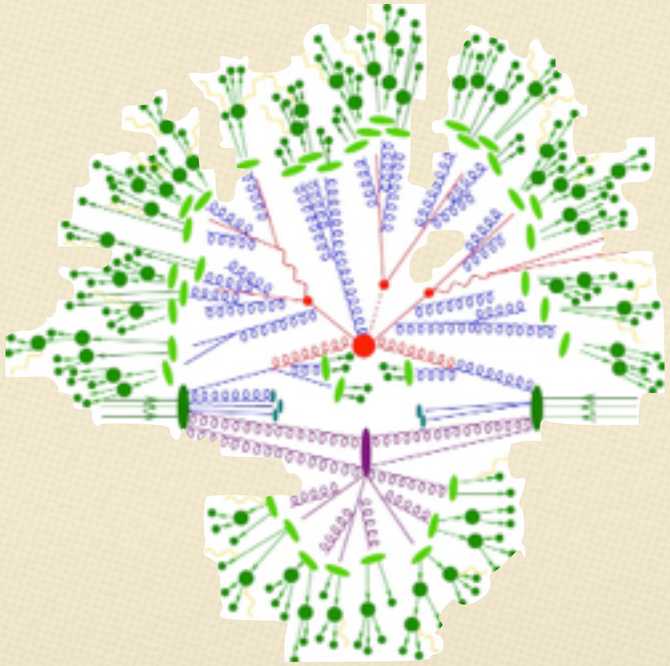
$$\frac{1}{4}\mathrm{Tr}[G^2] - \bar{\psi}(\not{D} - m)\psi$$

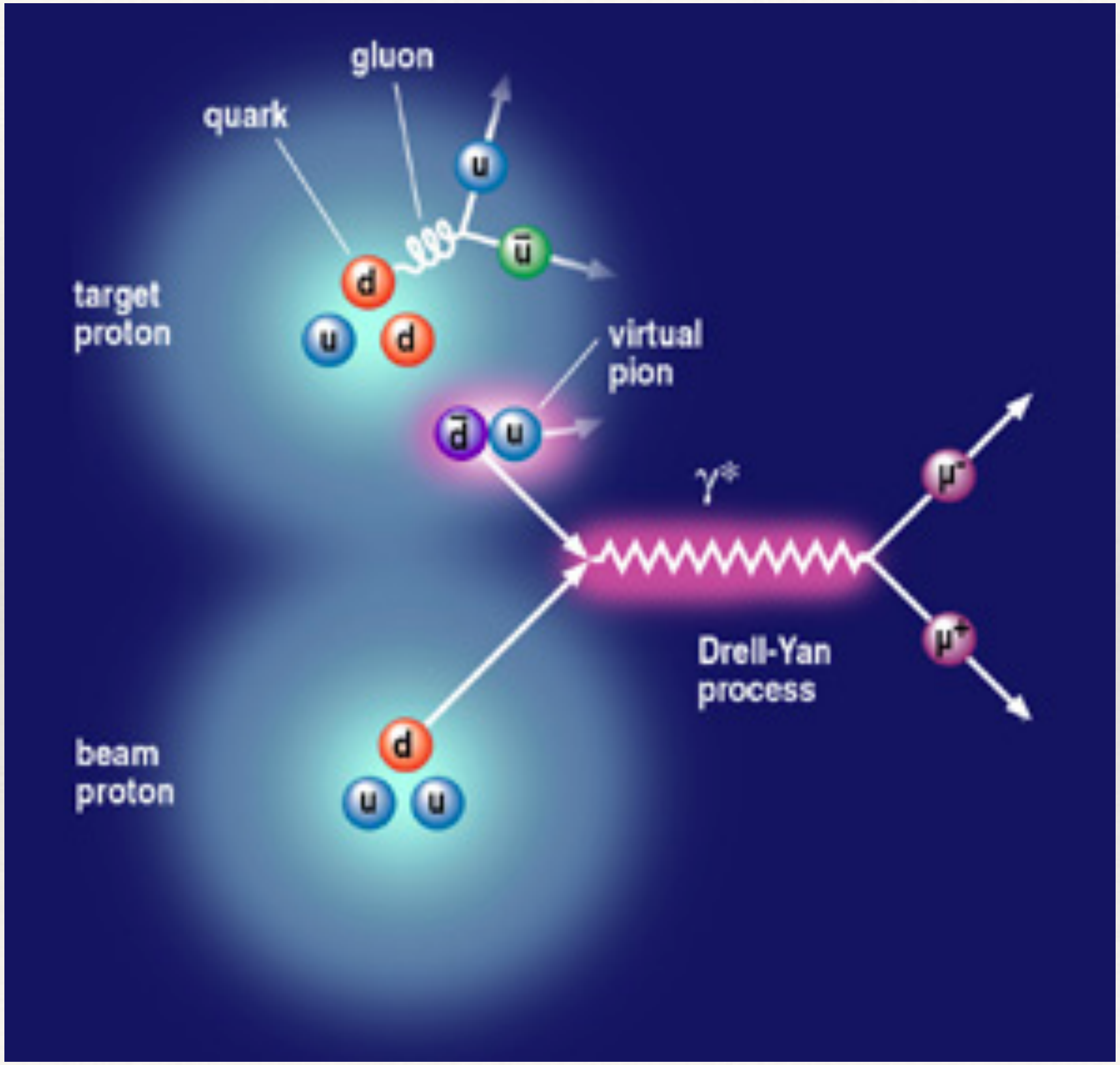
☞



QCD

Eric Laenen
CERN school 2014

Lecture 2: KLN, Drell-Yan and its lessons



Summary of last time

- We reviewed QCD and its symmetries
  - local (“gauge”), and the nice role played by the covariant derivative (which brings in the gluons)
  - global flavor symmetry
- We discussed its UV properties, and renormalization, and evidence for the hidden/confined color quantum number.
- We reviewed the strong/weak coupling paradox, and how the running QCD coupling solves it.
- We recalled the parton model, and introduced the parton distribution functions.

Parton distribution functions

- Before concentrating on the computing the partonic cross sections, let us discuss the PDF's. In the parton model they only depend on the momentum fraction. But we had seen that structure function depend logarithmically on Q, so we expect that PDF's might also. Indeed that is the case, as we'll see. How does one determine them?
- Crucial at hadron colliders, must be known very accurately. But they cannot be computed from first principles.
- Answer: use their universality, as follows.
  - We need to determine 11 PDF (5 quarks + antiquarks + gluon), *and their uncertainties*
  - Choose with care a set of measurements/observables [e.g. DIS structure functions, or hadron collider cross sections] Each is described as a PDF ⊗ partonic cross sections. We then have the set of equations

$$(O_n \pm \Delta O_n)^{\mathrm{exp}} = \sum_{j=1}^{n_f} \phi_{j/p} \otimes [\hat{\sigma}_{n,j} \pm \delta\sigma_{n,j}]^{\mathrm{th}}$$

  - From the comparison one fits the $\varphi_{j/P}(x,\mu)$.
    - ✓ Various groups, employing slightly different approaches
      - MSTW, CTEQ, NNPDF, GJR, HERAPDF, ABKM…
  - If the partonic calculation is LO, NLO, NNLO etc, then the PDF thus fitted are also labelled LO, NLO etc.
    - ✓ NLO PDF's must be used with NLO calculations. NNLO also ok, LO not

Aside: PDF's as operator matrix elements

- Although they cannot yet be fully computed from first principles, one can give a precise definition of PDF's, in terms of operators. Essentially, these are counting operators (cf $a^\dagger a$ in QM)

$$\phi_{q/p}(\xi) = \frac{1}{4\pi}\int_{-\infty}^{\infty} dy^- e^{-i\xi p^+ y^-} \langle p|\bar{q}(0,y^-,0_T)\gamma^+ q(0,0,0_T)|p\rangle$$

proton state

Quark field

$$p^\pm = \frac{p^0 \pm p^3}{\sqrt{2}}$$

$$p\cdot q = -p^+q^- - p^-q^+ + p_1q_1 + p_2q_2$$

- in a certain gauge. The non-perturbative part sits in the hadronic state in which this counting operator is inserted.
- Benefit: once you have an operator, one can compute its renormalization, and derive an RG equation for it (just like for the coupling constant). This is in fact the DGLAP equation
  - ✓ There are other ways of deriving it. We will see another method later.
- To do so, just replace the proton states with quark states (and keep the operator). At lowest order this is just

$$\delta(1-\xi)$$

- At next order it has the form

$$\frac{\alpha_s}{2\pi}C_F\frac{1}{\varepsilon}\left(\frac{1+\xi^2}{1-\xi}\right)_+ + \dots$$

quark-to-quark splitting function!

  - Plus distribution:

$$\int_0^1 dz\left[\frac{a(z)}{1-z}\right]_+ g(z) = \int_0^1 (g(z)-g(1))\left[\frac{a(z)}{1-z}\right]$$

Parton distribution functions

- The logic is thus very similar to running coupling, we now have "running functions":

$$\mu \frac{d}{d\mu} \phi_{i/H}(x,\mu) = \int_x^1 \frac{dz}{z} P_{ij}(z, \alpha_s(\mu))\, \phi_{j/H}\left(\frac{x}{z}, \mu\right) \qquad \left[\equiv P_{ij} \otimes \phi_{j/H}\right](x,\mu)$$

  - DGLAP equations (we derive them later). P_{ij} are the splitting functions, aka parton evolution kernels. They are now known to NNLO (3rd order)
  - Logic: determine the PDF's at some scale Q, then compute them at all other scales by solving the DGLAP equations.

