Non-perturbative effects for the BFKL equation in QCD and in $N = 4$ SUSY

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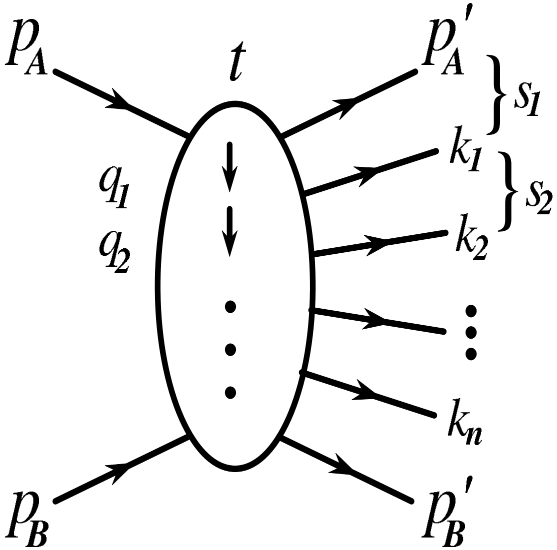
1 Amplitudes in multi-Regge kinematics



$$M_{2\to 2+n}^{FKL} = 2s\, g \frac{s_1^{\omega_1}}{q_1^2+m^2}\, gT^{d_1}_{c_2c_1} C_1^\mu e^1_\mu \,\frac{s_2^{\omega_2}}{q_2^2+m^2} ... gT^{d_n}_{c_{n+1}c_n}\, C^\sigma_n e^n_\sigma \,\frac{s_{n+1}^{\omega_{n+1}}}{q_{n+1}^2+m^2}\, g\,,$$

$$C_1 = -q_1^\perp - q_2^\perp - p_A\left(\frac{q_1^2+m^2}{p_Ak_1} - \frac{p_Bk_1}{p_Ap_B}\right) + p_B\left(\frac{q_2^2+m^2}{p_Bk_1} - \frac{p_Ak_1}{p_Ap_B}\right)\,,$$

$$\omega_r = j_r - 1 = -\frac{g^2N_c}{16\pi^3}\int \frac{d^2k\,\left(q_r^2+m^2\right)}{\left(k^2+m^2\right)\left((q_r-k)^2+m^2\right)}$$

2 Fadin-Kuraev-Lipatov equation (1975)

Total cross-section in the Higgs model at high energies in LLA

$$\sigma_t \sim s^{\Delta}\,, \;\; \Delta = -\frac{\alpha_s N_c}{2\pi}\, E_0 = \frac{4\alpha_s N_c}{\pi}\, \ln 2$$

FKL equation for the Pomeron wave function at $q = t = 0$

$$E f(r) = H f(r)\,, \;\; H = T(p) + V(r)\,, \;\; r = |x|$$

Kinetic energy related to two Regge trajectories

$$T(p) = \frac{2(|p|^2 + m^2)}{|p|\sqrt{|p|^2 + 4m^2}}\, \ln \frac{\sqrt{|p|^2 + 4m^2} + |p|}{\sqrt{|p|^2 + 4m^2} - |p|}\,, \;\; |p|^2 = -\frac{1}{r}\, \partial\, r\, \partial$$

Potential energy

$$V(r) = -4\, K_0(r\, m) + \frac{N_c^2 + 1}{N_c^2}\, \hat{P}\,, \;\; \hat{P}\, \phi(p) = \frac{m^2}{|p|^2 + m^2} \int \frac{d^2 p'}{\pi}\, \frac{\phi(p')}{|p'|^2 + m^2}$$

Semiclassical solution (Levin, Lipatov, Siddikov (2014))

3 BFKL equation in LLA (1978)

Balitsky-Fadin-Kuraev-Lipatov equation

$$E\,\Psi(\vec{\rho}_1,\vec{\rho}_2)=H_{12}\,\Psi(\vec{\rho}_1,\vec{\rho}_2)\ :\ \Delta=-\frac{\alpha_s N_c}{2\pi}\,E_0=\frac{4\alpha_s N_c}{\pi}\ \ln 2$$

Holomorphic separability (L.)

$$H_{12}=h_{12}+h^*_{12}\,,\ [h_{12},h^*_{12}]=0\,,\ E=\epsilon+\epsilon^*\,,\ \epsilon=\psi(m)+\psi(1-m)-2\psi(1)$$

