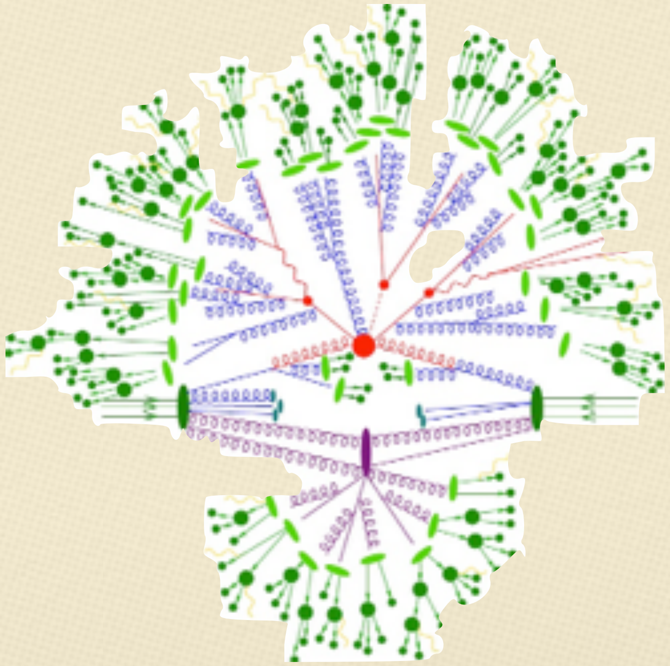
$$\frac{1}{4}\mathrm{Tr}[G^2] - \bar{\psi}(\not{D} - m)\psi$$ ☞ 

QCD

Eric Laenen
CERN school 2014

Lecture 3: modern methods in QCD

Summary of last time

- We looked at QCD final states in e+e- collision
  - Total cross section, event shapes, jets.
- We saw that IR and COL divergences occur at intermediate stages, but they cancel when adding virtual and real contributions
- Underlying this fortune is the KLN theorem (transitions between collections of degenerate states are finite)
- Drell-Yan, very important reaction. At NLO:
  - All IR divergences cancel (KLN)
  - All final state collinear divergences cancel (KLN)
  - Initial state collinear divergences cancel after "renormalization" of the PDF's (Factorization)
    - ✓ This also makes the PDF's scale dependent, leading to the DGLAP equations. Feature, not bug!
  - This holds in fact beyond NLO to all orders [Collins, Soper, Sterman]
- At the end we began to discuss some concepts of the parton shower
    - ✓ Already stressed the importance of the Sudakov form factor = the non-emission probability

IR and COL divergences for multi-differential cross sections

- Thanks to the KLN theorem and thanks to factorization, we can consistently cancel all IR and COL divergences for total cross sections. But we'd like to be able to compute more differential cross sections to higher order in QCD.

$$\frac{d^{3n}\sigma}{d^3p_1 \ldots d^3p_n}$$

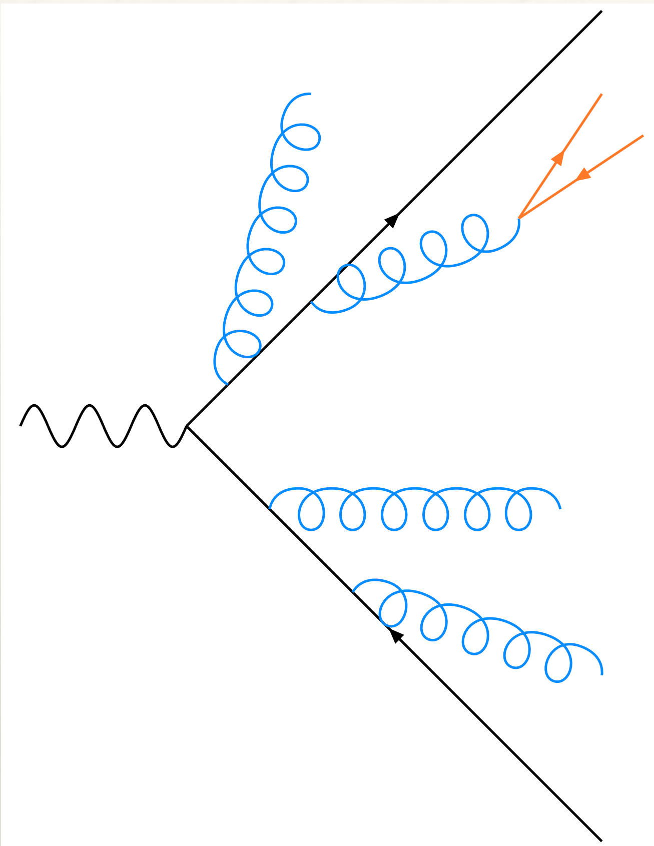
  - Can we get rid of IR and COL divergences here also?

$$d\sigma_{NLO} \quad = \quad \int_{d\Phi_{n+1}} \left(d\sigma^R_{NLO} \; - \; d\sigma^S_{NLO} \right) \; + \; \left[\int_{d\Phi_n} d\sigma^V_{NLO} \; + \; \int_{d\Phi_n} \left(\int_{d\Phi_1} d\sigma^S_{NLO} \right) \right]$$

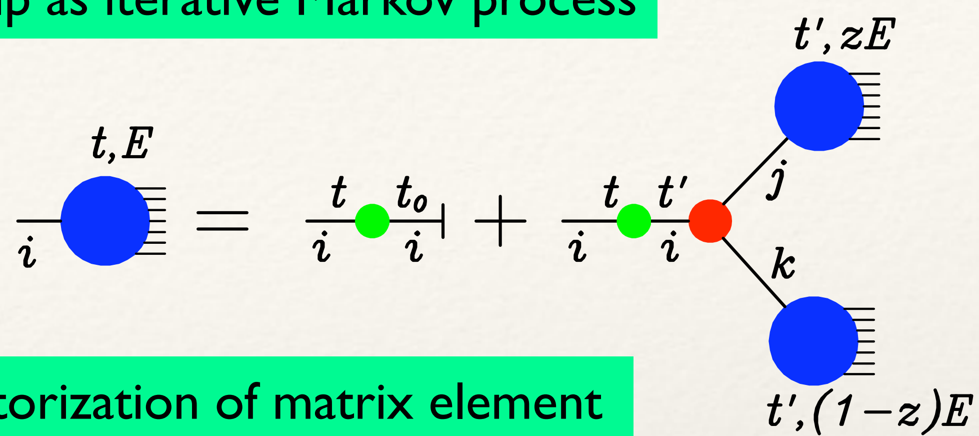
- Subtraction schemes (Ellis, Ross, Terrano): find clever $d\sigma^S_{NLO}$
  - Dipole (Catani, Seymour): emitter + spectator + soft-collinear parton
  - Antenna (Kosower): 2 radiators + soft-collinear parton
  - FKS (Frixione, Kunszt, Signer): isolate each singularity with projectors. Use plus prescriptions to define IR-finite part.
- Phase space slicing (Fabricius et al, Giele, Glover, Kosower; EL, Keller; Harris, Owens,..): define near-IR regions through small cut-off. Approximate integrands in these small regions, and handle them analytically ⟶ 1/ε poles.
  - Outside, do numerical integrals