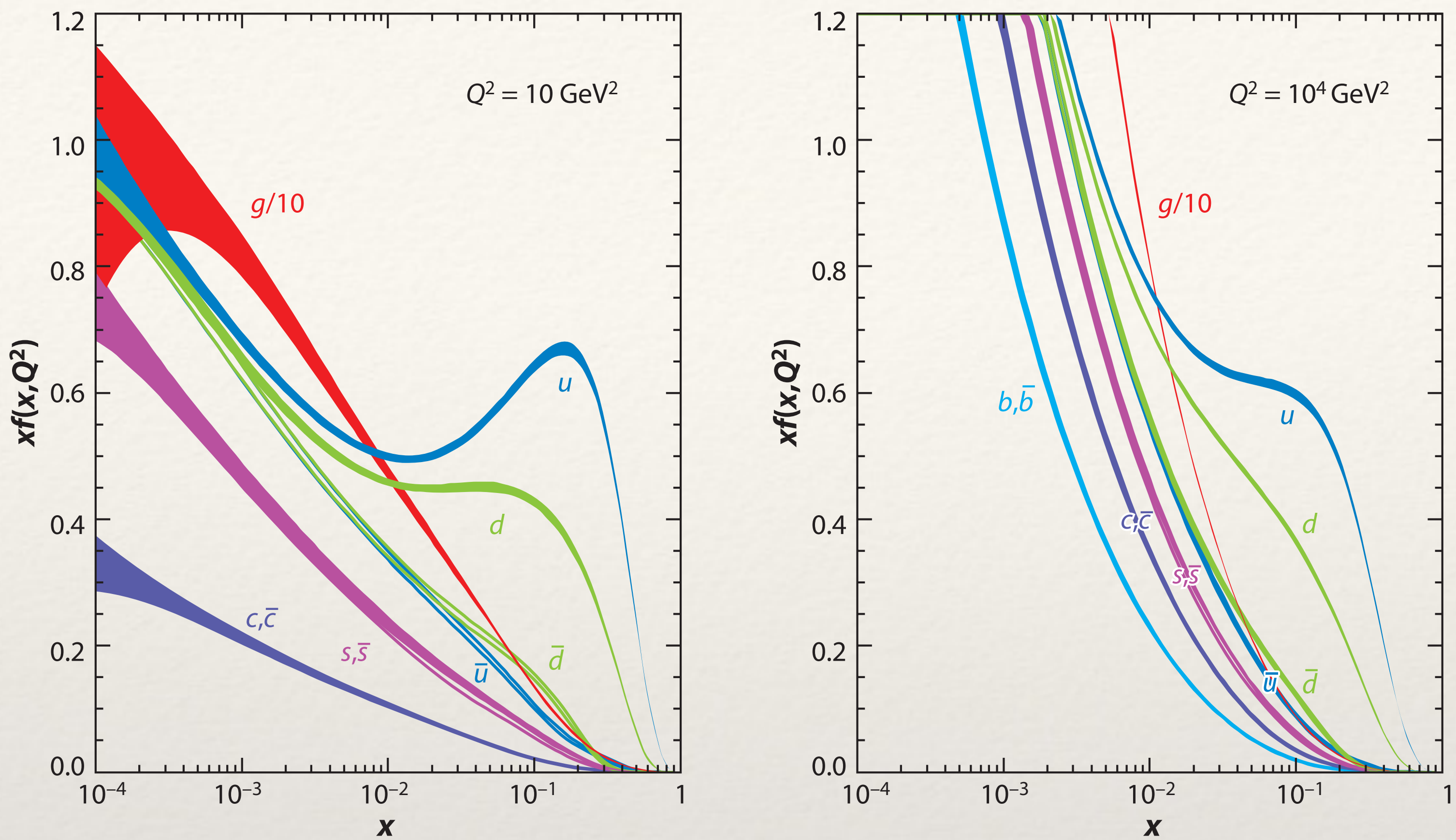
- Note:
  - for LO PDF's, use one-loop splitting and beta-function
  - for NLO PDF's use two-loop splitting and beta-function, etc.
  - in 2004 the three-loop splitting functions [Moch, Vermaseren, Vogt] were computed, so also NNLO sets are now available (NNLO partonic cross sections for DIS, Drell-Yan etc were already available).

- To determine the PDF's from the equation

$$(O_n \pm \Delta O_n)^{\text{exp}} = \sum_{j=1}^{n_f} \phi_{j/p} \otimes [\hat{\sigma}_{n,j} \pm \delta\sigma_{n,j}]^{\text{th}}$$

- one must choose the data on the lhs well.

Form of PDF's



MSTW08 at two values of Q^2

- Notice how evolving the sets to high scale narrows the uncertainty.
  - and how all PDF's grow towards small x: driven by the gluon density in the evolution
- Only u and d still show some bumps: a memory of them being partly valence quarks
- For hadronic collisions one often makes out of the two PDF's the parton luminosity [for "simple enough" cross sections]

$$\sigma_H(s, M^2) = \sum_{a,b} \int_\tau^1 \frac{dx}{x} \mathcal{L}_{ab}(x, M^2) \hat{\sigma}_{ab}(\frac{\tau}{x}, M^2, \alpha_s(M^2)) \qquad \tau = M^2/s$$

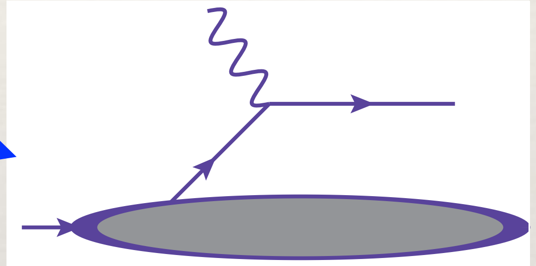
$$\mathcal{L}_{ab}(x, M^2) = \int_x^1 \frac{dz}{z} \phi_{a/p}(z, M^2) \phi_{b/p}(\frac{x}{z}, M^2)$$

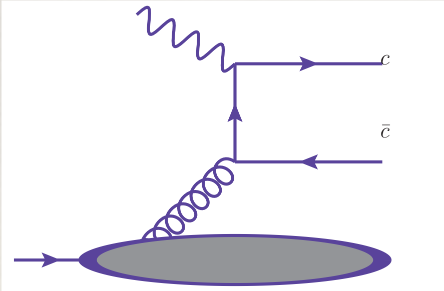
PDF input data

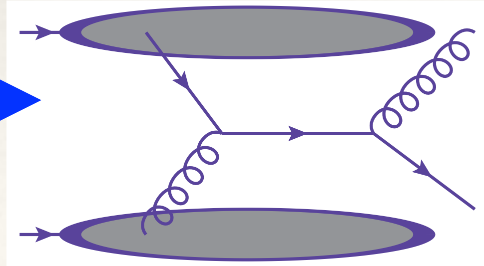
See e.g. Forte, Watt '13

- What data to choose as inputs to fit to?
  - Those that single out particular parton distributions
    - ✓ DIS structure functions most sensitive to valence (u-ū etc) quarks. Prompt photon production sensitive to gluon density etc.
  - Those that provide extra information in certain x ranges (e.g. jet production gives large-x gluon information)

Process	Subprocess	Partons	x range
$\ell^\pm\{p,n\} \to \ell^\pm X$	$\gamma^* q \to q$	$q, \bar{q}, g$	$x \gtrsim 0.01$
$\ell^\pm n/p \to \ell^\pm X$	$\gamma^* d/u \to d/u$	d/u	$x \gtrsim 0.01$
$pp \to \mu^+\mu^- X$	$u\bar{u},\, d\bar{d} \to \gamma^*$	$\bar{q}$	$0.015 \lesssim x \lesssim 0.35$
$pn/pp \to \mu^+\mu^- X$	$(u\bar{d})/(u\bar{u}) \to \gamma^*$	$\bar{d}/\bar{u}$	$0.015 \lesssim x \lesssim 0.35$
$\nu(\bar{\nu})\, N \to \mu^-(\mu^+)\, X$	$W^* q \to q'$	$q, \bar{q}$	$0.01 \lesssim x \lesssim 0.5$
$\nu\, N \to \mu^-\mu^+ X$	$W^* s \to c$	s	$0.01 \lesssim x \lesssim 0.2$
$\bar{\nu}\, N \to \mu^+\mu^- X$	$W^* \bar{s} \to \bar{c}$	$\bar{s}$	$0.01 \lesssim x \lesssim 0.2$
$e^\pm\, p \to e^\pm\, X$	$\gamma^* q \to q$	$g, q, \bar{q}$	$0.0001 \lesssim x \lesssim 0.1$
$e^+\, p \to \bar{\nu}\, X$	$W^+\{d,s\} \to \{u,c\}$	d, s	$x \gtrsim 0.01$
$e^\pm p \to e^\pm c\bar{c}\, X$	$\gamma^* c \to c,\, \gamma^* g \to c\bar{c}$	c, g	$0.0001 \lesssim x \lesssim 0.01$
$e^\pm p \to \text{jet} + X$	$\gamma^* g \to q\bar{q}$	g	$0.01 \lesssim x \lesssim 0.1$
$p\bar{p} \to \text{jet} + X$	$gg, qg, qq \to 2j$	g, q	$0.01 \lesssim x \lesssim 0.5$
$p\bar{p} \to (W^\pm \to \ell^\pm \nu)\, X$	$ud \to W,\, \bar{u}\bar{d} \to W$	$u, d, \bar{u}, \bar{d}$	$x \gtrsim 0.05$
$p\bar{p} \to (Z \to \ell^+\ell^-)\, X$	$uu,\, dd \to Z$	d	$x \gtrsim 0.05$







