# Poincaré Invariance in pNRQCD

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# Wilson coefficients in EFTs

Example: Expand QCD Lagrangian in 1/M

$$\mathcal{L}_{NRQCD} = \phi^\dagger \mathcal{O} \phi + \chi^\dagger \mathcal{O}' \chi + \cdots \tag{1}$$

- Operators $\mathcal{O}$ and $\mathcal{O}'$ come with Wilson coefficients.
- $\cdots$ includes terms other than bilinear ones.
- How do we fix or find the relations between the coefficients?

# Poincaré Invariance

The relations are found by imposing Poincaré invariance of the corresponding EFTs

- Construct the Poincaré generators of the EFTs.
- Impose the algebra conditions up to the desired order in $1/M$.
- Constraints yield relations between the coefficients.
- Calculations done up to $\mathcal{O}(1/M)$ in NRQCD.
  [Brambilla, Gromes, Vairo, 2003]

# Outline

One can go to higher orders in $1/M$ with a simpler method, the little group formalism in particular, rather than directly expanding the Poincaré generators in $1/M$.

- Little Group and Induced Representation
- Wilson Coefficients in NRQCD
- Poincaré Invariance in pNRQCD
- Wilson Coefficients in pNRQCD
- Open Issues

# Quantum Field Theory

A generic quantum field transforms under the Lorentz group as

$$\phi_a \rightarrow M(\Lambda)_{ab}\phi_b(\Lambda^{-1}x) \tag{2}$$

$M(\Lambda)$ being a representation of the Lorentz group. In the infinitesimal form

$$\delta\phi = i(a_0 h - \mathbf{a}\cdot\mathbf{p} - \boldsymbol{\theta}\cdot\mathbf{j} + \boldsymbol{\eta}\cdot\mathbf{k})\phi \tag{3}$$

and our interest lies upon the boost generator

$$\mathbf{k} = \mathbf{r}h - t\mathbf{p} \pm i\boldsymbol{\Sigma} \tag{4}$$

a generic quantum field transfroms under the spatial boost $\mathcal{B}$ as

$$\phi_a(x) \rightarrow (e^{\mp\boldsymbol{\eta}\cdot\boldsymbol{\Sigma}})_{ab}\phi_b(\mathcal{B}^{-1}x) \tag{5}$$

# Little group element

The little group element for the infinitesimal boost is given by

$$W(\mathcal{B}(\eta), p) \equiv L(\mathcal{B}(\eta)p)^{-1}\mathcal{B}(\eta)L(p)$$
$$= 1 + \frac{i}{2}\left[\frac{1}{M + v \cdot p}(\eta^\alpha p_\perp^\beta - p_\perp^\alpha \eta^\beta)\mathcal{J}_{\alpha\beta}\right] + \mathcal{O}(\eta^2) \quad (6)$$

where $p_\perp^\beta \equiv p^\beta - (v \cdot p)v^\beta$ and

$$\mathcal{J}_{1/2}^{\alpha\beta} = \frac{i}{4}[\gamma^\alpha, \gamma^\beta]$$
$$(\mathcal{J}^{\alpha\beta})_{\mu\nu} = i(g_\mu^\alpha g_\nu^\beta - g_\mu^\beta g_\nu^\alpha) \quad (7)$$

# Induced representation [Heinonen, Hill, Solon, 2012]

And we postulate the transformation of a massive field with mass $M$ through the induced representation

$$\phi_a(x) \to D[W(\Lambda, i\partial)]_{ab}\phi_b(\Lambda^{-1}x) \tag{8}$$

where $D$ is the representation of the little group, and $W$ is the little group element associated with the Lorentz transformation $\Lambda$. The transformation of the field in particular under the Lorentz boost is,

$$\phi_a(x) \to \exp\left[\mp\boldsymbol{\eta}\cdot\left(\frac{\boldsymbol{\Sigma}\times\boldsymbol{\partial}}{M+\sqrt{M^2-\boldsymbol{\partial}^2}}\right)\right]_{ab}\phi_b(\mathcal{B}^{-1}x) \tag{9}$$

when the reference frame is chosen $v = (1, 0, 0, 0)$.

# Non-relativistic expansion (1/3)

Up until now, we have figured out the boost transformation of a relativistic field in the little group formalism. Let us combine this with a non-relativistic expansion so that we can apply it later to NRQCD and pNRQCD. Extract the rest mass by

$$\phi_a(x) = e^{-iMt}\phi'_a(x) \tag{10}$$

and take the non-relativisitic field normalization

$$\phi_a(x) = e^{-iMt}\left(\frac{M^2}{M^2-\partial^2}\right)^{1/4}\phi''_a(x) \tag{11}$$

then how does this non-relativistic field $\phi''_a(x)$ transform under the Lorentz boost?