Parton branchings

An approximate description of full matrix elements when radiation is mostly collinear and soft



Set up as iterative Markov process



based on factorization of matrix element and phase

Sudakov form factor: probability of no emission between two emissions (related to virtual graphs)

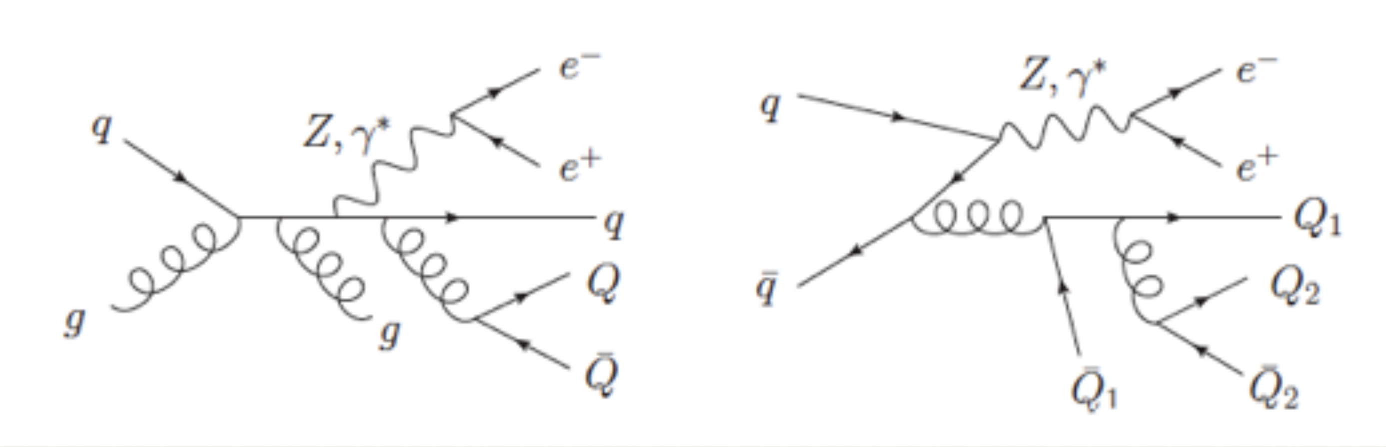
$$\Delta_i(t_1,t_2) = \exp\left[-\sum_{(jk)}\int_{t_2}^{t_1}\frac{dt}{t}\,\frac{\alpha_S(t)}{2\pi}\int dz\,P_{i,jk}(z)\int\frac{d\varphi}{2\pi}\right]$$

In contrast to NLO, NNLO, with parton showers the number of partons per event is not fixed. But it is still a unitary process

$$\sum_n P_n(t) = 1$$

Matrix element generators

Good description when no two partons are too collinear, or one is soft



- Calculate full tree-level matrix element using
  - Diagrams
    - ✓ Helicity amplitudes (MadEvent, Amegic++), squared amplitudes (Comphep)
  - Always fixed number of partons
    - ✓ MadEvent: $2 \longrightarrow 6$, HELAC, Alpgen, Amegic++ $2 \longrightarrow 8$

Matching NLO to PS

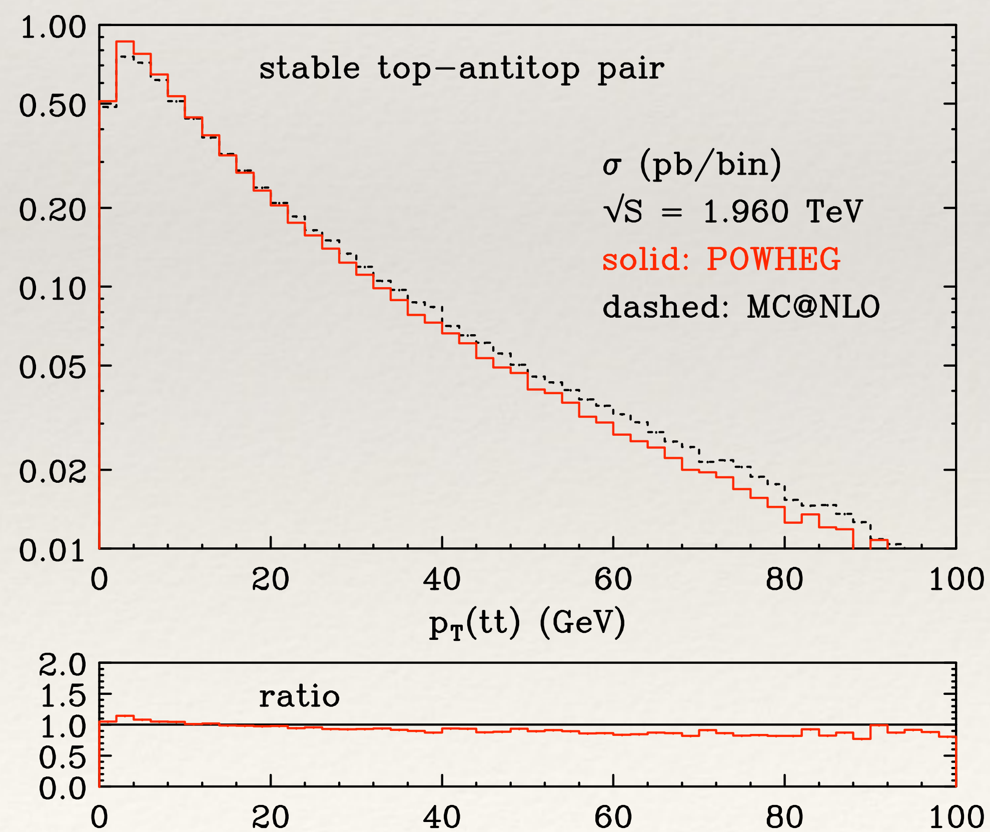
- Match to avoid double counting
  - 1 parton emission from NLO and PS should be counted once
    - virtual part of NLO and Sudakov should not overlap either
  - two main approaches:

Frixione, Webber; Nason

✓ MC@NLO exact to NLO (no NNLO “overshoot”), but can have negative weights

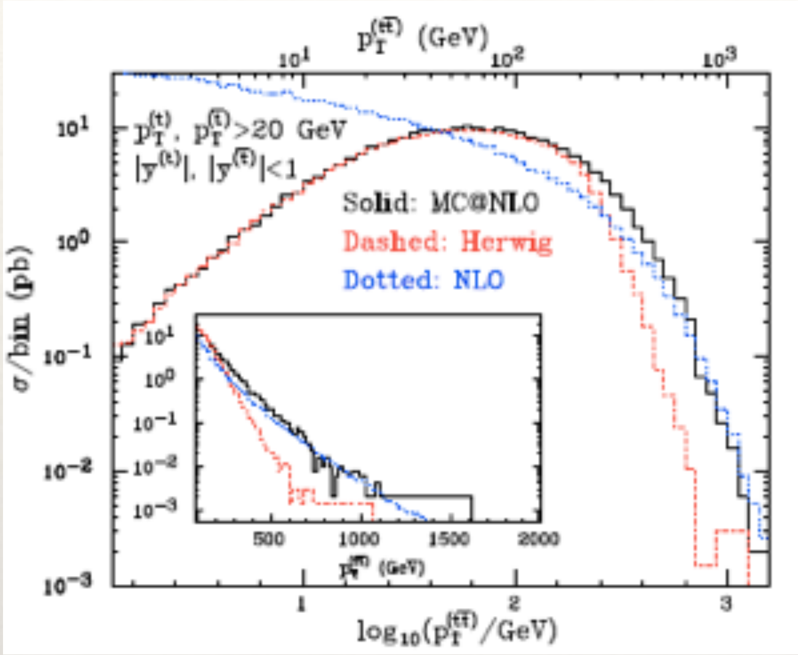
✓ POWHEG insists on having positive weights, exponentiates complete real matrix element.