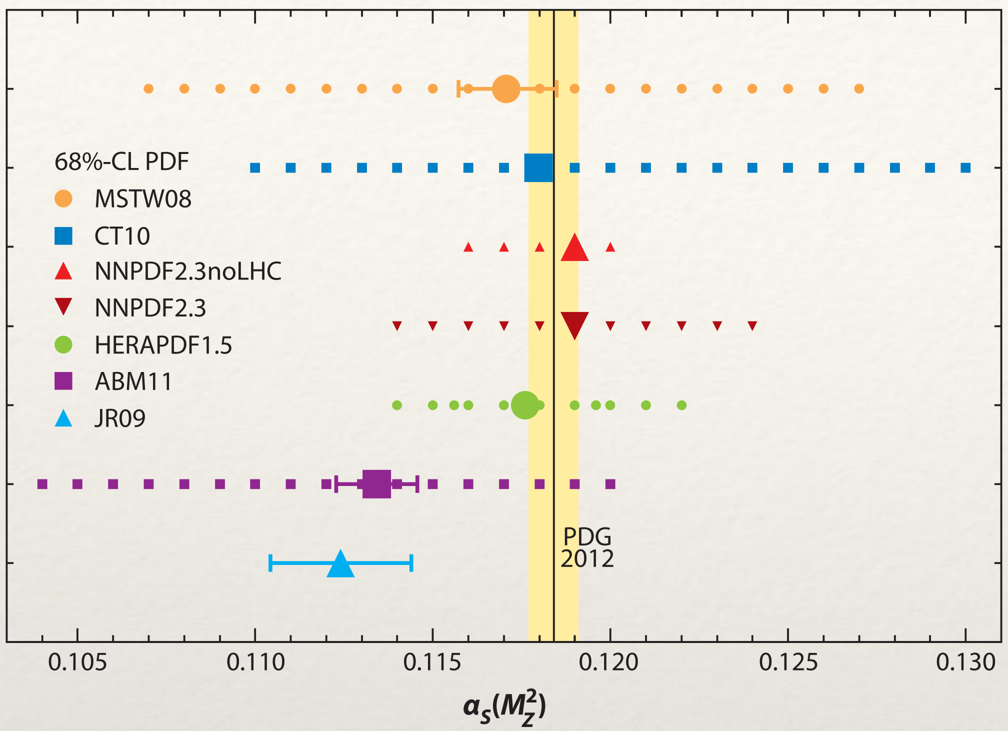
Theory of PDF set formation

- Some theoretical constraints: sum rules
  - Charge sum rule: $\int_0^1 dx(\phi_{i/p}(x,Q^2) - \phi_{\bar{i}/p}(x,Q^2)) = \{2,1,0,0,0\}, \quad i = \{u,d,s,c,b\}$
  - Momentum sum rule: $\sum_{i\in\{g,u,d,s,\dots\}} \int_0^1 dx\, x\, \phi_{i/p}(x,Q^2) = 1$
- In principle, must solve 7x7 matrix evolution equation. But one can cleverly arrange this to have five independent equations, and one 2x2 equation.
  - Subtle issue: how to think about charm and bottom PDF's? In principle they can be computed from the gluon and light flavor PDF's. Also here different approaches, but won't go into details.
- Fitting: not easy. Use χ2 as goodness-of-fit [D_i = data, T_i = theory, V=exp. covariance matrix]
- $$\chi^2 = \sum_{i=1}^{N_{\rm data}} \sum_{j=1}^{N_{\rm data}} (D_i - T_i)(V^{-1})_{ij}(D_j - T_j)$$
- We need a probability measure on the space of functions (in principle ∞-dimensional). To make things tractable, groups choose some parametrization for initial PDF. Many choose a physically motivated form with a limited set of parameters $$\phi_i(x,Q_0^2) = x^{\alpha_i}(1-x)^{\beta_i} g_i(x)$$
- Can also choose a (very redundant) set of unbiased functions, with hundreds of parameters. But then minimization difficult.

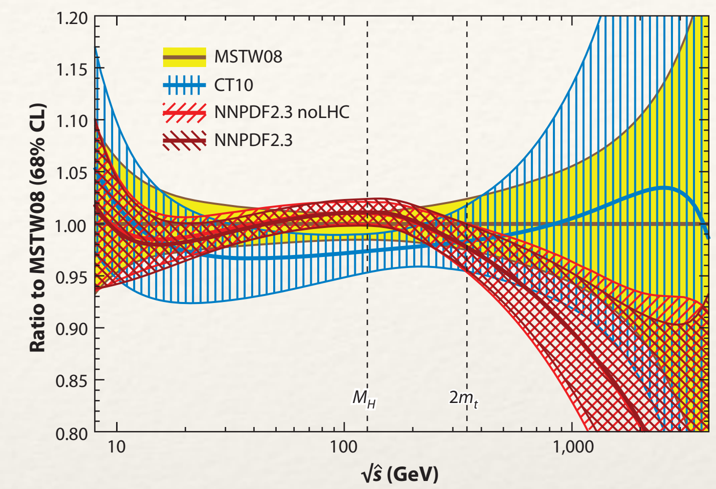
PDF uncertainties

- Two approaches to establish probability measure: 1) Hessian 2) Monte Carlo
  - Hessian: 1-σ confidence interval by moving parameters that make up χ^2 to χ^2_{min}+T. Note that "tolerance" T=1 is theoretically correct, but problematic in practice
    - ✓ Advantage: compact representation of uncertainties.
    - ✓ Product: S_0 central set, and then N_{par} 1-σ error S_i sets.
  - Monte Carlo: create a large number of replica sets
    - ✓ E.g. by constucting data replica's with the right average and covariance
    - ✓ Fit then PDF sets S^k to data replicas.
    - ✓ Now best fit is MC mean over sets S^k., also 1-σ straightforward
  - Both methods agree overall reasonably well. So far uncertainties based only on experimental ones.
- Let us compare some best, most modern NNLO sets

