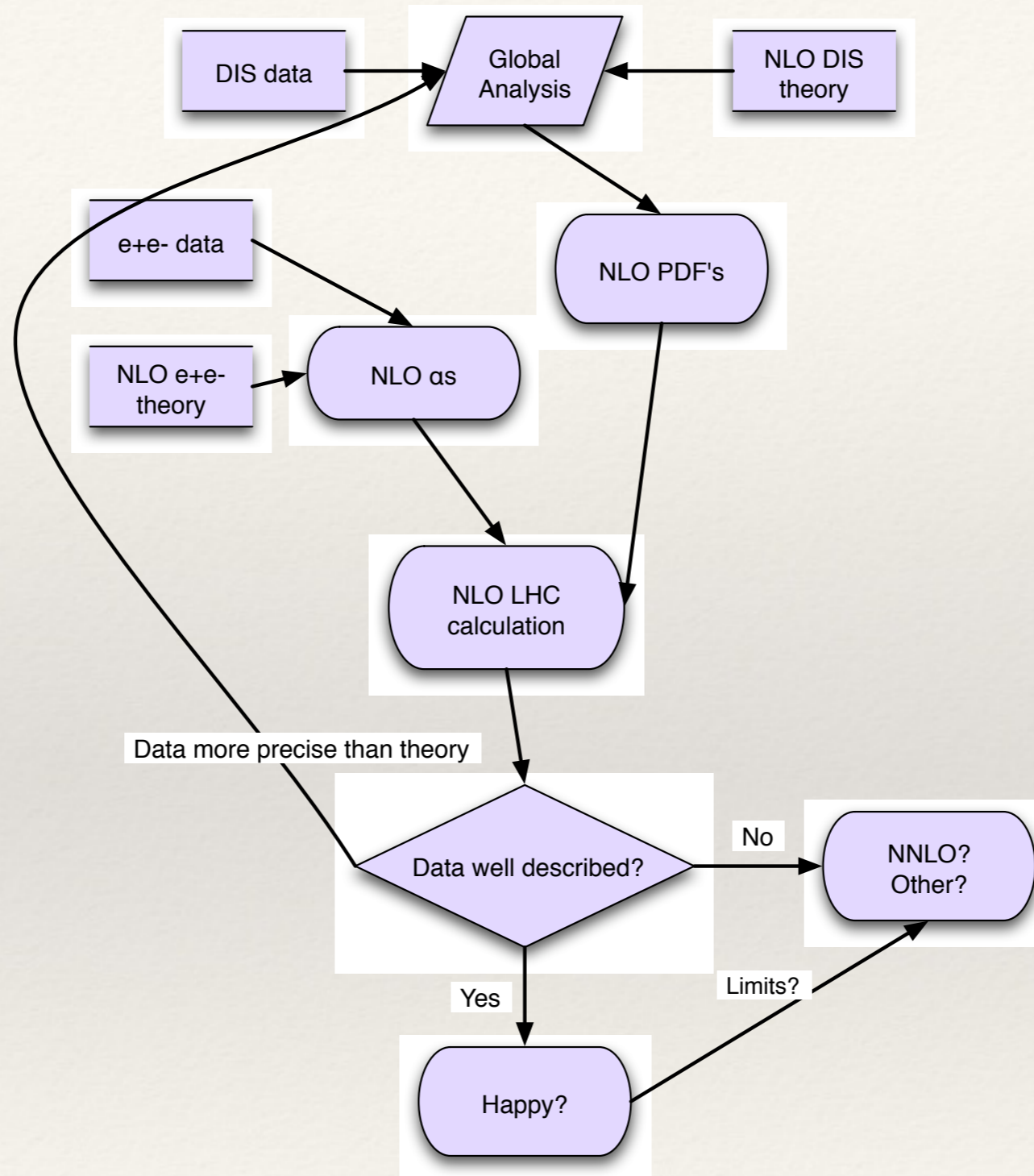
▶ =Number of colors squared, since amplitude must be squared.