Nason; Frixione, Oleari



PS plus NLO

- Here for $t\bar{t}$ production in MC@NLO, the distribution in the p_T of the top quark pair.
  - At small pT, the PS [here: HERWIG] gives the right description. At large pT the 2⟶3 diagrams ("matrix element") in the NLO part of the calculation.



- Good, consistent interpolatation between NLO and parton showers. Accuracy still NLO, the PS has not changed that.

- But not everything you can do with MC@NLO has NLO accuracy.
  - e.g. choosing the $t\bar{t}$ process in MC@NLO (or POWHEG), and then selecting $t\bar{t}$+1 jet events is LO
    - Exercise: convince yourself of this!

Parton showers in Monte Carlo's

- Great advances in Monte Carlo in recent years
  - Multipurpose: PYTHIA 8 (C++), HERWIG++, SHERPA
  - Matrix element based: Alpgen, Madgraph/Madevent, Comphep, Helac
  - NLO combined with parton showers: MC@NLO, POWHEG,
- Different program use slightly different versions of PS's:

PS	Splitting function	Recoil scheme	Construction
PYTHIA8	$1 \to 2$	Local	DGLAP
HERWIG++ (angular)	$1 \to 2$	Global	DGLAP
HERWIG++ (dipole)	$1 \to 2$	Local	CS Dipoles
SHERPA/CSSHOWER++	$1 \to 2$	Local	CS Dipoles
ARIADNE	$2 \to 3$	Local	Antenna
VINCIA	$2 \to 3$	Local	Antenna
SHERPA/ANTS	$2 \to 3$	Local	Antenna
KRKMC	$1 \to 2$	Global	DGLAP
DEDUCTOR	$n \to n+1$	Local	NS subtraction
$\vdots$			

M. Schoenherr, PSR workshop 2014

  - Recoil scheme: fixin' up momentum conservation in branchings

Intermediate summary

- So far, we've learned a lot about LO, NLO
    - ✓ And a little bit about parton showers
- For hadron colliders, we discussed the issues of and behind the PDF's
    - x-dependence cannot be computed, but the scale (μ_F) dependence can, through solving the DGLAP equation
- We now understand (right?) how divergences can occur, and how to get rid of them
    - UV renormalization, KLN theorem ["just add all diagrams, and define your observable with some degeneracy"], and factorization [for initial state collinear divergences in QCD]
- After this, NLO cross sections are finite, and ready for use.
    - Of course, if LO is ok, no divergence issues
- That was all fixed order. Yields curves to put on plots, or histograms with weighted event entries if one does any phase space integrals by Monte Carlo integration.
- Another description is Monte Carlo, especially the parton shower.
- Next, for some modern developments in QCD. We begin with spinor helicity methods.

Spinor helicity methods

- Powerful, very explicit - and fun - method for computing amplitudes. In general, n-parton amplitude (e.g. 2 partons in, n-2 out)

$$A_n^{tree}(p_1, \lambda_1, a_1, \ldots, p_n, \lambda_n, a_n)$$

  - ✓ p_i: momenta, λ_i helicities, a_i: color states of n partons
  - Notice already here: if everything is specified, then A is just a complex number. For the cross section, one computes $|A|^2$ numerically. We will see examples below.
- Helicity: spin projected along direction of motion $h = |\vec{S} \cdot \vec{p}/|\vec{p}|$
  - For massless particles, this is good quantum number, conserved under Lorentz-transformations
    - ✓ Quarks can have helicities +1/2, -1/2. Gluons +1 or -1.
- First, solutions to Dirac equation for massless fermions: $\not{p} u(p, \lambda) = 0$

$$u_-(p,\xi) = 2^{1/4} \begin{pmatrix} -\sqrt{p^-} e^{-i\phi_p} \\ \sqrt{p^+} \\ 0 \\ 0 \end{pmatrix}, \qquad u_+(p,\xi) = 2^{1/4} \begin{pmatrix} 0 \\ 0 \\ \sqrt{p^+} \\ \sqrt{p^-} e^{i\phi_p} \end{pmatrix} \qquad e^{i\phi_p} = \frac{p^T}{\sqrt{2p^+p^-}}$$

  - v-spinor solutions can be expressed in these, so we only work with u-spinors

Spinor conventions

- Conventions:

Charge conjugation matrix

$$\gamma^k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \quad \gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \; \gamma_5 = -\mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \; C = \mathrm{i}\gamma^1\gamma^3 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}$$

- Note that u- is eigenvector of $(1+\gamma_5)/2$ and similarly for u+: lefthanded and righthanded. Dirac-conjugate helicity spinors

$\bar{\psi} = \mathrm{i}\psi^\dagger\gamma^0$

$$\overline{u_+} = \mathrm{i}2^{1/4}(\sqrt{p^+}, \sqrt{p^-}e^{-\mathrm{i}\phi_p}, 0, 0) \qquad \overline{u_-} = \mathrm{i}2^{1/4}(0, 0, \sqrt{p^-}e^{-\mathrm{i}\phi_p}, -\sqrt{p^+})$$

- Now we can define spinor product, and "state" notation $u_\lambda(\vec{p}) := |\, p; \lambda\rangle \qquad \bar{u}_\lambda(\vec{p}) := \langle p; \lambda\,|$

$$\overline{u_-}(k)u_+(p) \equiv \langle k-|p+\rangle \equiv \langle kp\rangle \qquad \overline{u_+}(k)u_-(p) \equiv \langle k+|p-\rangle \equiv [kp]$$

  - Using explicit forms for u's this becomes

$$\langle kp\rangle = -\mathrm{i}(e^{\mathrm{i}\phi_p}\sqrt{2k^+p^-} - e^{\mathrm{i}\phi_k}\sqrt{2k^-p^+}) \qquad [kp] = \mathrm{i}(e^{-\mathrm{i}\phi_p}\sqrt{2k^+p^-} - e^{-\mathrm{i}\phi_k}\sqrt{2k^-p^+})$$

  - One can check that the following identities hold

$$\langle kp\rangle = -\langle pk\rangle, \quad [kp] = -[pk], \quad \langle kp\rangle^* = [kp] \qquad \langle kp\rangle[kp] = -2k\cdot p$$

  - So spinor products are sort of a "square-root" of regular vector dot-products

Vectors in terms of spinors

- The real power comes when expressing vectors in terms of spinors. For external gluons one has an polarization vector $\varepsilon^{\mu}(p,\lambda)$

$$\varepsilon_{+}^{\mu}(k,p) = A_{+}\overline{u_{+}}(k,+)\gamma^{\mu}u(p,+) \equiv A_{+}\langle k+|\gamma^{\mu}|p+\rangle \qquad A_{+} = \frac{\mathrm{i}}{\sqrt{2}\langle kp\rangle}$$

  - notice the extra momentum label p (on-shell, $p^2=0$). We can choose it freely (related to gauge invariance), and is known as a “reference momentum”. Choose it cleverly to simplify calculations.