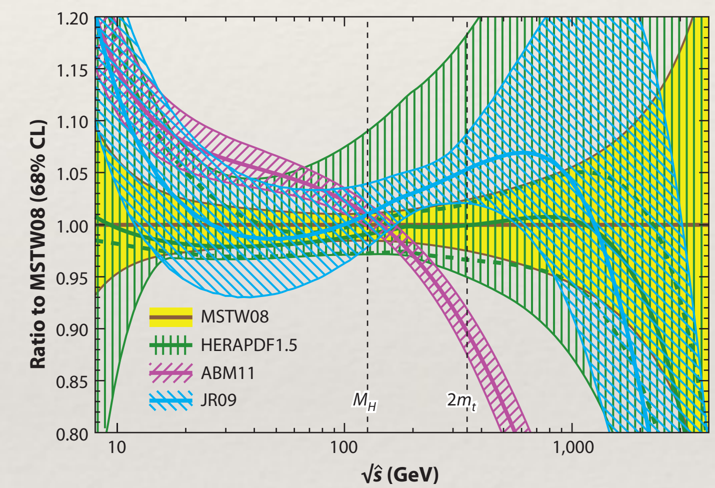
Comparing NNLO PDF sets

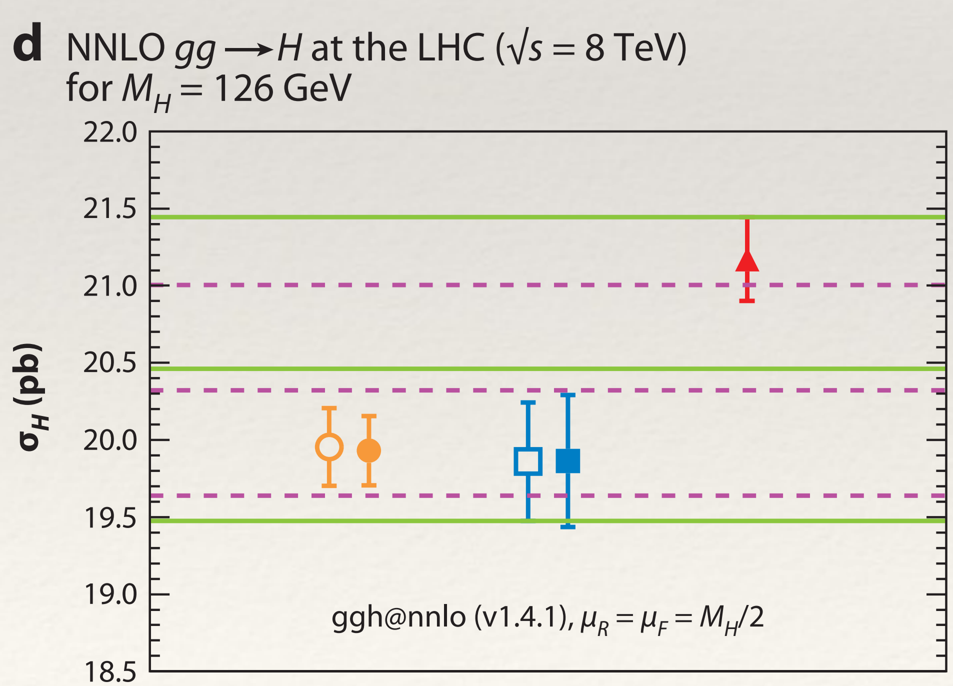


α_s values in NNLO sets



gg luminosities at 8 TeV, relative to MSTW08



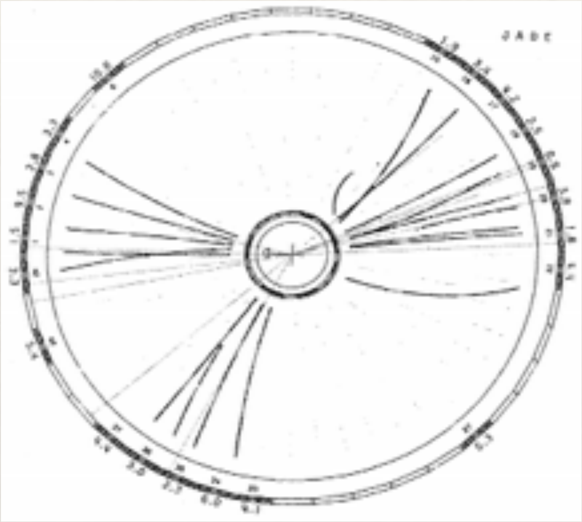


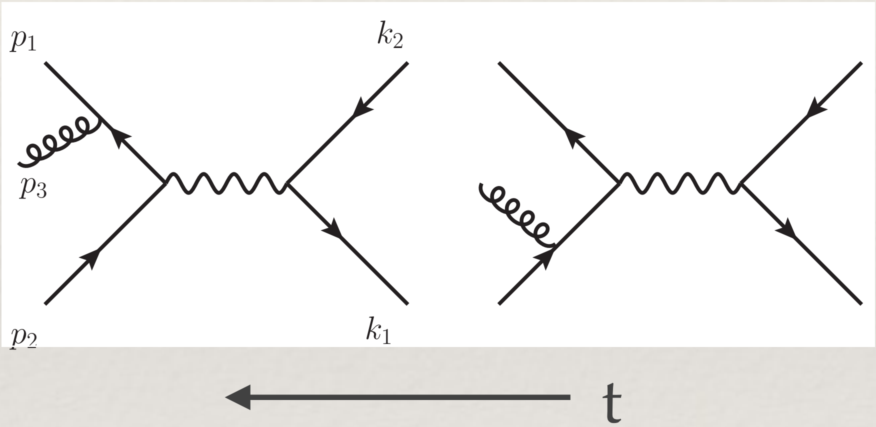
MSTW08 CT10 NNPDF2.3

Impact from LHC

QCD and e^+e^- collisions

- But before turning to hadronic collisions in more details, let us review what QCD does in a simpler setting.
- The cleanest place to study and test QCD is at a e^+e^- collider, where QCD is only active in the final state. We saw already the importance of the R ratio in establishing the number of colors.
- But the R ratio just involves a total cross section: nothing is asked of the final state. It often has an interesting structure, possibly reflecting certain diagrams.





- Two classes of observables do take structure into account
  - Jet cross sections (more on these later)
  - Event shapes

Event shapes - Thrust

- There are *many*. A famous one is Thrust (maximum directed momentum)

$$T = \max_{\hat{n}} \frac{\sum_i |\vec{p}_i \cdot \vec{\hat{n}}|}{\sum_i |\vec{p}_i|}$$

  - Exercise: show that T=1/2 for spherical final states, and T=1 for two very narrow jets.

$$s_{ij} = -(p_i + p_j)^2$$

$$x_i = E_i / E$$

$$x_1 + x_2 + x_3 = 2$$

- Reaction

$$e^+(k_1) + e^-(k_2) \to \gamma(q) \to q(p_1) + \bar{q}(p_2) + g(p_3)$$

- Phase space measure

$$\frac{1}{(2\pi)^5} \int \frac{d^3 p_1}{2E_1} \int \frac{d^3 p_2}{2E_2} \int \frac{d^3 p_3}{2E_3} = \frac{1}{(2\pi)^5} \int \frac{1}{32 q^2} ds_{13} ds_{23} d\phi d\sin\theta d\chi$$

- Squaring the two diagrams and integrating over ϕ and χ

$$\frac{d^3\sigma}{ds_{13} ds_{23} d\sin\theta} = \frac{\alpha_e^2}{8} \frac{\alpha_s}{q^2} \left({x_1}^2 + {x_2}^2\right) \left(2 + \cos^2\theta\right) \frac{1}{s_{13} s_{23}}$$

- Integrating over θ

$$\sigma_T^{-1} \frac{d^2\sigma}{dx_1 dx_2} = \frac{2}{3\pi} \alpha_s \frac{x_1^2 + x_2^2}{(1 - x_1)(1 - x_2)}, \qquad \sigma_T = \frac{4\pi\alpha^2}{3s}$$

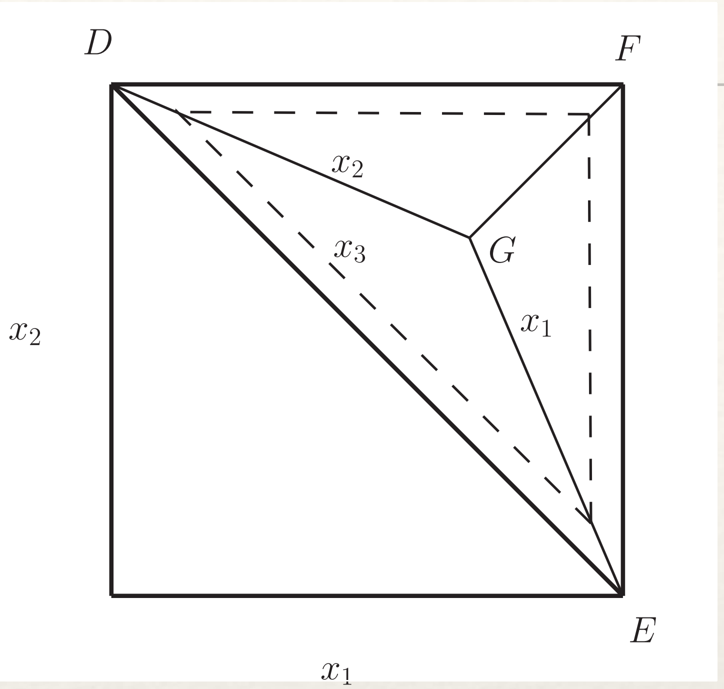
  - ✓ Notice divergences near x_1 or x_2 near 1. But it is not always possible to reach there.

Thrust

For this 3-parton final state, we have $T = \mathrm{max}_i x_i$

Picture of available phase space

- ✓ In each subtriangle, one x is the largest

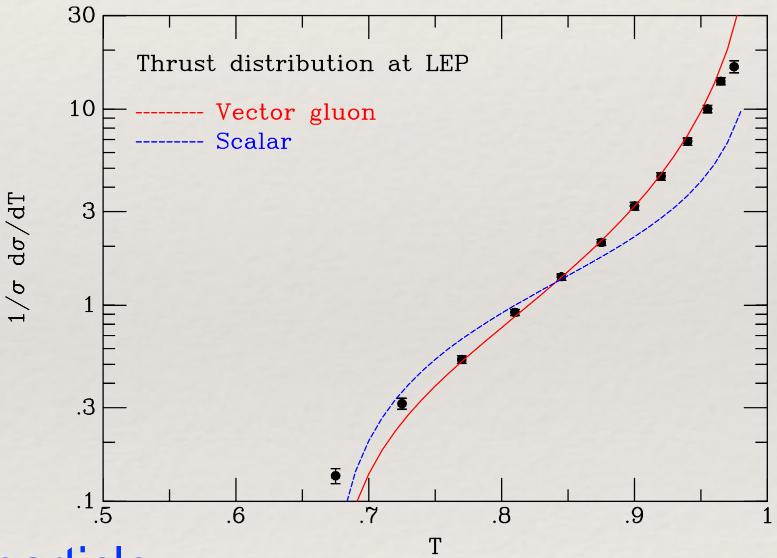


Consider first $T=x_2$ (x_1 is identical)

$$\sigma_T^{-1}\frac{d\sigma}{dT} = \frac{2\alpha_s}{3\pi}\int dx_1 dx_2 \delta(T-x_2)\theta(T-x_1)\theta(T-x_3)$$
$$\times \frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}$$
$$= \frac{2\alpha_s}{3\pi}\left\{\frac{1+T^2}{1-T}\ln\frac{2T-1}{1-T}+\frac{3T^2-14T+8}{2(1-T)}\right\}$$

For $T=x_3$ one finds

$$\sigma_T^{-1}\frac{d\sigma}{dT} = \frac{4\alpha_s}{3\pi}\left\{\frac{1+(1-T)^2}{T}\ln\frac{2T-1}{1-T}+2-3T\right\}$$



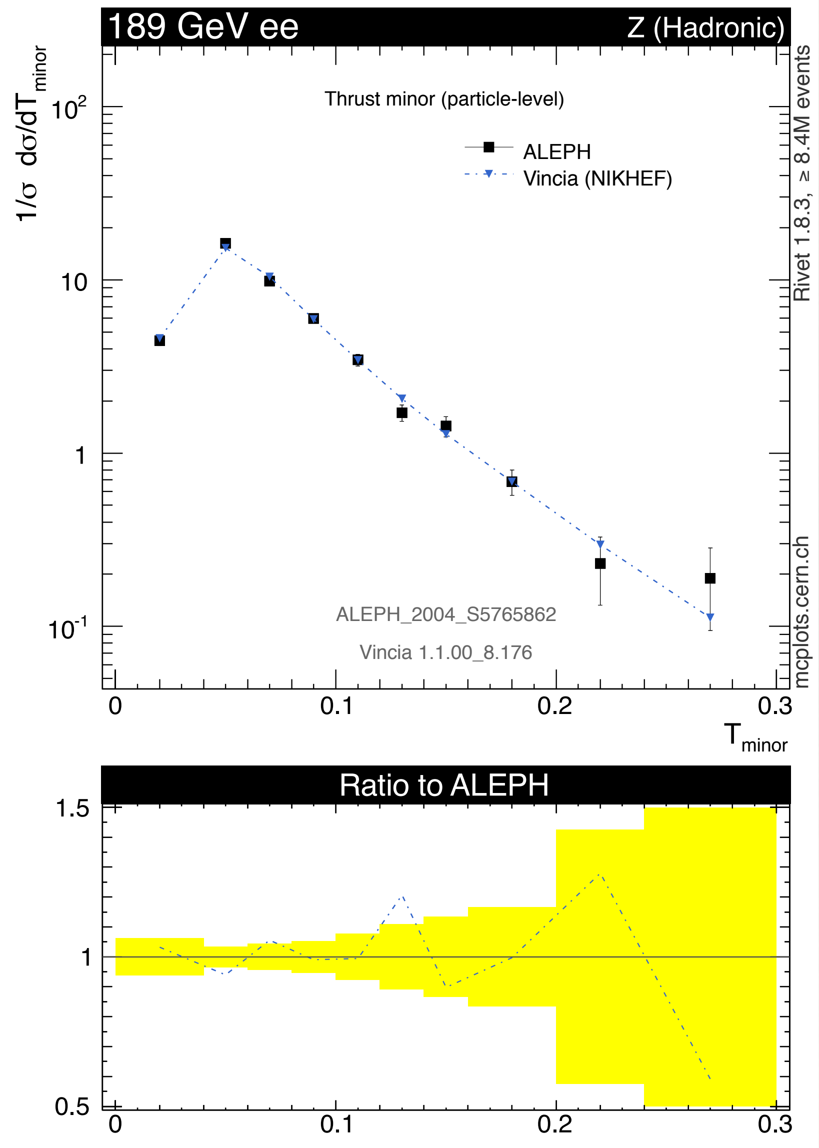
- Integrate this from T=2/3 to 1, find probability that gluons is the most energetic particle