# QCD in practice, simplified

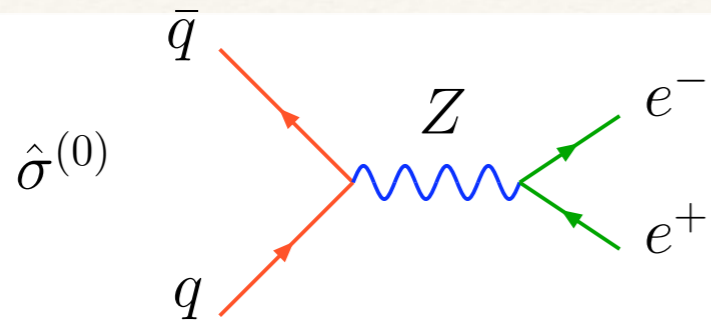


# How to use QCD in practice, less simplified



# LO and higher order amplitudes

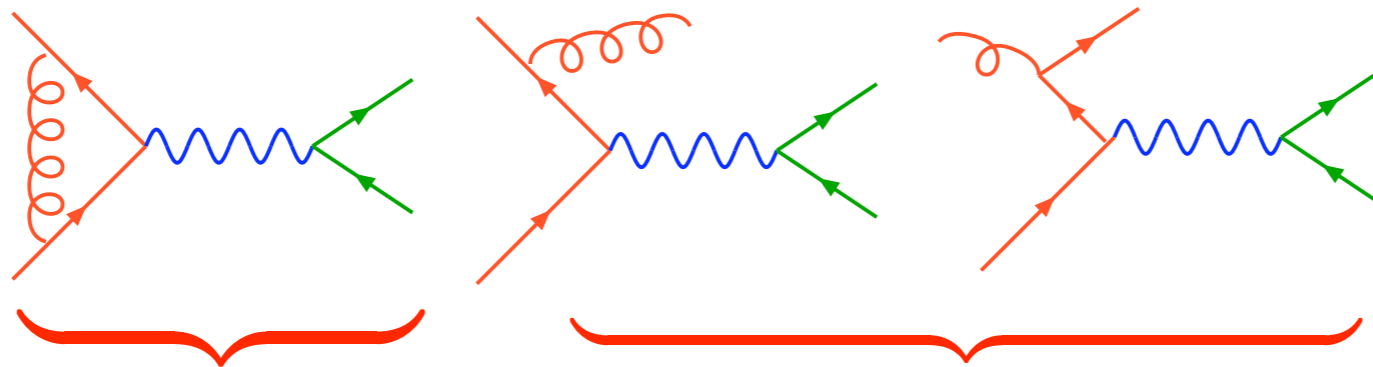
LO



Calculate all this  
in  $D=4-2\epsilon$  dimensions

NLO

$\hat{\sigma}^{(1)}$

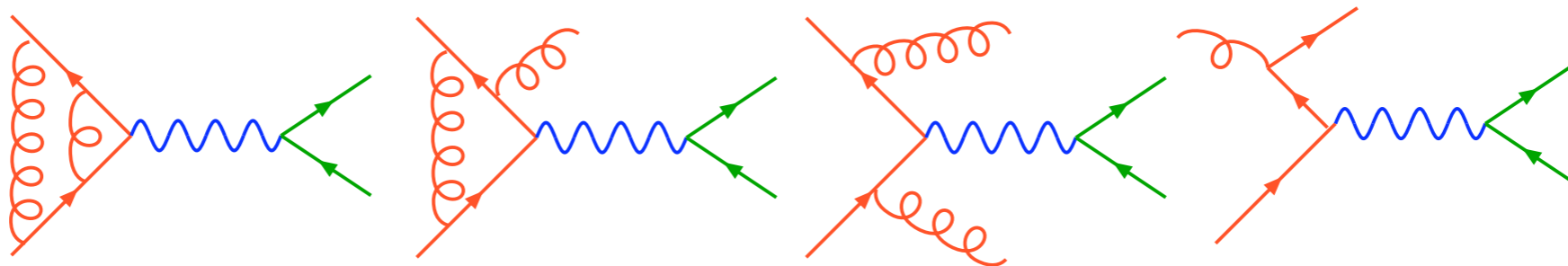


1 loop

1 extra parton

NNLO

$\hat{\sigma}^{(2)}$



2 loop

1 loop +  
1 extra parton

2 extra partons

# General structure of LO, NLO,.. cross sections

Multi-differential hadronic NLO cross section

NLO PDF's

$$\frac{d\sigma^{pp \rightarrow X}}{d^3p_1 \dots d^3p_n} = \sum_{a,b} \int dx_1 dx_2 \phi_{a/p}(x_1, \mu_F) \phi_{b/p}(x_2, \mu_F)$$

$$\times \hat{\sigma}_{ab}(p_a + p_b \rightarrow p_X, \alpha_s(\mu_R), \mu_R, \mu_F) + \mathcal{O}\left(\frac{\Lambda^2}{Q^2}\right)$$

Multi-differential partonlevel NLO cross section

Power corrections. Hard!

Renormalization and Factorization scale

For NNLO, add "N" in front of every occurrence of "NLO"..