# Non-relativistic expansion (2/3)

From the "inverse" non-relativistic normalization

$$\phi_a''(x) = \left(\frac{M^2}{M^2 - \boldsymbol{\partial}^2}\right)^{-1/4} e^{iMt} \phi_a(x) \quad (12)$$

we can extract the Lorentz boost of the (free-) field

$$\begin{aligned} \phi_a''(x) \quad \to \quad & \left(\frac{M^2}{M^2 - \boldsymbol{\partial}^2}\right)^{-1/4} e^{iMt} \\ & \times \exp\left[\mp \boldsymbol{\eta} \cdot \left(\frac{\boldsymbol{\Sigma} \times \boldsymbol{\partial}}{M + \sqrt{M^2 - \boldsymbol{\partial}^2}}\right)\right]_{ab} \phi_b(\mathcal{B}^{-1}x) \\ = \quad & \left(\frac{M^2}{M^2 - \boldsymbol{\partial}^2}\right)^{-1/4} e^{iMt} \exp\left[\mp \boldsymbol{\eta} \cdot \left(\frac{\boldsymbol{\Sigma} \times \boldsymbol{\partial}}{M + \sqrt{M^2 - \boldsymbol{\partial}^2}}\right)\right]_{ab} \\ & \times e^{-iMt'} \left(\frac{M^2}{M^2 - \boldsymbol{\partial}'^2}\right)^{1/4} \phi_b''(x') \end{aligned} \quad (13)$$

where $x' \equiv \mathcal{B}^{-1} x$.

# Non-relativistic expansion (3/3)

Therefore, the Lorentz transformation (boost) of the non-relativistic field in 1/M expansion is given by

$$\phi_a''(x) \to \Big\{ 1 + iM\boldsymbol{\eta}\cdot\mathbf{x} - \frac{i\boldsymbol{\eta}\cdot\boldsymbol{\partial}}{2M} - \frac{i\boldsymbol{\eta}\cdot\boldsymbol{\partial}\boldsymbol{\partial}^2}{4M^3} + \frac{(\boldsymbol{\Sigma}\times\boldsymbol{\eta})\cdot\boldsymbol{\partial}}{2M}\left[1 + \frac{\boldsymbol{\partial}^2}{4M^2}\right] + \mathcal{O}(1/M^4) \Big\} \phi_a''(\mathcal{B}^{-1}x) \quad (14)$$

and this is the transformation of the non-interacting and non-relativistic field. Can this be implemented into the interacting theory?

# Transformation of the interacting theory

It is natural to postulate the transformation of the field just by promoting $\partial$ to $D$

$$\phi_a(x) \quad \to \quad D[W(\Lambda, iD)]_{ab}\phi_b(\Lambda^{-1}x) \tag{15}$$

and the Lorentz boost is thereby given

$$\phi_a(x) \quad \to \quad \exp\left[\mp\boldsymbol{\eta}\cdot\left(\frac{\boldsymbol{\Sigma}\times\mathbf{D}}{M+\sqrt{M^2-\mathbf{D}^2}}\right)+\mathcal{O}(g)\right]_{ab}\phi_b(\mathcal{B}^{-1}x) \tag{16}$$

in which $\mathcal{O}(g)$ contains all quantum corrections which vanish in the free-theory, so that the non-relativistic expansion is

$$\phi''_a(x) \to \left\{1 + iM\boldsymbol{\eta}\cdot\mathbf{x} - c_1\frac{i\boldsymbol{\eta}\cdot\mathbf{D}}{2M} - c_2\frac{i\boldsymbol{\eta}\cdot\mathbf{D}\mathbf{D}^2}{4M^3} + c_3\frac{(\boldsymbol{\Sigma}\times\boldsymbol{\eta})\cdot\mathbf{D}}{2M}\right.$$
$$\left. + c_4\frac{(\boldsymbol{\Sigma}\times\boldsymbol{\eta})\cdot\mathbf{D}}{2M}\frac{\mathbf{D}^2}{4M^2} + \mathcal{O}(g, 1/M^4)\right\}\phi''_a(\mathcal{B}^{-1}x) \tag{17}$$