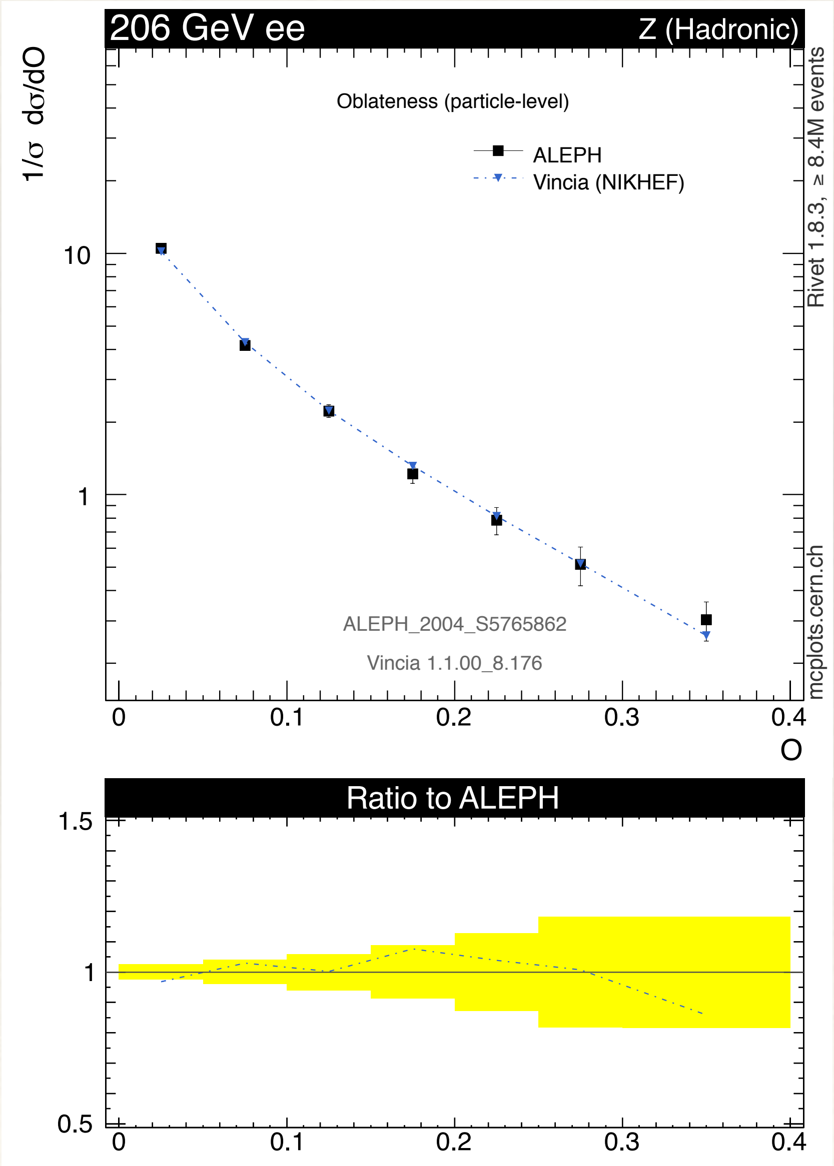
$$\sigma_T^{-1}\int_{2/3}^{1}\frac{d\sigma}{dT}dT = 0.61\frac{\alpha_s}{\pi}$$

- Decreases with increasing Q^2. Probability for (anti)quark to be most energetic is $1-0.61\frac{\alpha_s}{\pi}$

Some other event shapes

- Define extra two axes orthogonal to thrust axis: major (max. energy flow perp) and minor
  - In thrust-major plane: looks like 3 jet event. In thrust-minor plane: looks more like 2-jet event
  - Oblateness: difference of energy flow along major and minor axes
  - there are many others. Check out: http://mcplots.cern.ch





Back to e^+e^-

- Recall the formula for the 3-parton ($qg\bar{q}$) final state

$$\frac{d^2\sigma}{dx_1 dx_2} = \sigma_T \frac{2}{3\pi}\alpha_s \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

- If we wish to compute the NLO QCD correction to the total cross section, we must integrate this over x_1 and x_2 ($=E_1/E$, E_2/E).
  - but there is an obvious problem if these x's are near 1. What kinematic limit does that correspond to?
  - $x_1=1$ means that the quark takes half the cm energy, leaving only half the anti-quark plus gluon. It would work out well if the gluon wasn't there. The gluon can imitate "not being there" by having either zero energy and momentum (infrared), or by being perfectly collinear with the massless antiquark

$$p_2^2 = 0,\ p_3^2 = 0, \qquad (p_2 + p_3)^2 = 2 p_2 \cdot p_3 = 0 \ \text{ iff } \ p_3^\mu = z p_2^\mu$$

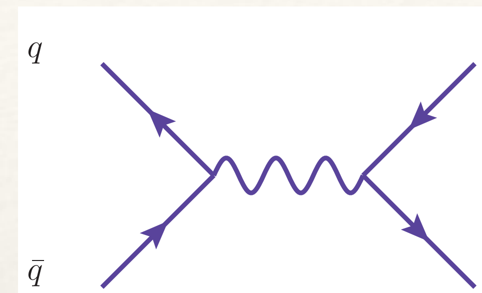
  - Clearly these are divergent situations
    - ✓ Infrared divergence ($p_3^\mu \longrightarrow 0$) and collinear divergence ($p_3^\mu \longrightarrow z p_2^\mu$)
  - Let us see how the occur in practice. We regularized UV divergence using dimensional regularization
  - It turns out DimReg can also be used for IR and COL divergences

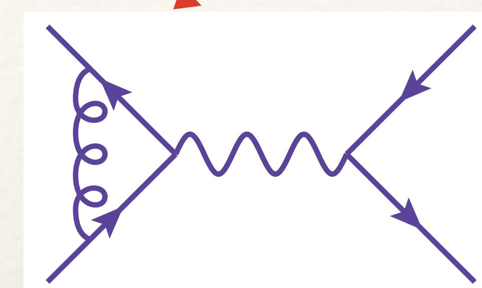
General comment about LO and NLO

- For the cross section one must compute |M|2. How do the one-loop and the one-emission graph fit into this? Consider a process with a 2-particle final state, e.g. $e^+e^- \to q\bar{q}$

- Then we have

$$M = M_2^{(0)} + g^2 M_2^{(2),V} + \mathcal{O}(g^4)$$



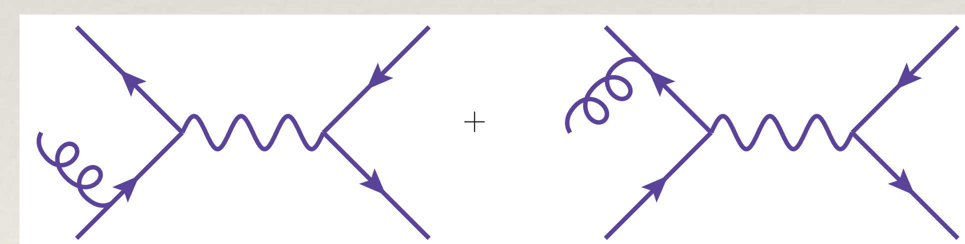


- At NLO, the loop amplitude enters *through the interference* with the lowest order amplitude:

$$|M|^2 = |M_2^{(0)}|^2 + g^2\left((M_2^{(0)})^* M_2^{(2),V} + c.c.\right) + \mathcal{O}(g^4)$$

- For the radiative contribution (aka bremstrahlung) we have a 3-particle final state, enters cross section as a pure qure

$$M = gM_3^{(1),R} + \mathcal{O}(g^2) \qquad |M|^2 = g^2|M_3^{(1),R}|^2 + \mathcal{O}(g^4)$$



- Total contribution at order g^2 enters as a pure square

$$|M|^2 = |M_2^{(0)}|^2 + g^2\left[2\mathrm{Re}\left\{M_2^{(0)})^* M_2^{(2),V}\right\} + |M_3^{(1),R}|^2\right]$$

- Good to keep in mind.