Holomorphic Hamiltonian

$$h_{12}=\frac{1}{p_1}(\ln\rho_{12})p_1+\frac{1}{p_2}(\ln\rho_{12})p_2+\ln(p_1p_2)-2\psi(1)$$

Möbius-invariant solution (L.)

$$\Psi=\left(\frac{\rho_{12}}{\rho_{10}\rho_{20}}\right)^m\left(\frac{\rho^*_{12}}{\rho^*_{10}\rho^*_{20}}\right)^{\widetilde{m}},\ m=i\nu+\frac{1+n}{2}\,,\ \widetilde{m}=i\nu+\frac{1-n}{2}$$

4 BKP equation in LLA

Bartels-Kwiecinski-Praszalowicz equation

$$E\,\Psi(\vec{\rho}_1, ..., \vec{\rho}_n) = H\,\Psi(\vec{\rho}_1, ..., \vec{\rho}_n)$$

Holomorphic separability at large N_c (L. (1988))

$$H = \frac{1}{2}\,(h + h^*)\,,\;\; h = \sum_{k=1}^{n} h_{k,k+1}\,,$$

Monodromy matrix

$$\prod_{k=1}^{n} L_k = \left(\begin{array}{cc} A(u) & B(u) \\ C(u) & D(u) \end{array} \right),\;\; L_k = \left(\begin{array}{cc} u + \rho_k\, p_k & p_k \\ -\rho_k^2\, p_k & u - \rho_k\, p_k \end{array} \right)$$

Transfer matrix and integrability (L. (1993))

$$T(u) = A(u) + D(u)\,,\;\; [T(u), T(v)] = [T(u), h] = 0$$

5 Pomeron in next-to-leading order

Eigenvalue of BFKL kernel at QCD in NLO (F., L. (1998))

$$\omega = 4\,\hat{a}\;\chi(n,\gamma) + 4\;\hat{a}^2\,\Delta(n,\gamma)\,,\;\;\hat{a} = g^2 N_c/(16\pi^2)\,,\;\;\gamma = 1/2 + i\nu$$

Hermitian separability in $N = 4$ SUSY (K.,L. (2000))

$$\Delta(n,\gamma) = \phi(M) + \phi(M^*) - \frac{\rho(M) + \rho(M^*)}{2\hat{a}/\omega}\,,\;M = \gamma + \frac{|n|}{2}\,,$$

$$\rho(M) = \beta'(M) + \frac{1}{2}\zeta(2)\,,\;\beta'(z) = \frac{1}{4}\left[\Psi'\Big(\frac{z+1}{2}\Big) - \Psi'\Big(\frac{z}{2}\Big)\right]$$

Maximal transcendentality (2002) and integrability (1997)

$$\phi(M) = 3\zeta(3) + \Psi^{''}(M) - 2\Phi(M) + 2\beta^{'}(M)\Big(\Psi(1) - \Psi(M)\Big),$$

$$\Phi(M) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k+M}\left(\Psi'(k+1)\;-\;\frac{\Psi(k+1)-\Psi(1)}{k+M}\right)$$

6 Equation with running coupling

Simplified BFKL equation in QCD at $q=0$

$$\omega f_\omega(t) = \frac{1}{ct}\,\chi(\hat{\nu})\,f_\omega(t)\,,\ t = \ln\frac{k^2}{\Lambda^2_{QCD}}\,,\ \hat{\nu} = -i\frac{\partial}{\partial t}\,,\ c = \frac{11}{12} - \frac{n_f}{18}$$

Pomeron wave functions falling at large t

$$f_\omega(t) = \int_{-\infty}^{\infty} d\nu\, e^{it\nu}\, g_\omega(\nu)\,,\ g_\omega(\nu) = e^{-\frac{2i\nu}{c\omega}\,\psi(1)}\left(\frac{\Gamma(1/2+i\nu)}{\Gamma(1/2-i\nu)}\right)^{\frac{1}{c\omega}}$$

Equation for saddle points $\nu = \pm\widetilde{\nu}_\omega(t)$

$$\omega = \frac{1}{ct}\chi(\widetilde{\nu}_\omega(t))\,,\ \chi(\nu) = 2\psi(1) - \psi(1/2+i\nu) - \psi(1/2-i\nu)$$