# Non-relativistic QCD (NRQCD)

Let us apply this method to NRQCD, in particular. Integrating out the hard scale $M$ from the full QCD Lagrangian, one obtains

$$\mathcal{L}_{NRQCD} = \mathcal{L}_{\text{heavy}}(\phi, \chi) + \mathcal{L}_{\text{light}}(\psi) \tag{18}$$

in which the light and heavy quarks are decoupled at the leading order, and we consider the heavy part of the Lagrangian, which contains the Pauli spinors of heavy quark $\phi$ and antiquark $\chi$. Apply the Lorentz boost for the NR field and we observe

$$0 = \delta\mathcal{L}_{heavy} = \frac{1}{M}\delta\mathcal{L}_1 + \frac{1}{M^2}\delta\mathcal{L}_2 + \frac{1}{M^3}\delta\mathcal{L}_3 + \cdots \tag{19}$$

up to total derivatives.

# Wilson Coefficients in NRQCD

There exist other constraints

$$[k^i, k^j] = -i\epsilon^{ijk} J^k \tag{20}$$

which fix the parameters in the boost generator of the non-relativistic field. Our results of fixing NRQCD boost coefficients ($c_i = 1$, for $i \in \{1, 2, 3, 4\}$) coincide with the literature [Heinonen, Hill, Solon, 2012], which simply implies that their ansatz on the interacting theory works. The reason why such ansatz works up to this order still remains to be answered.

# pNRQCD - Energy Scale & Power Counting

Potential non-relativistic QCD (pNRQCD) is derived by integrating out the relative momentum between quark and antiquark, $Mv \sim 1/\text{r}$ (with $v << 1$ and $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$), from NRQCD; i.e., multipole expansion. Let $\mathbf{R} = (\mathbf{x}_1 + \mathbf{x}_2)/2$ (c.o.m. frame), then we have the following power counting scheme:

$$\boldsymbol{\nabla}_r, \frac{1}{r} \sim Mv \tag{21}$$

$$\partial_0, \boldsymbol{\nabla}_R, A_\mu \sim Mv^2 \tag{22}$$

$$\mathbf{E}, \mathbf{B} \sim M^2 v^4 \tag{23}$$

# pNRQCD - Degrees of freedom

Field contents:

I Quark-antiquark colour singlet $S$ and octet $O^a$ configuration

II $2 \times 2$ spin matrix $S_{ij}, O^a_{ij}$, for quark spin i and antiquark spin j

III Degrees of freedom depend on relative and COM coordinates as well as time; $S = S(t, \mathbf{R}, \mathbf{r})$, $O^a = O^a(t, \mathbf{R}, \mathbf{r})$

IV Multipole expanded gluon fields $A^a_\mu(t, \mathbf{R})$

Writing $S$ and $O^a$ as $3 \times 3$ matrices in colour space

$$S \to \frac{1}{\sqrt{3}} S\, I_3, \quad O^a \to O = \sqrt{2} O^a T^a \tag{24}$$

so that $\mathrm{Tr}[S^\dagger S] = S^\dagger S$, $\mathrm{Tr}[O^\dagger O] = O^{a\dagger} O^a$, $\mathrm{Tr}[S^\dagger \mathbf{E} O] = S^\dagger \mathbf{E}^a O^a$.

# pNRQCD - Lagrangian

And the Lagrangian up to order $Mv^3$ (+ c.o.m. kinetic term):

$$\begin{aligned}\mathcal{L}_{pNRQCD} &= \mathrm{Tr}\Big[S^\dagger\Big(i\partial_0 + \frac{\boldsymbol{\nabla}_R^2}{4M} + \frac{\boldsymbol{\nabla}_r^2}{M} - V_S^{(0)}(r) + \frac{1}{M}V_S^{(1)}(r)\Big)S \\ &\quad + O^\dagger\Big(iD_0 + \frac{\mathbf{D}_R^2}{4M} + \frac{\boldsymbol{\nabla}_r^2}{M} - V_O^{(0)}(r) + \frac{1}{M}V_O^{(1)}(r)\Big)O \\ &\quad + V_A(r)(S^\dagger \mathbf{r}\cdot\mathbf{E}O + \text{h.c.}) \\ &\quad + \frac{V_B(r)}{2}(O^\dagger\mathbf{r}\cdot\mathbf{E}O + O^\dagger O\mathbf{r}\cdot\mathbf{E})\Big] \end{aligned} \tag{25}$$