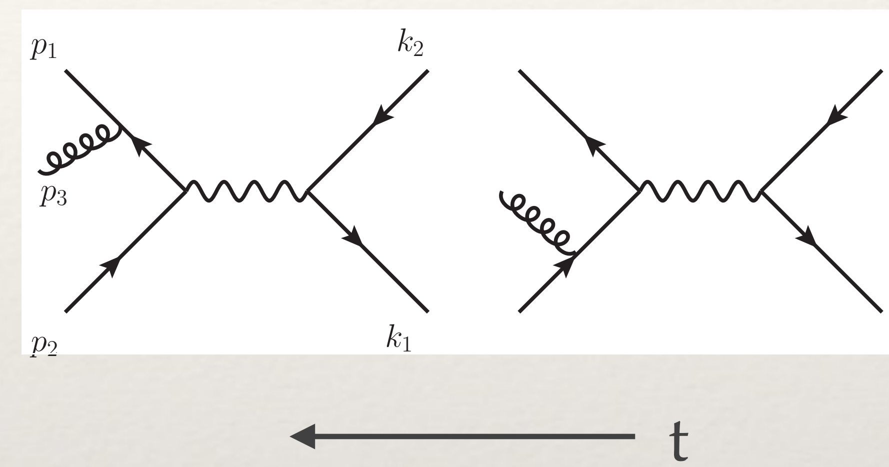
Final state IR and COL divergences

- To use DimReg, we should really have written the final state phase space measure also in n=4-2ε dimensions

$$\int \frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2}\frac{d^3p_3}{2E_3} \rightarrow \int \frac{d^{n-1}p_1}{2E_1}\frac{d^{n-1}p_2}{2E_2}\frac{d^{n-1}p_3}{2E_3}$$

- Then we find

$$\sigma_{qg\bar{q}}(\varepsilon) = \sigma_T 3\frac{\alpha_s}{2\pi}\sum_f Q_f^2 H(\varepsilon)\int dx_1 dx_2 \frac{x_1^2+x_2^2-\varepsilon(2-x_1-x_2)}{(1-x_1)^{1+\varepsilon}(1-x_2)^{1+\varepsilon}}$$



- which yields

$$\sigma_{qg\bar{q}}(\varepsilon) = \sigma_T 3\frac{\alpha_s}{2\pi}\sum_f Q_f^2 H(\varepsilon)\left[\frac{2}{\varepsilon^2}+\frac{3}{\varepsilon}+\frac{19}{2}+\mathcal{O}(\varepsilon)\right]$$

- Double and single poles in ε!! From IR and COL regios of phase space. How do they cancel? Do they cancel?
    - Spoiler: no fixing by renormalization of couplings etc.

Virtual contribution

- But this is not the only contribution to NLO, we also need the virtual contribution. The result of the doing the loop integral in n-dimensions is

$$\sigma_{q\bar{q},V}(\varepsilon) = \sigma_T 3 \frac{\alpha_s C_F}{2\pi} \sum_f Q_f^2 H(\varepsilon) \left[-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + \mathcal{O}(\varepsilon) \right]$$

- We just found

$$\sigma_{qg\bar{q},R}(\varepsilon) = \sigma_T 3 \frac{\alpha_s}{2\pi} \sum_f Q_f^2 H(\varepsilon) \left[\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + \frac{19}{2} + \mathcal{O}(\varepsilon) \right]$$

- Add up and add the LO contribution

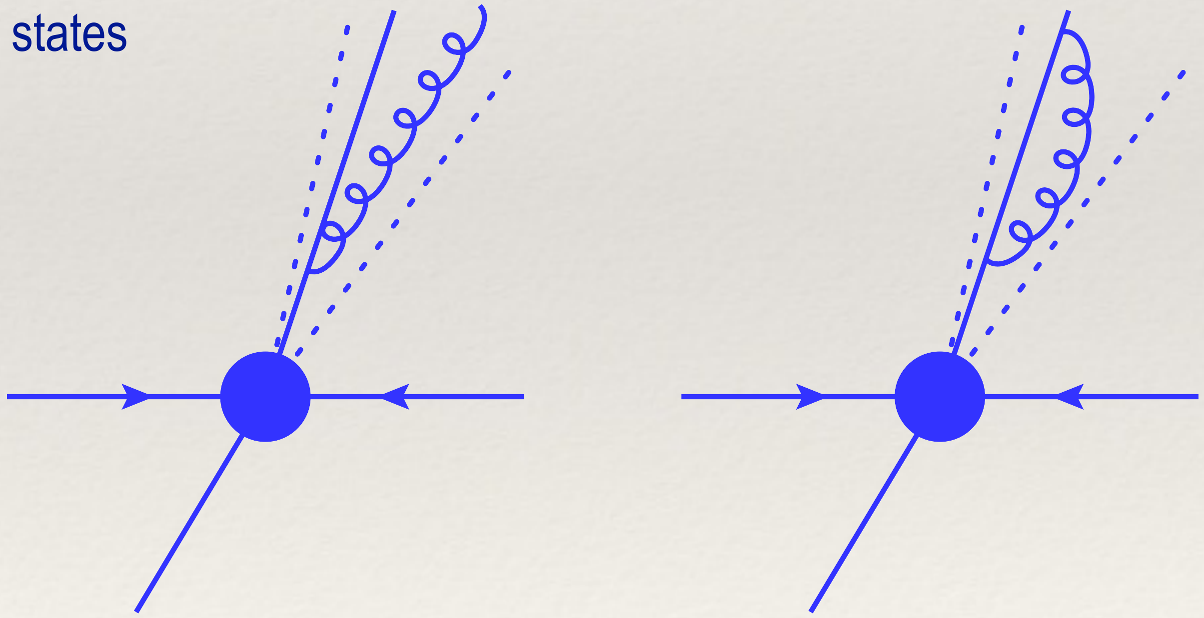
$$\sigma_{NLO} = \sigma_T 3 \sum_f Q_f^2 \left[1 + \frac{\alpha_s}{\pi} \right]$$

- The IR and COL divergent just cancelled! All we had to do was add the real and virtual contributions.

- This is in fact a very general phenomenon, and it known as the KLN theorem.

Kinoshita-Lee-Nauenberg theorem

- Theorem not only for QCD, but very generally for quantum mechanical transition probabilities
- In essence it says that if one computes the transition probability not just to one very specific state, but to a collection of degenerate states [E-ΔE, E+ΔE] one gets a finite answer.
    - Clearly, a state of just 2 quarks and a state with 2 quarks plus a soft or collinear gluon are degenerate.
    - This is why inclusive, or semi-inclusive cross sections are finite
    - But is also why we look at jets.
        - ✓ A quark with a correction and a quark with a soft of collinear gluon are part of the same jet
            - so a jet defines a collection of degenerate states
            - also event shapes are ***infrared-safe***

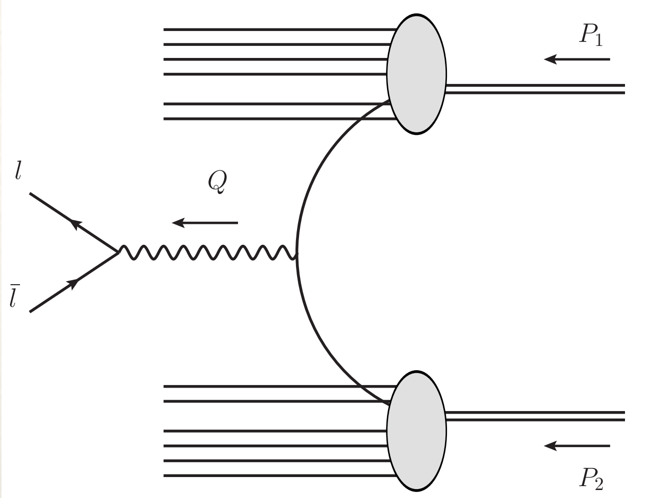


- Now we turn to hadronic collisions.

Drell-Yan

- Production of lepton pair in hadronic collision, either through photon, W or Z

$$p + \bar{p}/p \to l + \bar{l} + X$$



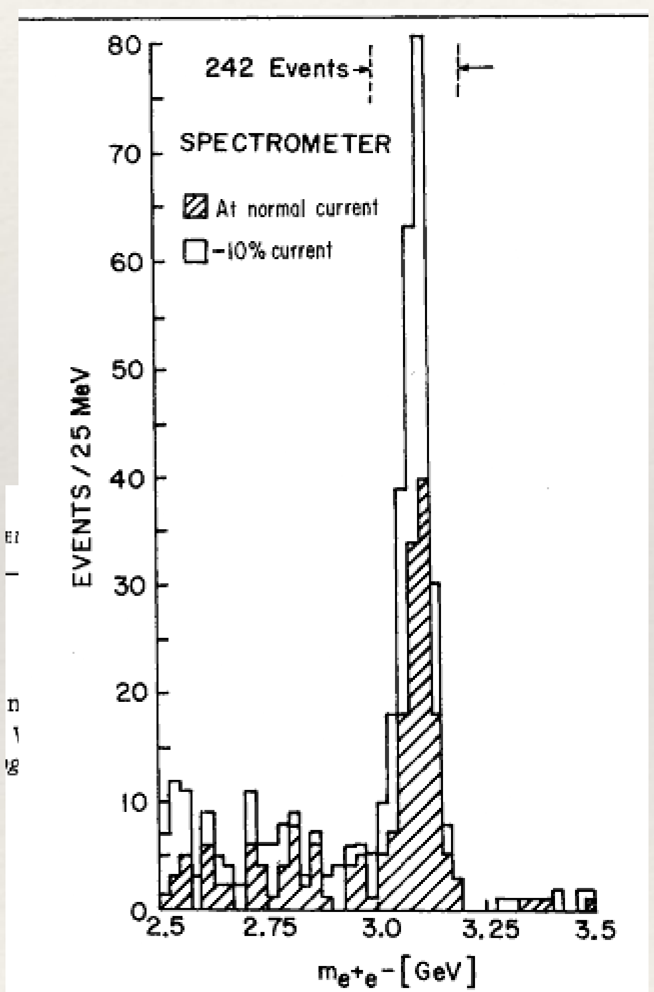
- Storied physics background (next slide)

- These days: often a "theory" laboratory. All the key complications without many external legs. Higgs production is just "Drell-Yan with initial state gluons".

- To illustrate typical issues in QCD higher-order calculations, we shall compute Drell-Yan to NLO.
    - Infrared and collinear divergences, KLN theorem, factorization

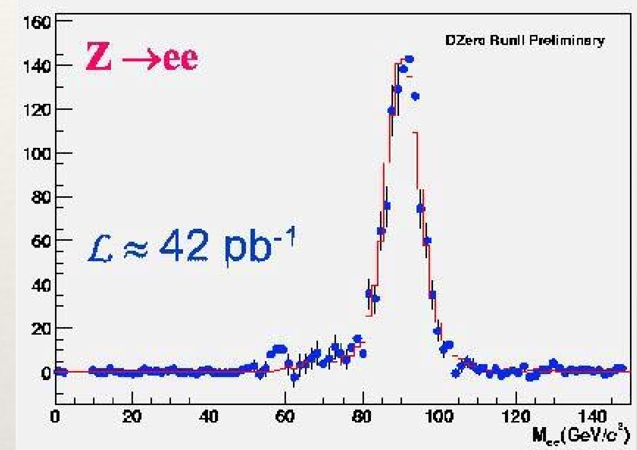
Drell-Yan history

To predict DY cross section
could use the PDF's from DIS.
This worked well.