---

# Parton distribution functions

---

- ✦ Before concentrating on the computing the partonic cross sections, let us discuss the PDF's. In the parton model they only depend on the momentum fraction. But we had seen that structure function depend logarithmically on  $Q$ , so we expect that PDF's might also. Indeed that is the case, as we'll see. How does one determine them?
- ✦ Crucial at hadron colliders, must be known very accurately. But they cannot be computed from first principles.
- ✦ Answer: use their universality, as follows.

- ▶ We need to determine 11 PDF (5 quarks + antiquarks + gluon), *and their uncertainties*
- ▶ Choose with care a set of measurements/observables [e.g. DIS structure functions, or hadron collider cross sections] Each is described as a PDF  $\otimes$  partonic cross sections. We then have the set of equations


$$(O_n \pm \Delta O_n)^{\text{exp}} = \sum_{j=1}^{n_f} \phi_{j/p} \otimes [\hat{\sigma}_{n,j} \pm \delta\sigma_{n,j}]^{\text{th}}$$

- ▶ From the comparison one fits the  $\phi_{j/p}(x,\mu)$ .
  - ✓ Various groups, employing slightly different approaches
    - MSTW, CTEQ, NNPDF, GJR, HERAPDF, ABKM...
- ▶ If the partonic calculation is LO, NLO, NNLO etc, then the PDF thus fitted are also labelled LO, NLO etc.
  - ✓ NLO PDF's must be used with NLO calculations. NNLO also ok, LO not


# Aside: PDF's as operator matrix elements

Although they cannot yet be fully computed from first principles, one can give a precise definition of PDF's, in terms of operators. Essentially, these are counting operators (cf  $a^\dagger a$  in QM)

$$\phi_{q/p}(\xi) = \frac{1}{4\pi} \int_{-\infty}^{\infty} dy^- e^{-i\xi p^+ y^-} \langle p | \bar{q}(0, y^-, 0_T) \gamma^+ q(0, 0, 0_T) | p \rangle$$



proton state



Quark field

$$p^\pm = \frac{p^0 \pm p^3}{\sqrt{2}}$$

$$p \cdot q = -p^+ q^- - p^- q^+ + p_1 q_1 + p_2 q_2$$

- ▶ in a certain gauge. The non-perturbative part sits in the hadronic state in which this counting operator is inserted.
- ▶ Benefit: once you have an operator, one can compute its renormalization, and derive an RG equation for it (just like for the coupling constant). This is in fact the DGLAP equation
- ✓ There are other ways of deriving it. We will see another method later.
- ▶ To do so, just replace the proton states with quark states (and keep the operator). At lowest order this is just

$$\delta(1 - \xi)$$

▶ At next order it has the form

$$\frac{\alpha_s}{2\pi} C_F \frac{1}{\varepsilon} \left( \frac{1 + \xi^2}{1 - \xi} \right)_+ + \dots$$

quark-to-quark splitting function!

- Plus distribution:

$$\int_0^1 dz \left[ \frac{a(z)}{1-z} \right]_+ g(z) = \int_0^1 (g(z) - g(1)) \left[ \frac{a(z)}{1-z} \right]$$



---

# Parton distribution functions

---

- ◆ The logic is thus very similar to running coupling, we now have “running functions”:

$$\mu \frac{d}{d\mu} \phi_{i/H}(x, \mu) = \int_x^1 \frac{dz}{z} P_{ij}(z, \alpha_s(\mu)) \phi_{j/H}\left(\frac{x}{z}, \mu\right) \quad [\equiv P_{ij} \otimes \phi_{j/H}](x, \mu)$$

- ▶ DGLAP equations (we derive them later).  $P_{ij}$  are the splitting functions, aka parton evolution kernels. They are now known to NNLO (3rd order)
- ▶ Logic: determine the PDF's at some scale  $Q$ , then compute them at all other scales by solving the DGLAP equations.
- ◆ Note:
  - ▶ for LO PDF's, use one-loop splitting and beta-function
  - ▶ for NLO PDF's use two-loop splitting and beta-function, etc.
  - ▶ in 2004 the three-loop splitting functions [Moch, Vermaseren, Vogt] were computed, so also NNLO sets are now available (NNLO partonic cross sections for DIS, Drell-Yan etc were already available).

- ◆ To determine the PDF's from the equation

$$(O_n \pm \Delta O_n)^{\text{exp}} = \sum_{j=1}^{n_f} \phi_{j/p} \otimes [\hat{\sigma}_{n,j} \pm \delta\sigma_{n,j}]^{\text{th}}$$

- ◆ one must choose the data on the lhs well.