- Now some more very useful identities.

$$\begin{aligned}
&\langle k+|p+\rangle = \langle k-|p-\rangle = 0\,,\\
&\langle k+|\gamma^{\mu}|p-\rangle = \langle k+|\gamma_5|p+\rangle = 0\,,\\
&\langle k+|\gamma^{\mu}|k+\rangle = -2\mathrm{i}k^{\mu}\,,\\
&\langle k+|\gamma^{\mu}|p+\rangle = \langle p-|\gamma^{\mu}|k-\rangle
\end{aligned}$$

  - Can all be checked by substituting explicit formulas for u’s

  - Fierz identity $\langle 1+|\gamma^{\mu}|2+\rangle\langle 3-|\gamma_{\mu}|4-\rangle = 2\langle 1+|4-\rangle\langle 3-|2+\rangle$

  - Schouten identity $\langle ij\rangle\langle kl\rangle = \langle il\rangle\langle kj\rangle + \langle ik\rangle\langle lj\rangle$

  - Normalization of polarization vectors $\varepsilon^{+}(k,p)\cdot\varepsilon^{-}(k,p) = A_{+}A_{-}\langle k+|\gamma^{\mu}|p+\rangle\langle k-|\gamma_{\mu}|p-\rangle = 1$

  - Polarization sum for fermions $-\mathrm{i}\not{k} = |k+\rangle\langle k+| + |k-\rangle\langle k-|$

  - Polarization sum for vectors, and relation of reference momentum to gauge tranformation

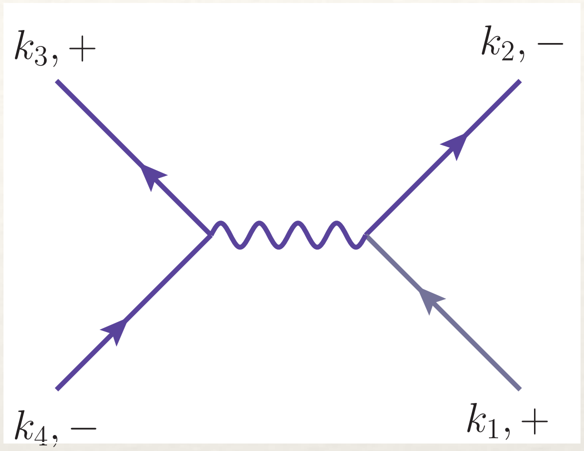
$$\sum_{\lambda=\pm}\varepsilon_{\lambda}^{\mu}(k,p)(\varepsilon_{\lambda}^{\nu}(k,p))^{*} = \eta^{\mu\nu} - \frac{p^{\mu}k^{\nu} + p^{\nu}k^{\mu}}{p\cdot k}$$

$$\varepsilon_{+}^{\mu}(k,p) - \varepsilon_{+}^{\mu}(k,q) = \frac{-\mathrm{i}\sqrt{2}\langle pq\rangle}{\langle kp\rangle\langle kq\rangle}k^{\mu}$$

Different reference momenta
= gauge transformation: no physical consquence

Process: $e^+e^- \rightarrow \mu^+\mu^-$

- Amplitude (4 external fermions, one propagator), take all momenta outgoing, all masses zero.
  - if charge flow = momentum: fermion, otherwise anti-fermion
  - outgoing external fermions with helicity ±: $\langle k\pm|$
  - outgoing external anti-fermions with helicity ±: $|k\mp\rangle$



t

- Reaction $e^-(k_1) + e^+(k_2) \to \mu^-(k_3) + \mu^+(k_4)$
- Amplitude $\mathcal{M}(1^{\lambda_1}, 2^{\lambda_2}, 3^{\lambda_3}, 4^{\lambda_4})$
  - Just using the rules: $\mathcal{M}(1^+, 2^-, 3^+, 4^-) = (\mathrm{i}e)^2\langle 2-|\gamma^\mu|1-\rangle \frac{-\mathrm{i}}{s_{12}}\langle 3+|\gamma_\mu|4+\rangle$
  - Using Fierzing: $\mathcal{M}(1^+, 2^-, 3^+, 4^-) = 2\mathrm{i}e^2\frac{[24]\langle 31\rangle}{\langle 12\rangle[12]}$

Each a just complex number!
Easy to square by computer.

  - Using momentum conservation:

$$\mathcal{M}(1^+, 2^-, 3^+, 4^-) = 2\mathrm{i}e^2\frac{[13]^2}{[12][34]} \quad \text{Likewise} \quad \mathcal{M}(1^-, 2^+, 3^-, 4^+) = 2\mathrm{i}e^2\frac{\langle 13\rangle^2}{\langle 12\rangle\langle 34\rangle}$$

  - Get others from charge conjugation
  - $$\mathcal{M}(1^+, 2^-, 3^-, 4^+) = 2\mathrm{i}e^2\frac{\langle 23\rangle^2}{\langle 12\rangle\langle 34\rangle}$$
  - These are the only non-zero amplitudes! $\mathcal{M}(1^+, 2^+, 3^-, 4^+) = 0$

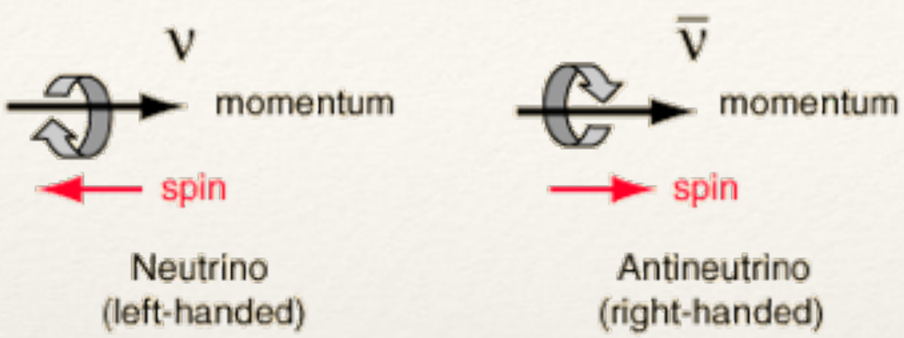
$$\frac{[24]\langle 31\rangle}{\langle 12\rangle[12]} \times \frac{-[43]}{[34]}$$

$$\langle 24\rangle[43] = \langle 2-|\not{4}|3-\rangle = -\langle 2-|\not{1}|3-\rangle$$

$$= -\langle 21\rangle[13]$$

Helicity and chirality

- The two words are often used interchangeably. For fermions that is ok.