J/Psi discovery
at BNL AGS and SLAC in '74

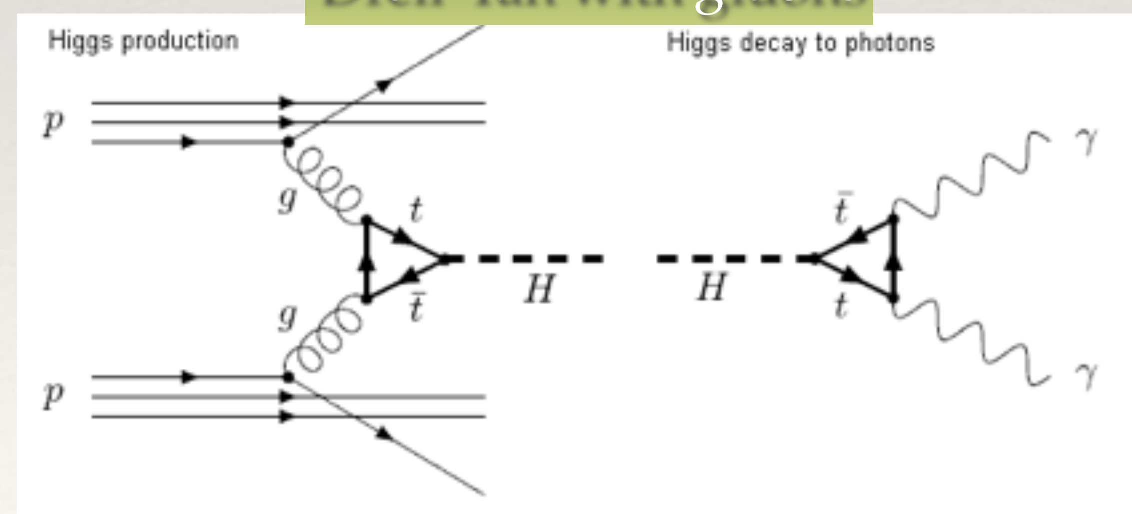
$p+N \to \Upsilon\,(b\bar{b})+X$
bottom discovery '77
Fermilab E288 exp.



Not discovery but a nice peak!

$p+\bar{p} \to W/Z+X$
W/Z discovery '83
at CERN UA1/UA2

Last but not least:
Drell-Yan with gluons

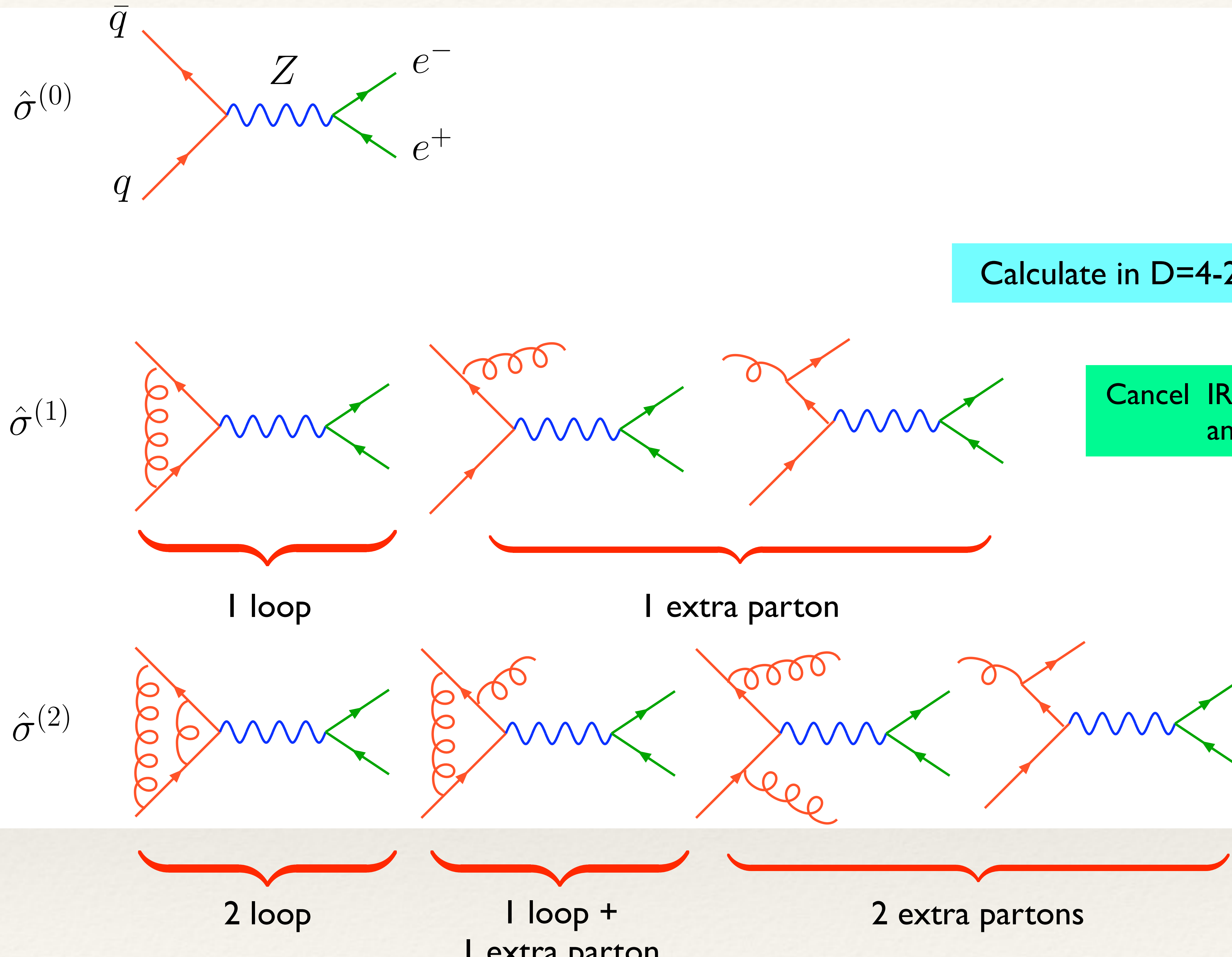


Recall: LO and higher order amplitudes

LO

NLO

NNLO



Calculate in D=4-2ε dimensions

Cancel IR poles $1/\varepsilon^2$ before anything else

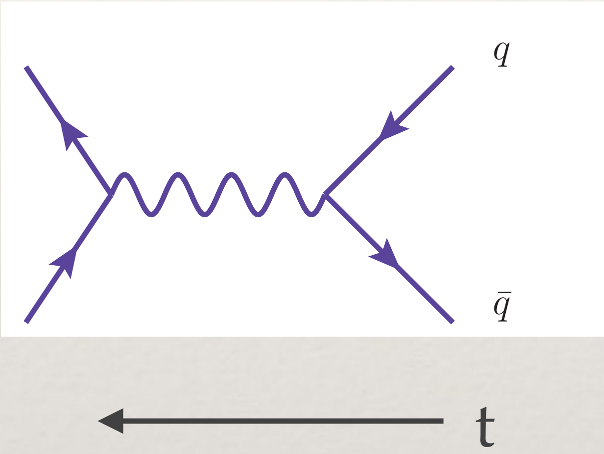
Cancel IR poles $1/\varepsilon^4$ etc before anything else; hard!

Drell-Yan at LO

- Process: production of lepton pair of invariant mass Q^2, plus anything else. Leading order partonic cross section

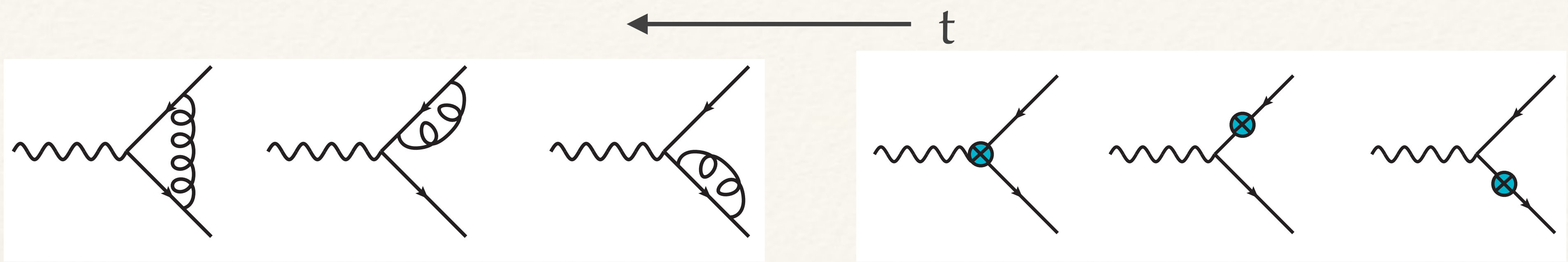
$$\frac{d\sigma^{(0)}_{q\bar{q}}(Q^2)}{dQ^2} = \left[\frac{4\pi\alpha^2}{3N_c(Q^2)^2}\right]\delta\left(1-\frac{Q^2}{\hat{s}}\right).$$

  - Comes from one diagram



    - Exercise: can you motivate each element of this formula?