Oscillations of eigenfunctions at small t

$$f_\omega \sim \cos\delta^p_\omega(t)\,,\ \delta^p_\omega(t) = \frac{\pi}{4} + t\widetilde{\nu}_\omega(t) - \frac{2\psi(1)}{c\omega}\widetilde{\nu}_\omega(t) + \frac{\Im}{c\omega}\ln\frac{\Gamma(1/2+i\widetilde{\nu}_\omega(t))}{\Gamma(1/2-i\widetilde{\nu}_\omega(t))}$$

7 Matching to non-perturbative physics

Green function of the BFKL equation

$$G_0^\omega(t,t') = -\frac{it'}{\omega}\int_{-\infty}^{\infty}\frac{d\nu}{2\pi}\,e^{it\nu}g_\omega(\nu)\int_{-\infty}^{\infty}\frac{d\nu'}{2}\,\epsilon(\nu-\nu')\,e^{it'\nu'}g_\omega(\nu')$$

Integral operator of evolution in rapidity Y

$$G_0^Y(t,t') = \int_{a-i\infty}^{a+i\infty}\frac{d\omega}{2\pi i}e^{Y\omega}\,G_0^\omega(t,t')\,,\;\;\lim_{Y\to 0}G_0^Y(t,t') = \delta(t-t')$$

Oscillations of G_0 at small t'

$$G_0^\omega(t,t') \approx \frac{t'}{\omega}\,f_\omega(t)\,\left(2\pi(\chi'(-\widetilde{\nu}_\omega)\right)^{-1/2}\,\sin\delta_\omega^p(t')$$

Green function with a non-perturbative phase

$$G^\omega(t,t') = G_0^\omega(t,t') + \frac{t'\cot\phi_\omega}{4\pi\omega}\,f_\omega(t)\,f_\omega(t')\,,\;\;\delta_\omega^{np}(t) = \phi_\omega + \delta_\omega^p(t)$$

8 Spectrum of Pomerons in QCD

Dispersion representation for the Green function

$$G^{\omega}(t,t') = \sum_{n=1}^{\infty} \frac{c_n(t)}{\omega - \omega_n}\,,\ \ \phi(\omega_n) \approx \frac{a}{\omega_n} - \eta = \pi n$$

Spectrum of Pomerons and physics BSM (KLR)

$$\omega_n \approx \frac{0.5}{1+0.95n}\,,\ \ \bar{k}_n \approx \Lambda_{QCD}\, e^{4n}\,,\ \ \bar{k}_3 \approx 10\,Tev$$

DGLAP equation for the Pomeron wave function

$$-i\frac{d}{dt} f_{\omega}(t) = \widetilde{\nu}_{\omega}(t)\, f_{\omega}(t)\,,\ \ \chi(\widetilde{\nu}_{\omega}(t)) = c\omega t$$

BFKL singularity and the "turning" point t_c

$$\widetilde{\nu}_{\omega}(t) \sim \sqrt{\omega - \frac{4\alpha_c(t)N_c}{\pi}\ln 2}\,,\ \ \frac{4\alpha_c(t_c)N_c}{\pi}\ln 2 = \omega$$

9 Phase ϕ_ω in the Higgs model

Pomeron wave function at the Higgs model (LLS)

$$\lim_{k^2\to\infty} f(\nu, \frac{k^2}{m^2}) \approx \sin\left(\nu \ln \frac{6.456\, k^2}{m^2}\right)$$

Running coupling and phase δ^p_ω at Higgs scale

$$\alpha(t_m) = \frac{\pi}{3c\ln t_m}\,,\;\; \delta^p_{\omega,m} = \delta^p_\omega(t_m)\,,\;\; t_m = \ln\frac{m^2}{6.456\,\Lambda_s^2}$$

Matching of perturbative and "non-perturbative" phases

$$\phi_\omega = \frac{\pi}{2} - \delta^p_{\omega,m} = \pi n$$

Quantization of Pomeron intercepts

$$\omega_n \approx \frac{3.68}{11 - \frac{2}{3}\, n_f}\,\frac{1}{n+\frac{1}{4}}$$

10 Pomeron at a thermostat

Gluon coordinates and momenta at a non-zero temperature

$$\rho_r = x_r + iy_r \,,\ 0 < y_r < 1/T_t \,;\ p_r = p_r^x + ip_r^y \,,\ p_r^y = 2\pi T_t k \,,\ k = 0, \pm 1, \pm 2, ...$$