- Difference:
  - chirality is, formally speaking, an eigenstate of γ_5, thus something for spin-1/2 particles
  - helicity can be defined for any spin $h = |\vec{S} \cdot \vec{p}/|\vec{p}|$
- For the fermion-antifermion amplitude

$$\mathcal{M}(1^+, 2^-, 3^-, 4^+) = 2ie^2 \frac{\langle 23 \rangle^2}{\langle 12 \rangle \langle 34 \rangle}$$

- the helicity labels are equivalent to LR $\longrightarrow$ RL

Relations between amplitudes

- One can save *a lot of* work by using relations between helicity amplitudes, using parity (P) and charge conjugation (C) transformations
  - Parity flips helicities <ab> ⟷ [ab]
  - Charge conjugation fermion(p,λ) ⟷ antifermion (p,-λ): $(u_+(p))^c = C^{-1}(\overline{u_+})^T = u_-(p)$
  - For the fermion line formed by the initial state charge conjugation leads to

$$\mathcal{M}(1^+, 2^-, \ldots) \xrightarrow{C_{k_1,k_2}\text{ line}} \mathcal{M}(2^+, 1^-, \ldots)$$

  - We just saw it helped, we needed just one helicity amplitude.

- Square all 4 amplitudes and add (using $|\langle ij\rangle|^2 = s_{ij}$).

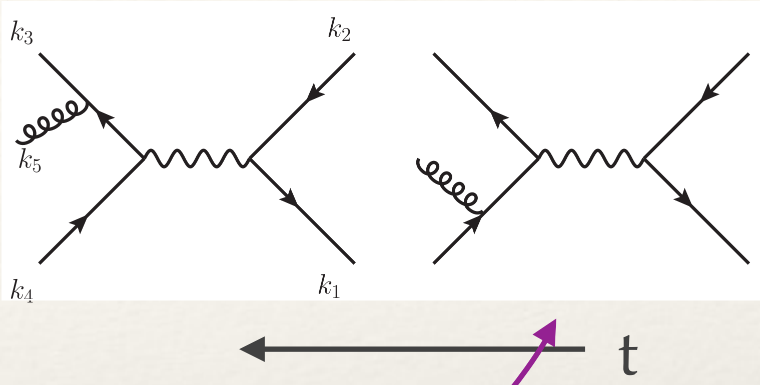
$$\sum_{\lambda_i} |\mathcal{M}(1^{\lambda_1}, 2^{\lambda_2}, 3^{\lambda_3}, 4^{\lambda_4})|^2 = 2(4e^2)\frac{s_{13}^2 + s_{23}^2}{s_{12}^2} = 2(4e^2)\frac{t^2+u^2}{s^2}$$

$$= \frac{1}{2}(4e^2)\left[(1-\cos\theta)^2 + (1+\cos\theta)^2\right] = (4e^2)\left[1+\cos^2\theta\right]$$

  - so easy to get angular dependence of squared matrix element out.

Process: $e^+e^- \to q g \bar{q}$

- Amplitude (4 external fermions, 1 external gluon, two propagators), take again all momenta outgoing
- Reaction $e^-(k_1) + e^+(k_2) \to q(k_3) + \bar{q}(k_4) + g(k_5)$
- Let compute the amplitude $\mathcal{M}(1^{\lambda_1}, 2^{\lambda_2}, 3^{\lambda_3}, 4^{\lambda_4}, 5^{\lambda_5})$



  - In principle 2^5=32 amplitudes. But with P and C relations only 1 to compute
  - Consider then

$$\mathcal{M}(1^+, 2^-, 3^+, 4^-, 5^+)$$

    - ✓ Choose k_4 as reference momentum for the outgoing gluon
      - Then the rightmost diagram is zero!!
  - One remaining diagram gives, using the rules we just saw

$$\mathcal{M}(1^+, 2^-, 3^+, 4^-, 5^+) = 2\sqrt{2}e^2 g T_a \frac{\langle 24\rangle^2}{\langle 12\rangle\langle 35\rangle\langle 45\rangle}$$

  - With this one expression, a pure complex number, we can completely reconstruct all 32 amplitudes

Process: $e^+e^- \to qg\bar{q}$ some properties

- The result

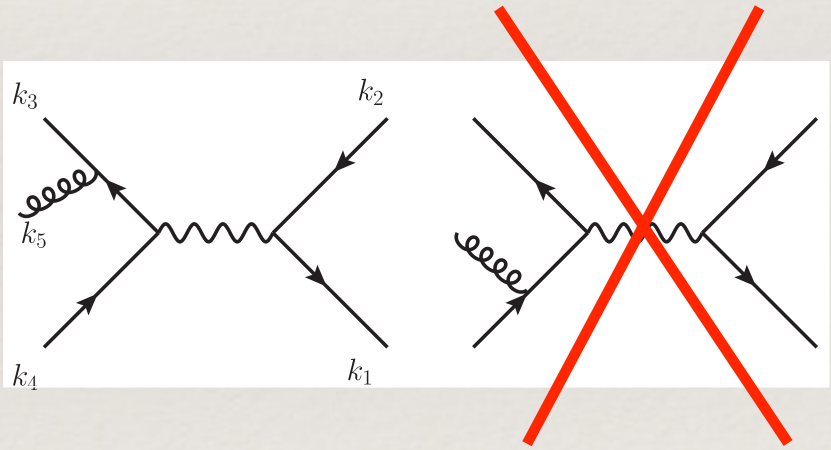
$$\langle kp\rangle = -\mathrm{i}(e^{\mathrm{i}\phi_p}\sqrt{2k^+p^-} - e^{\mathrm{i}\phi_k}\sqrt{2k^-p^+})$$

$$\mathcal{M}(1^+,2^-,3^+,4^-,5^+) = 2\sqrt{2}e^2 g T_a \frac{\langle 24\rangle^2}{\langle 12\rangle\langle 35\rangle\langle 45\rangle}$$

- has some nice limits.
  - $k^\mu_5 \longrightarrow 0$ (IR limit). Because each spinor product is just a complex number we can write

$$\mathcal{M}(1^+,2^-,3^+,4^-,5^+) = 2\sqrt{2}e^2 g T_a \frac{\langle 24\rangle^2}{\langle 12\rangle\langle 35\rangle\langle 45\rangle} = \frac{\langle 24\rangle^2}{\langle 12\rangle\langle 34\rangle} \times \frac{\langle 34\rangle}{\langle 35\rangle\langle 45\rangle}$$

    - ✓ Structure: LO amplitude x “eikonal factor” (more in next lecture)
  - Collinear limit: $k_5 = zk_P$, $k_3 = (1-z)\, k_P$



$$\mathcal{M}(1^+,2^-,3^+,4^-,5^+) \propto \frac{\langle 24\rangle^2}{\langle 12\rangle\langle 35\rangle\langle 45\rangle} \simeq \frac{\langle 24\rangle^2}{\langle 12\rangle\langle P4\rangle}\frac{1}{\sqrt{1-z}\langle 35\rangle}$$

    - ✓ Structure: rescaled lower order x splitting amplitude
  - Can recover usual soft factors and/or splitting functions after squaring this amplitudes
- In general, spinor helicity methods very clean, short expressions, nice to work with

General spinor helicity amplitudes

- So these methods are very powerful indeed. For scattering amplitudes in QCD (with massless partons) with many external lines, but at tree-level (i.e. no loops) the real power comes in.