NLO Drell-Yan: virtual diagrams



- Time here from right to left (apologies). 6 diagrams, but we are in luck
  - Sum of three “counterterm contributions” = 0
    - because QCD corrections should not affect the electric charge of the quark
  - Self-energy diagrams = 0, leaves only triangle graph (leftmost one). We suspect (from the e+e- case) that the loop integral will produce IR and COL divergences/
    - ✓ Indeed we find

$$\frac{d\sigma^{(1)}_{q\bar{q}}}{dQ^2}\Big|_{\text{virtual}} = \sigma^{(0)}_{\gamma} Q_f^2 \frac{1}{2\pi} C_2(R) \left(\frac{4\pi\mu^2}{\hat{s}}\right)^{-\varepsilon/2} \frac{\Gamma(1+\varepsilon/2)}{\Gamma(1+\varepsilon)}$$
$$\times \left[-\frac{8}{\varepsilon^2} + \frac{6}{\varepsilon} - 8 + \frac{2\pi^2}{3} + O(\varepsilon)\right] \delta(1-x) \,, \qquad x = \frac{Q^2}{s}$$

    - ✓ Observe again double and single pole

NLO Drell-Yan: real diagrams

- Now there are two diagrams, with a gluon radiated of either incoming quark. Result

$$\frac{d\sigma^{(1)}_{q\bar{q}}}{dQ^2}\Big|_{\rm real} = \sigma^{(0)}_{\gamma} Q_f^2 \frac{1}{2\pi} C_2(R) \left(\frac{4\pi\mu^2}{\hat{s}}\right)^{-\varepsilon/2} \frac{\Gamma(1+\varepsilon/2)}{\Gamma(1+\varepsilon)} \frac{4}{\varepsilon} \times \left[2x^{1-\varepsilon/2}(1-x)^{-1+\varepsilon} + x^{-\varepsilon/2}(1-x)^{1+\varepsilon}\right]$$

- We see a single pole, but no double pole! Trouble with KLN?
- No. To see this, express the functions of x in terms of “plus-distributions”

$$\frac{1}{(1-x)^{1-\varepsilon}} = \frac{1}{\varepsilon}\delta(1-x) + \left[\frac{1}{1-x}\right]_+ + \varepsilon\left[\frac{\ln(1-x)}{1-x}\right]_+ + O(\varepsilon^2)$$

  - Now do get double pole

Proof: use a test function F

$$\int_0^1 dx\, \frac{F(x)}{(1-x)^{1-\varepsilon}} = \frac{1}{\varepsilon}\int_0^1 dx\, F(x)\delta(1-x) + \int_0^1 dx\, \frac{F(x)-F(1)}{1-x} + \varepsilon \int_0^1 dx [F(x)-F(1)]\frac{\ln(1-x)}{1-x} + O(\varepsilon^2)$$

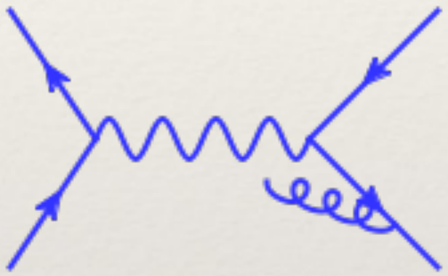
- Use, and add to virtual. Result

$$\frac{d\sigma^{(1)}_{q\bar{q}}}{dQ^2} = \sigma^{(0)}_{\gamma} Q_f^2 \frac{1}{2\pi} C_2(R) \left(\frac{4\pi\mu^2}{\hat{s}}\right)^{-\varepsilon/2} \frac{\Gamma(1+\varepsilon/2)}{\Gamma(1+\varepsilon)} \times \left\{\frac{4}{\varepsilon}\left((1+x^2)\left[\frac{1}{1-x}\right]_+ + \frac{3}{2}\delta(1-x)\right) + 4(1+x^2)\left[\frac{\ln(1-x)}{1-x}\right]_+ - 2(1+x^2)\frac{\ln x}{1-x} + (4\zeta(2)-8)\delta(1-x) + O(\varepsilon)\right\}$$

NLO Drell-Yan: sum of real and virtual

- Again, now expressed in terms of the splitting function $P_{qq}(x)$.

$$\frac{d\sigma^{(1)}_{q\bar{q}}}{dQ^2} = \sigma^{(0)}_{\gamma} \frac{Q_f^2}{2\pi} C_\varepsilon \times \left\{ \frac{4}{\varepsilon} P_{qq}(x) + 4(1+x^2) \left[\frac{\ln(1-x)}{1-x}\right]_+ - 2(1+x^2)\frac{\ln x}{1-x} + (4\zeta(2) - 8)\delta(1-x) \right\}$$



- Even with KLN helping, there is a remaining divergence!
  - ✓ Initial state collinear divergence
- How to get rid of it?
- Answer: very analogous to use of Z-factor for renormalization of coupling. Renormalize the PDF's as

$$\phi_{q/A}(\xi) = \int_0^1 dz \int_0^1 dy\, \phi_{q/A}(y, \mu_F) T^{-1}_{qq}(z, \mu_F) \delta(\xi - zy)$$

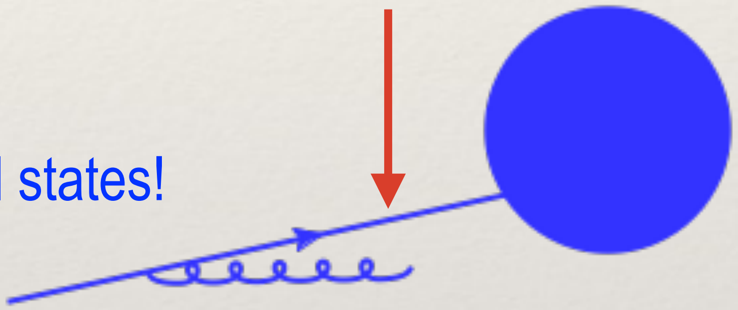
- To first order

$$\phi_{q/A}(\xi) = \phi_{q/A}(\xi, \mu_F) - \int_\xi^1 \frac{dz}{z} \phi_{q/A}\left(\frac{\xi}{z}, \mu_F\right) \times \left\{ \frac{\alpha_s}{2\pi} \frac{1}{\varepsilon} P_{qq}(z) \right\}$$

  - ✓ This new divergence cancels the above one.
- Notice: this new contribution shows no information about this being the Drell-Yan process

QCD Factorization

- What you just witnessed is called "factorization". It turns out:
  - For any process this removes the remaining initial state collinear divergence!
    - ✓ Works to all orders [Collins, Soper Sterman]
    - ✓ KLN theorem helps cancel all IR and all final state collinear divergences
- As a result, the "renormalized" PDF depends on μF. How? It obeys now the DGLAP equation.
- Why does KLN not solve this?
  - Answer: the initial state is precisely defined, there is no set of degenerate initial states!



- What is the physical picture behind this?

$$\phi_{q/A}(\xi) = \phi_{q/A}(\xi,\mu_F) - \int_\xi^1 \frac{dz}{z}\phi_{q/A}\left(\frac{\xi}{z},\mu_F\right) \times \left\{\frac{\alpha_s}{2\pi}\frac{1}{\varepsilon}P_{qq}(z)\right\}$$

- Consider the indicated propagator. If the gluon is very collinear, the virtuality of that line is very small.
  - Therefore, that state could be very long-lived: the gluon could have been radiated off long, long before the hard scattering. The very collinear gluon thus should be grouped with the proton.

Upshot and NLO status

- Congratulations, you have now really understood (?) hadronic collisions.
- For other reactions the story is precisely the same! [The formula's are a lot longer]
- The whole NLO calculational approach has been automatized
  - Tremendous progress
  - Efforts/codes: aMC@NLO, POWHEG-Box, many others…

Fixed order as Monte Carlo integral

- Monte Carlo integration $$I = \int_0^1 dx\, f(x) = \frac{1}{N}\sum_{i=1}^{N} f(x_i) + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right)$$
  - directly extendable to a multi-dimensional (order 5-30)

$$\sigma = K \int d\mathrm{PS}(p_1, \ldots, p_n)\, |\mathcal{M}(p_1, \ldots, p_n)|^2 = \sum_{i=1}^{N} W(\{x_i\})$$

- xi must be random numbers from uniform distribution
  - For each event (“x”) the weight changes
  - Weight W = |M|2 times jacobians → Fill histogram for each “event” {xi}
    - ✓ Likely final states have large weight and v.v.
- “Event generation”: can we similate such functions where all events have weight 1, but more likely ones occur often, etc.
  - Just like Nature.
- Two unweightings
  - Hit-and-miss, and the veto-algorithm

Elementary MC

- Unweighting (sampling a distribution)
  - Hit and miss : an exercise.
  - Veto algorithm
- Consider a process in which branchings take place (radioactive decays, or parton showers).
  - f(t): chance of branching for time t. Then probability for branching at time t

$$P(t) = -\frac{d\Delta(t)}{dt} = f(t)\Delta(t)$$

  - Δ(t): probability that no branching has occurred until t. [“Sudakov form factor”]

$$P(t) = f(t) \exp\left\{ -\int_0^t dt' f(t') \right\}$$

  - Prototype for parton shower! How to imitate this function with random points? Depends:
    - ✓ If I can find the primitive F(t) of f(t), then
      - pick a random number between 0 and 1
      - compute $t = F^{-1}(F(0) - \ln R)$
    - ✓ If I cannot, find an upper bound g(t) > f(t), and use the veto algorithm

Standard Veto Algorithm

- Example
  - $f(t) = t$, $F^{-1}(x) = (2x)^{(1/2)}$, $g(t) = t+1$, G the primitive of g
  - Algorithm
    1. start with i=0, $t_0=0$
    2. i++, then select t_i according to $t = G^{-1}(\, G(t_{i-1}) - \ln R\,)$, $t_i > t_{i-1}$.
    3. compare a new R with $f(t_i)/g(t_i)$. If $f(t_i)/g(t_i) < R$, return to 2
    4. otherwise accept t_i.
  - Result: nice agreement between analytical and veto-algorithm result