BFKL Hamiltonian at a thermostat with $T_t \neq 0$ (de Vega, Lipatov)

$$h = \sum_{s=1,2} \left(\psi\left(1 + i\frac{p_s}{2\pi T_t}\right) + \psi\left(1 - i\frac{p_s}{2\pi T_t}\right) - 2\psi(1) + \frac{2}{p_s} \ln(2 \sinh(\pi T_t \rho_{12}))\, p_s \right)$$

Conformal transformation to the zero temperature and integrability

$$\rho_r = \frac{1}{2\pi T_t} \ln \rho'_s$$

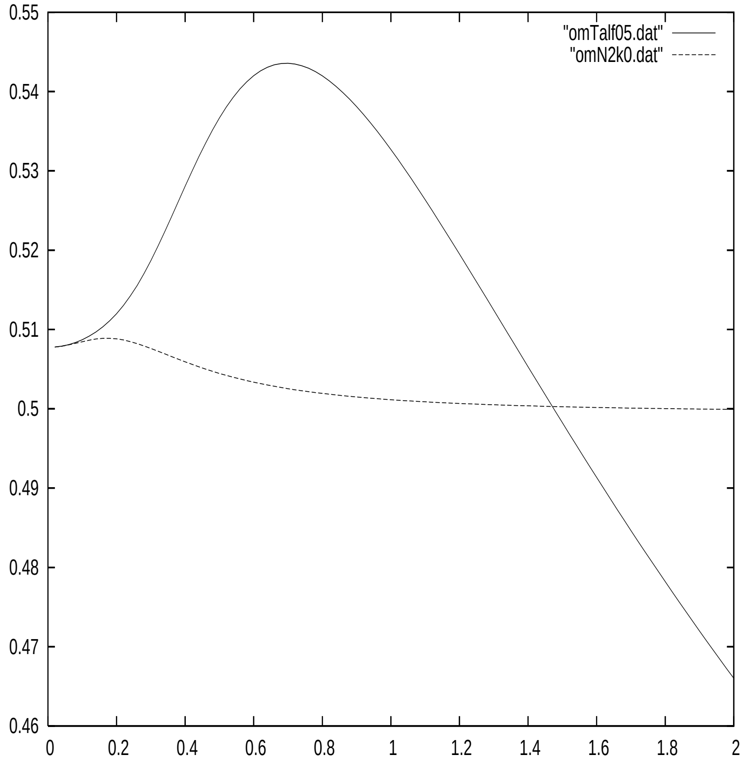
Running α_s and Pomeron trajectories for $T_t \neq 0$

$$\omega_n(\vec{q}) = \alpha_s(q)\, \chi(\nu_n) \,,\ q = q_x + 2\pi N i \,,\ \delta_\omega^{np}(q, T) - \delta_{\omega,q}^p = \pi n$$

11 T-dependence of Regge trajectories

Leading trajectories $\omega(\vec{q}, T)$ at $\alpha_s(q) = 0.5$ for $N = 0$ and N=2

(H. de Vega, L. Lipatov (2013))



Gluon attraction for increasing T and anti-Meissner effect

Confinement can be imitated by a cylinder-type topology (a bag)

12 BFKL Pomeron in a rectangle bag

BFKL hamiltonian on a torus

$$H = T + V\,,\; T = \ln|\widetilde{p}_1|^2 + \ln|\widetilde{p}_2|^2\,,\; V = v + v^*$$

Regge trajectory on the torus

$$\ln|\widetilde{p}|^2 = \sum_{n_x=-\infty}^{\infty}\sum_{n_y=-\infty}^{\infty} \frac{T_x^{-1}T_y^{-1}\pi^{-1}}{\frac{n_x^2}{T_x^2}+\frac{n_y^2}{T_y^2}} \frac{p_x^2+p_y^2}{(p_x-2\pi\frac{n_x}{T_x})^2+(p_y-2\pi\frac{n_y}{T_y})^2}$$

Potential energy and Green function

$$v = \frac{1}{p_1 p_2^*}\,G(\vec{\rho}_{12})\,p_1 p_2^*\,,\; G(\vec{\rho}) = \sum_{n_x=-\infty}^{\infty}\sum_{n_y=-\infty}^{\infty} \frac{T_x^{-1}T_y^{-1}(2\pi)^{-1}}{\frac{n_x^2}{T_x^2}+\frac{n_y^2}{T_y^2}}\, e^{2\pi i(\frac{x n_x}{T_x}+\frac{y n_y}{T_y})}$$