- First, make the QCD colors manageable, by writing the full amplitude in terms of "color ordered" ones

$$\mathcal{M}_n(p_i^{\lambda_i}, a_i) = g^{n-2} \sum_{\sigma \in S_n/Z_n} \mathrm{Tr}\left(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} \dots T^{a_{\sigma(n)}}\right) \mathcal{M}_n(\sigma(1^{\lambda_1}), \dots \sigma(n^{\lambda_n}))$$

  - E.g. for n=6, reduction from 34300 to 501 amplitudes

Parke, Taylor

- So one must compute color-ordered amplitudes for various helicities combinations. One can show that for any n the "all-plus" and the "all-plus-but-one" are zero.

$$\mathcal{M}_n(1^+, \dots, n^+) = 0, \qquad \mathcal{M}_n(1^-, 2^+, \dots, n^+) = 0,$$

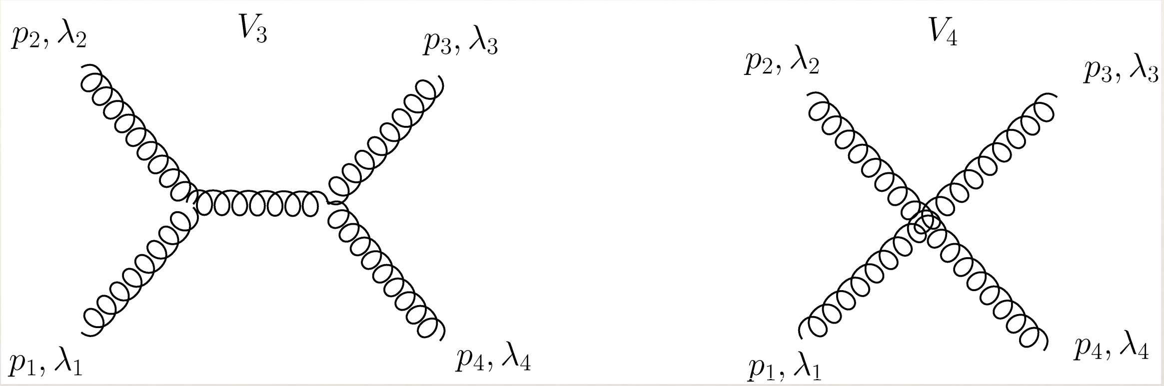
- The first non-zero ones are have two minus helicities. These are "maximal helicity violating", or MHV. They have a shockingly simple form (there can be thousands of diagrams for larger n)

$$\mathcal{M}_n^{\mathrm{MHV}}(1^+, 2^+, \dots, i^-, \dots, j^-, \dots, (n-1)^+, n^+) \propto i \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle (n-1), n \rangle \langle n1 \rangle}$$

  - Note the nice cyclic structure in the denominator
  - note also: they only involve <..> spinor product, not the [..] spinor product. For all-but-two minus the inverse is true
    - "Holomorphic structure"

Example: MHV gluon-gluon scattering

- Need only to compute color-order amplitudes (n=4) $\mathcal{M}_n(1^{\lambda_1},\ldots,n^{\lambda_n})$
  - in principle 16 amplitudes to compute, but thanks to P, C and cyclicity only 4 needed
    - ✓ (++++), (-+++), (—++), (-+-+)
    - ✓ 4 external polarization vectors $\varepsilon^{\mu_i}_{\lambda_i}(p_i,q_i)$
      - connected by either 4- or 3-gluon vertices
    - ✓ Use identities



$$\varepsilon^{\pm}(p,q)\cdot\varepsilon^{\pm}(p,q)=0,\qquad \varepsilon^{\pm}(p_1,q)\cdot\varepsilon^{\mp}(p_2,p_1)=0$$

  - By choosing the reference momenta cleverly, all 4-gluon vertex contributions can be set to zero

$$\begin{aligned}
&\mathcal{M}_{V_4}(1^+,2^+,3^+,4^+)=0 \quad \text{if} \quad q_1=q_2=q_3=q_4\\
&\mathcal{M}_{V_4}(1^-,2^+,3^+,4^+)=0 \quad \text{if} \quad q_3=q_4=q_2\\
&\mathcal{M}_{V_4}(1^-,2^-,3^+,4^+)=0 \quad \text{if} \quad q_1=q_2=p_3 \quad ; \quad q_3=q_4=p_1\\
&\mathcal{M}_{V_4}(1^-,2^+,3^-,4^+)=0 \quad \text{if} \quad q_1=q_3=p_2 \quad ; \quad q_2=q_4=p_1
\end{aligned}$$

  - For the 3-gluon vertices 2 of the amplitudes are zero again

$$\begin{aligned}
&\mathcal{M}_{V_3}(1^+,2^+,3^+,4^+)=0 \quad \text{if} \quad q_1=q_2=q_3=q_4\\
&\mathcal{M}_{V_3}(1^-,2^+,3^+,4^+)=0 \quad \text{if} \quad q_1=p_1, q_3=q_4=q_2
\end{aligned}$$

MHV amplitudes

- For the (-,-,+,+) amplitude we then have

$$\begin{aligned}
\mathcal{M}_{V_3}(1^-,2^-,3^+,4^+) &= \frac{i}{4p_1p_2}4\epsilon^-(2)\epsilon^+(3)p_2\epsilon^-(1)p_3\epsilon^+(4) = \\
&= \frac{-2i}{[12]\,\langle 12\rangle}\frac{\langle 2-|\gamma^\mu|\,4-\rangle\langle 3+|\gamma_\mu|\,1+\rangle\langle 1-|\not{2}|\,4-\rangle\langle 4+|\not{3}|\,1+\rangle}{4\,[24]\,\langle 31\rangle\,[14]\,\langle 41\rangle} = \\
&= \frac{-i}{2}\frac{2\langle 3+|4-\rangle\langle 2-|1+\rangle\langle 1-|2+\rangle\langle 2+|4-\rangle\langle 4+|2-\rangle\langle 2-|1+\rangle}{\langle 1+|2-\rangle\langle 1-|2+\rangle\langle 2+|4-\rangle\langle 3-|1+\rangle\langle 1+|4-\rangle\langle 4-|1+\rangle} = \\
&= -i\frac{\langle 12\rangle^3\langle 3+|4-\rangle\langle 4+|2-\rangle}{-\langle 2-|3+\rangle\langle 3+|4-\rangle\langle 3-|4+\rangle\langle 4+|2-\rangle\langle 41\rangle} = \\
&= i\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}
\end{aligned}$$

  - Exercise..

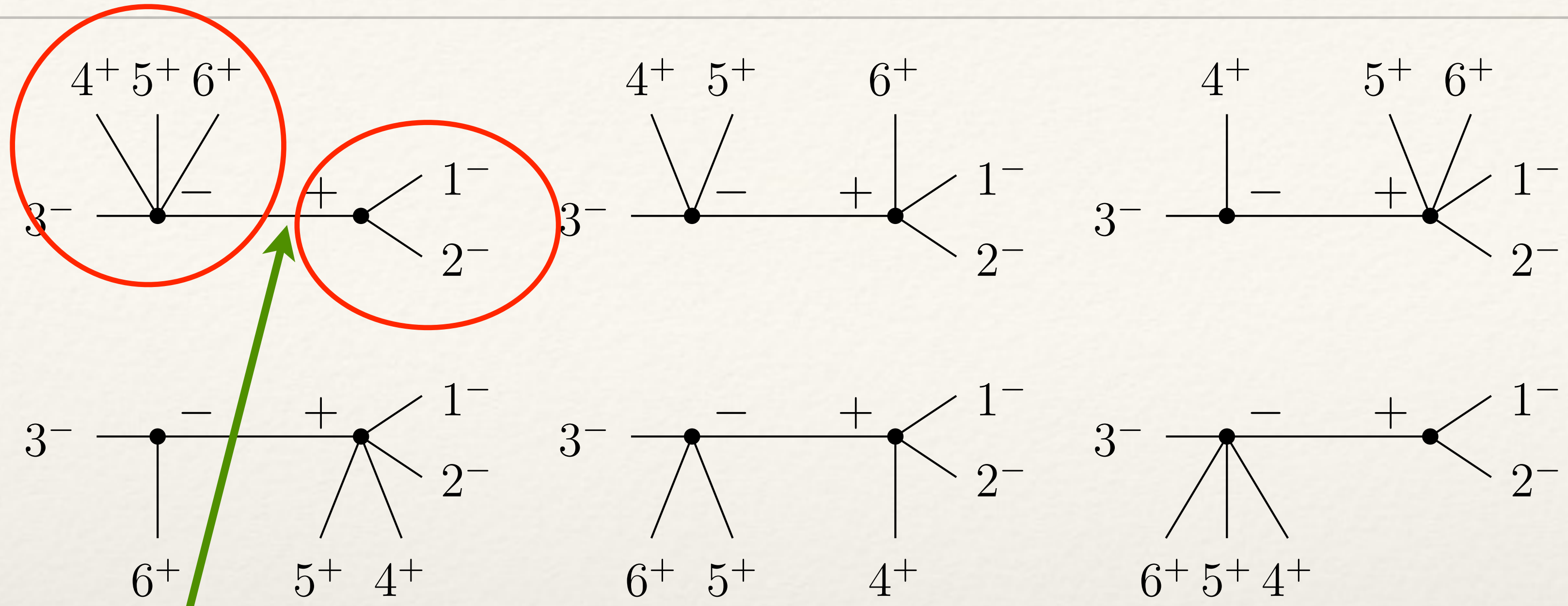
Leading order

- MHV amplitudes are very nice, but they don't cover all possible helicity combinations. How to handle large-n amplitudes in LO in general?
- The problem here is not handling divergences, but handling complexity. For gg $\to$ng number of diagrams grows factorially

n	2	3	4	5	6	7	8
diagrams	4	25	220	2485	34300	559405	10525900

- Solutions do exist already in code up to large n (order 10):
  - Madgraph/Madevent (helicity amplitudes), Sherpa/Amegic++, Helac/Phegas, Alpgen (recursion), Comphep (matrix elements)
- Especially Alpgen works fast, because it uses recursion relations, developed in the 80's and 90's.
- A few years ago approach Witten proposed not using Feynman rules as building blocks, but MHV amplitudes!

Recursion in terms of MHV building blocks



- Construct helicity amplitude by sewing together MHV building blocks using 1 propagator.

Cachazo, Svrcek, Witten

- Six diagrams only! (Was 220..)

On-shell recursion for non-MHV 6-point amplitude

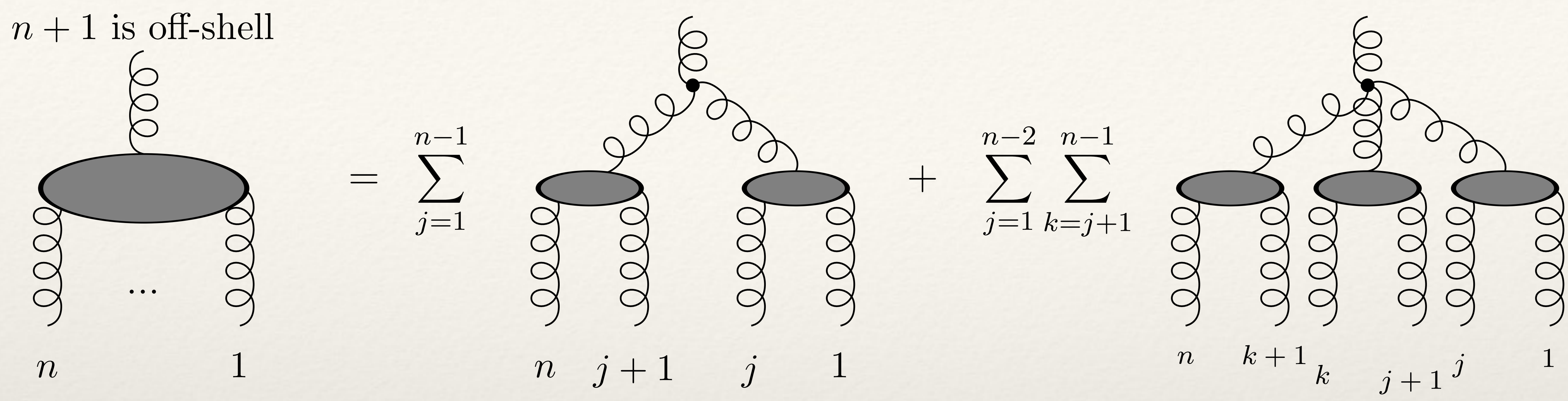
$$\mathcal{M}_6(1^-,2^-,3^-,4^+,5^+,6^+)$$

- Standard methods: 220 diagrams
- After all tricks, and BCF recursion: 2 diagrams
- Analytic form remarkably compact:

$$\frac{\langle 6+|1+2|3+\rangle^3}{[61][12]\langle 34\rangle s_{126}\langle 2+|1+6|5+\rangle}+\frac{\langle 4+|5+6|1+\rangle^3}{[23][34]\langle 56\rangle s_{156}\langle 2+|1+6|5+\rangle}$$

Pre-twistor recursion

Berends, Giele



- Define “currents”: one gluon off-shell, n off-shell. Obey indicated recursion.
- This method computes non-parton amplitudes on-the-fly for each event, does not use a long formula’s based on diagrams
- Analytically elegant, numerically efficient, used in Alpgen Monte Carlo
    - ✓ Earlier in the VECBOS MC for W+n jets, important for top quark discovery.