Equation for the Pomeron trajectories

$$\omega_n(\vec{q}) = \alpha_s(q)\,\chi(\nu_n)\,,\; \delta_\omega^{np}(q,T_x,T_y) - \delta_{\omega,q}^p = \pi n$$

13 Conformal transformations

Pomeron function on the impact parameter plane

$$\Psi = \left(\frac{\rho'_{12}}{\rho'_{10}\rho'_{20}}\right)^{m} \left(\frac{\rho'^{*}_{12}}{\rho'^{*}_{10}\rho'^{*}_{20}}\right)^{\widetilde{m}}$$

Transformation to a rectangle $|\Re\rho| < a,\ 0 < \Im\rho < b$

$$\rho' = sn\left(\frac{\rho K}{a}; k\right) = sn\left(\frac{\rho K'}{ib}; k'\right)\,,\ k^2 + k'^2 = 1$$

Inverse transformation

$$\rho = \frac{a}{K}\int_0^{\rho'} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}\,,\ K = \int_0^1 \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

Pomeron trajectory quantization

$$\omega_n(\vec{q}) = \alpha_s(q)\,\chi(\nu_n)\,,\ \delta^{np}_{\omega}(q,a,b) - \delta^{p}_{\omega,q} = \pi n$$

14 Pomeron and graviton in N=4 SUSY

Eigenvalue of the BFKL kernel in a diffusion approximation

$$j = 2 - \Delta - \Delta\,\nu^2\,,\ \gamma = 1 + \frac{j-2}{2} + i\nu$$

AdS/CFT relation for the graviton Regge trajectry

$$j = 2 + \frac{\alpha'}{2}\,t\,,\ t = E^2/R^2\,,\ \alpha' = \frac{R^2}{2}\,\Delta$$

Large coupling expansion of Δ (KLOV, BPST, KL)

$$\Delta = 2\lambda^{-1/2} + \lambda^{-1} - 1/4\,\lambda^{-3/2} - 2(1+3\zeta_3)\lambda^{-2} + \ldots\,,\ \lambda = \frac{\alpha N_c}{2\pi}$$

Exact expression for the slope of γ at $j = 2$ (KLOV, V., Basso)

$$\gamma'(2) = -\frac{\lambda}{24} + \frac{1}{2}\frac{\lambda^2}{24^2} - \frac{2}{5}\frac{\lambda^3}{24^2} + \frac{7}{20}\frac{\lambda^4}{24^4} - \frac{11}{35}\frac{\lambda^5}{24^5} + \ldots = -\frac{\sqrt{\lambda}}{4}\,\frac{I_3(\sqrt{\lambda})}{I_2(\sqrt{\lambda})}$$

15 Maximal helicity violation

BDS amplitudes in $N=4$ SUSY at $N_c \gg 1$ (2005)

$$A^{a_1,\ldots,a_n} = \sum_{\{i_1,\ldots,i_n\}} Tr\, T^{a_{i_1}} T^{a_{i_2}} \ldots T^{a_{i_n}} \, f(p_{i_1}, p_{i_2}, \ldots, p_{i_n})\,, \;\; f = f_B\, M_n$$

Invariant amplitudes

$$\ln M_n = \sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) \left(-\frac{1}{2\epsilon^2} \sum_{i=1}^{n} \left(\frac{\mu^2}{-s_{i,i+1}} \right)^{\epsilon} + F_n^{(1)}(0) \right) + C^{(l)} \right)\,,$$

$$a = \frac{\alpha N_c}{2\pi} \left(4\pi e^{-\gamma}\right)^{\epsilon}\,, \;\; C^{(1)} = 0\,, \;\; C^{(2)} = -\zeta_2^2/2\,, \;\; f^{(l)}(\epsilon) = \sum_{k=0}^{2} \epsilon^k\, f_k^{(1)}$$

Cusp anomalous dimension

$$f_0^{(l)} = \frac{1}{4}\gamma_K^{(l)}\,, \;\; f_1 = \beta(f) = -a\zeta_3/2 + a^2(2\zeta_5 + 5\zeta_2\zeta_3/3) + \ldots$$

16 Mandelstam cuts in j_2-plane

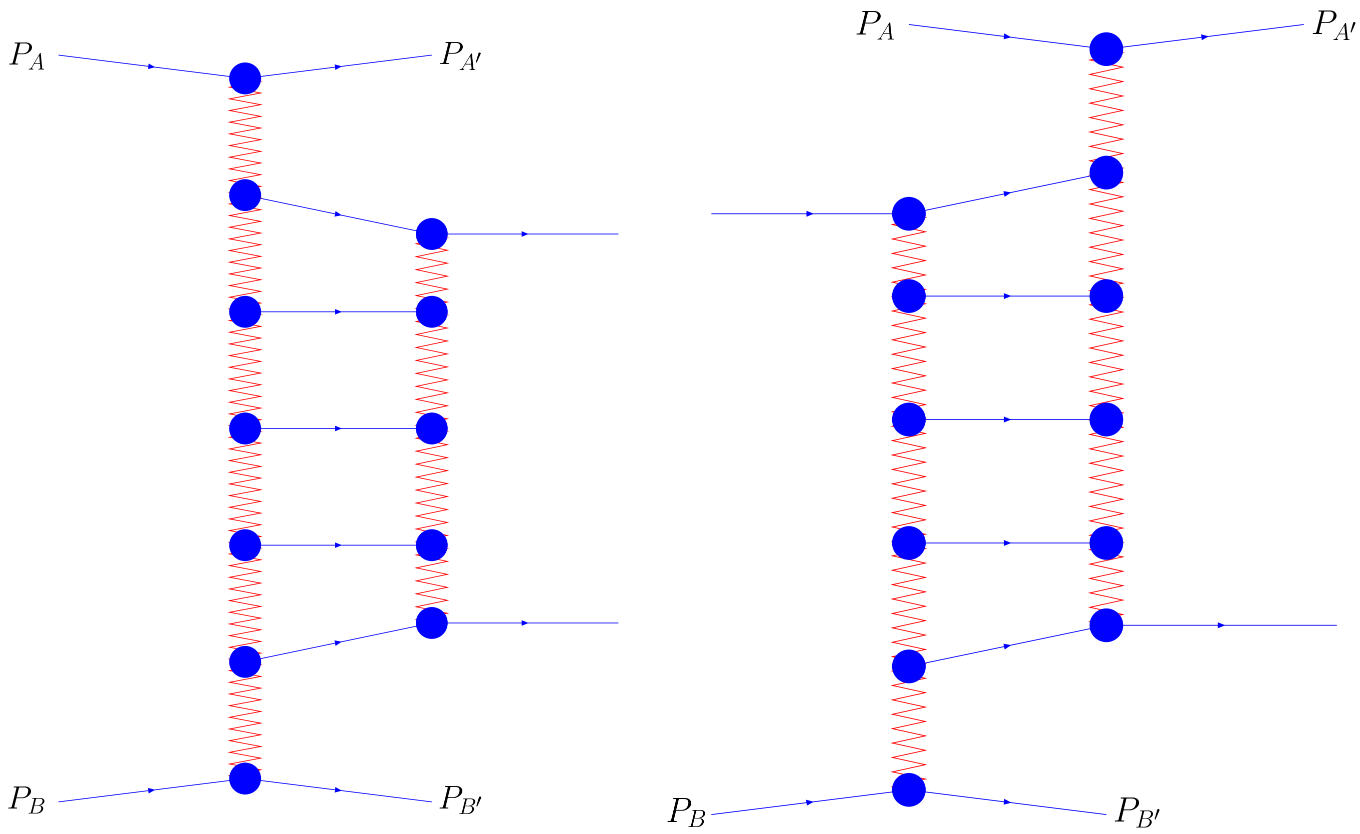


Figure 1: BFKL ladders in $M_{2\to 4}$ and $M_{3\to 3}$

17 Remainder function R_6

Physical regions with cut contributions

$$s, s_2 > 0\,,\ s_1, s_3 < 0\ ;\ \ s, s_2 < 0\,,\ s_{012}, s_{123} > 0$$

BDS amplitude and remainder function (B,L,S)

$$A_6 = A_6^{BDS}\, R_6(u_1, u_2, u_3)$$

Conformal invariance of R_6 in momentum space

$$p_r = X_r - X_{r-1}\,,\ X_r \to \frac{X_r}{X_r^2}$$

Anharmonic ratios for 6-point amplitude

$$u_1 = \frac{s s_2}{s_{012} s_{123}}\,,\ u_2 = \frac{t_1 s_3}{s_{123} t_2}\,,\ u_3 = \frac{t_3 s_1}{s_{012} t_2}$$