Recursion match. BG vs MHV-stuff

Duhr, Hoche, Maltoni

Final State	BG		BCF		CSW	
	CO	CD	CO	CD	CO	CD
$2g$	0.24	0.28	0.28	0.33	0.31	0.26
$3g$	0.45	0.48	0.42	0.51	0.57	0.55
$4g$	1.20	1.04	0.84	1.32	1.63	1.75
$5g$	3.78	2.69	2.59	7.26	5.95	5.96
$6g$	14.2	7.19	11.9	59.1	27.8	30.6
$7g$	58.5	23.7	73.6	646	146	195
$8g$	276	82.1	597	8690	919	1890
$9g$	1450	270	5900	127000	6310	29700
$10g$	7960	864	64000	-	48900	-

Dinsdale, Ternick, Weinzierl

n	4	5	6	7	8	9	10	11	12
Berends-Giele	0.00005	0.00023	0.0009	0.003	0.011	0.030	0.09	0.27	0.7
Scalar	0.00008	0.00046	0.0018	0.006	0.019	0.057	0.16	0.4	1
MHV	0.00001	0.00040	0.0042	0.033	0.24	1.77	13	81	—
BCF	0.00001	0.00007	0.0003	0.001	0.006	0.037	0.19	0.97	5.5

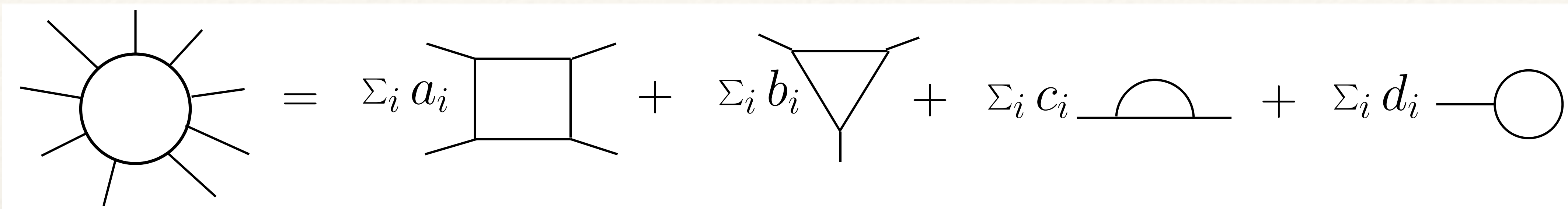
New things are not (yet) always better

The NLO "revolution"

- In the last 8 years or so new methods have led to a breakthrough in NLO calculations. Particular the calculation of the one-loop diagrams has become very tractable, and even automatizable.
- Results now in codes such as aMC@NLO, Powheg Box. But also many processes in MCFM.
- There is no time to discuss this in detail, but let's look at one important ingredient: how to find n-point one-loop amplitudes.
- Basic notion:
  - all one-loop amplitudes can written as a sum of boxes, triangle, bubbles and tadpoles
    - ✓ In essence because we live in 4 dimensions: every vector can be decomposed in at maximum four independent vectors

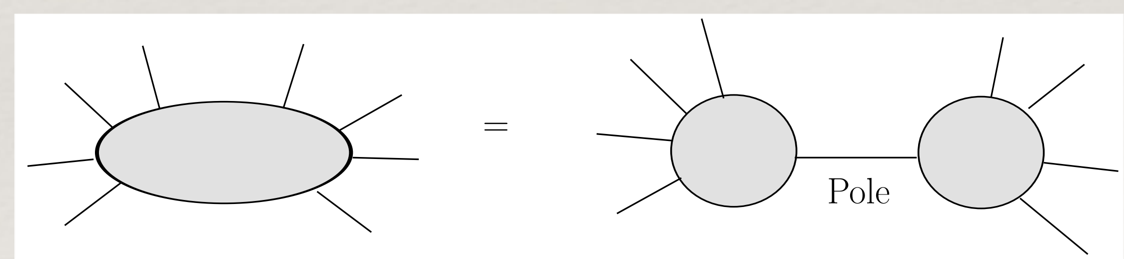
One-loop ideas

Vermaseren, van Neerven; Bern, Dixon, Kosower,...



$$\mathcal{M} = \sum_i a_i(D)\mathrm{Boxes}_i + \sum_i b_i(D)\mathrm{Triangles}_i + \sum_i c_i(D)\mathrm{Bubbles}_i + \sum_i d_i(D)\mathrm{Tadpoles}_i$$

- Since boxes etc are standard: find coefficients
- Many ideas based on unitarity: construct a function from its poles and branch cuts
  - ✓ poles: lower # of external lines
  - ✓ branch cuts: lower # of loops



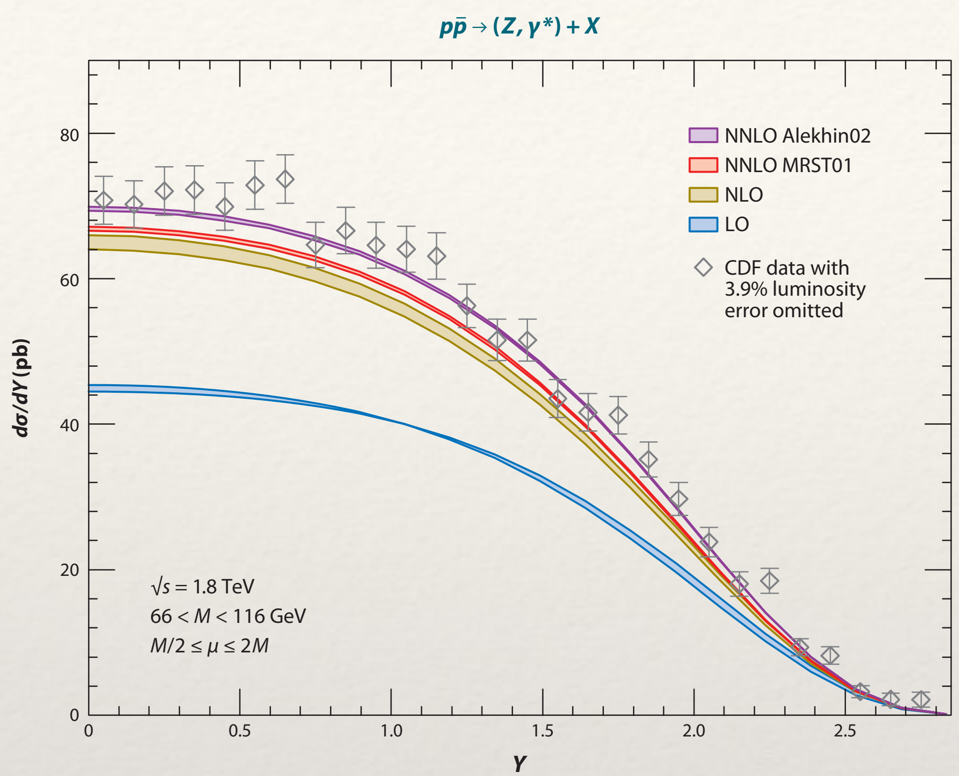
Status of higher order calculations in QCD

Order	$2 \to 1$	$2 \to 2$	$2 \to 3$	$2 \to 4$	$2 \to 5$	$2 \to 6$
1	LO					
α_s	NLO	LO				
α_s^2	NNLO	NLO	LO			
α_s^3	NNNLO	NNLO	NLO	LO		
α_s^4					LO	
α_s^5						LO

- LO well-understood, now more efficient than ever
- NLO: automatized, a flood of results
  - ✓ aMC@NLO (many processes), Blackhat (pp $\longrightarrow$ W/Z + 5 jets)
- NNLO: top quark total cross section, dijet production
- NNNLO: for F2(x,Q), almost for Higgs (and Drell-Yan) production

Beyond NLO

- There are times when NLO is not enough



Anastasiou, Dixon, Melnikov, Petriello

- either because NLO correction is too large for desired precision ⟶ go to NNLO
    - ✓ No time to discuss this. It is the present fixed order frontier..
- or because no finite order correction seems to be sufficient ⟶ go to all orders
    - ✓ either analytically ("resummation") or numerically ("parton showers")