18 Amplitude $A_{2\to 4}$ in $N=4$ SUSY

Remainder factor in next-to-leading LLA (F.,L. (2011))

$$R\, e^{i\pi\delta} = \cos\,\pi\omega_{ab} + i\,\frac{a}{2}\sum_{n=-\infty}^{\infty}(-1)^n e^{i\phi n}\int_{-\infty}^{\infty}\frac{|w|^{2i\nu}d\nu}{\nu^2+\frac{n^2}{4}}\,\Phi(\nu,n)\left(\frac{-1}{\sqrt{u_2u_3}}\right)^{\omega(\nu,n)},$$

$$\cos\phi = \frac{1-u_1-u_2-u_3}{2\sqrt{u_2u_3}}\,,\;\; \Phi(\nu,n) = 1-a\left(\frac{E_{\nu n}^2}{2}+\frac{3}{8}\,n^2/(\nu^2+\frac{n^2}{4})^2+\zeta(2)\right),$$

Spectrum of the eigenvalues of the BFKL kernel

$$\omega(\nu,n) = -aE_{\nu,n}-a^2(\epsilon_{\nu n}^{FL}+3\zeta(3))\,,\;\; E_{\nu n} = -\frac{|n|/2}{\nu^2+\frac{n^2}{4}}+2\Re\psi(1+i\nu+\frac{|n|}{2})-2\psi(1)\,,$$

Eigenvalue in the next-to-leading order (F.,L. (2011))

$$\epsilon_{\nu n}^{FL} = -\frac{\Re}{2}\left(\psi''(1+i\nu+\frac{|n|}{2})-\frac{2i\nu\psi'(1+i\nu+\frac{|n|}{2})}{\nu^2+\frac{n^2}{4}}\right)-\zeta(2)\,E_{\nu n}-\frac{1}{4}\,\frac{|n|\,\left(\nu^2-\frac{n^2}{4}\right)}{\left(\nu^2+\frac{n^2}{4}\right)^3}$$

19 Integrability of n-reggeon dynamics

Holomorphic separability of H for adjoint states

$$H = h + h^* \,, \; [h, h^*] = 0$$

Holomorphic hamiltonian at large N_c

$$h = \ln(Z_1^2 \partial_1) - 2\psi(1) + \ln \partial_{n-1} + \sum_{k=1}^{n-2} h_{k,k+1}$$

Pair hamiltonian

$$h_{1,2} = \ln(Z_{12}^2 \partial_1) + \ln(Z_{12}^2 \partial_2) - 2 \ln Z_{12} - 2\psi(1)$$

Integrals of motion for an open spin chain (L. (2009))

$$D(u) = \sum_{k=0}^{n-1} u^{n-1-k} q'_k \,, \; [D(u), h] = 0$$

20 Non-Fredholm property of kernels

Divergency of the Fredholm integral for the BFKL kernel

$$\int d^2p d^2p' |K(p^2, p'^2, (p-p')^2)|^2 = \infty$$

Asymptotic freedom at large p and confinement at large ρ

$$\lim_{p\sim p'\to\infty} |K(p^2, p'^2, (p-p')^2)|^2 \sim f(p/p') \frac{1}{|p|^2|p'|^2} \frac{1}{\ln^2(\max(p^2, p'^2))}$$

Divergency at small $p - p'$ and a continuous spectrum at $\omega < 0$

$$|K(p^2, p'^2, (p-p')^2)|^2 \sim \left(\omega(p^2)\delta^2(p-p')\right)^2$$

Semiclassical prediction for the BFKL spectrum in $N = 4$ SUSY

$$\lim_{|m|\to\infty} \omega^{(0)}(n,\nu) = -\gamma_K(a) \ln|m| \,, \quad \lim_{|m|\to\infty} \omega^{(8)}(n,\nu) = -\frac{1}{2}\gamma_K(a) \ln|m|$$

21 Discussion

1. Reggeized gluons and BFKL equation
2. Integrability of the BKP equation
3. Running coupling and infrared boundary conditions
4. Spectrum of Pomerons in QCD and confinement
5. Pomeron and reggeized graviton in N=4 SUSY
6. BDS amplitude and Mandelstam cuts
7. Integrability of equations for amplitudes
8. Non-Fredholm kernels and their semiclassical eigenvalues