We match this to NRQCD by the interpolating fields (for $U$ being a Wilson line and $x_{1/2} = R \pm \frac{1}{2}r$):

$$\chi_j^\dagger(x_2)U(x_2,x_1)\phi_i(x_1) \to Z_S^{(1)}S_{ij} + Z_S^{(2)}r\mathbf{r}\cdot g\mathbf{E}^aO_{ij}^a + \cdots$$

$$\chi_j^\dagger(x_2)U(x_2,\mathbf{R})T^aU(\mathbf{R},x_1)\phi_i(x_1) \to Z_O^{(1)}O_{ij}^a + Z_O^{(2)}r\mathbf{r}\cdot g\mathbf{E}^aS_{ij} + \cdots$$

# Boost transformations in pNRQCD - 1/3

Free quark and antiquark fields transform under the little group as

$$\begin{aligned}\chi(t,\mathbf{r}_2) &\rightarrow \left(1 - iM\boldsymbol{\eta}\cdot\mathbf{r}_2 + \frac{i}{2M}\boldsymbol{\eta}\cdot\boldsymbol{\nabla}_2 - \frac{1}{4M}\boldsymbol{\eta}\cdot(\boldsymbol{\nabla}_2\times\boldsymbol{\sigma})\right) \\ &\quad \chi(t-\boldsymbol{\eta}\cdot\mathbf{r}_2, \mathbf{r}_2-\boldsymbol{\eta}t) \\ \phi(t,\mathbf{r}_1) &\rightarrow \left(1 + iM\boldsymbol{\eta}\cdot\mathbf{r}_1 - \frac{i}{2M}\boldsymbol{\eta}\cdot\boldsymbol{\nabla}_1 + \frac{1}{4M}\boldsymbol{\eta}\cdot(\boldsymbol{\nabla}_1\times\boldsymbol{\sigma})\right) \\ &\quad \phi(t-\boldsymbol{\eta}\cdot\mathbf{r}_1, \mathbf{r}_1+\boldsymbol{\eta}t) \end{aligned} \tag{26}$$

Time discrepancy is solved by expanding the time arguments of the $Q\bar{Q}$ pair (define $t' = t - \boldsymbol{\eta}\cdot\mathbf{R}, \mathbf{R}' = \mathbf{R} - \eta t$, and $\mathbf{r}' = \mathbf{r}$)

$$\begin{aligned} &\chi(t-\boldsymbol{\eta}\cdot\mathbf{r}_2, \mathbf{r}_2-\boldsymbol{\eta}t)^\dagger\phi(t-\boldsymbol{\eta}\cdot\mathbf{r}_1, \mathbf{r}_1+\boldsymbol{\eta}t) \\ = &\left[\left(1+\frac{1}{2}\boldsymbol{\eta}\cdot\mathbf{r}'\partial_{t'}\right)\chi(t', \mathbf{R}'-\frac{1}{2}\mathbf{r}')\right]^\dagger \\ &\left(1-\frac{1}{2}\boldsymbol{\eta}\cdot\mathbf{r}'\partial_{t'}\right)\phi(t', \mathbf{R}'+\frac{1}{2}\mathbf{r}') \end{aligned} \tag{27}$$

# Boost transformations in pNRQCD - 2/3

As replacing the temporal derivatives by spatial derivatives via equations of motion, the transformed $Q\bar{Q}$ pair is given by

$$\chi'^{\dagger}(t,\mathbf{r}_2)\phi'(t,\mathbf{r}_1) = \Big(1 + iM\boldsymbol{\eta}\cdot(\mathbf{r}_1+\mathbf{r}_2) - \frac{i}{2M}\boldsymbol{\eta}\cdot(\boldsymbol{\nabla}_1+\boldsymbol{\nabla}_2)$$
$$-\frac{i}{4M}(\boldsymbol{\eta}\cdot\mathbf{r})(\boldsymbol{\nabla}_1^2-\boldsymbol{\nabla}_2^2) + \frac{1}{4M}\boldsymbol{\eta}\cdot(\boldsymbol{\nabla}_1\times\boldsymbol{\sigma}^{(1)}+\boldsymbol{\nabla}_2\times\boldsymbol{\sigma}^{(2)})$$
$$+\mathcal{O}(1/M^3)\Big)\chi^{\dagger}(t',\mathbf{r}'_2)\phi(t',\mathbf{r}'_1) \tag{28}$$

Thereafter, the singlet field transforms under the little group as

$$S'(t,\mathbf{R},\mathbf{r}) = \Big(1 + 2iM\boldsymbol{\eta}\cdot\mathbf{R} - \frac{i}{4M}\boldsymbol{\eta}\cdot\boldsymbol{\nabla}_R - \frac{i}{4M}\{\boldsymbol{\eta}\cdot\mathbf{r},\boldsymbol{\nabla}_R\cdot\boldsymbol{\nabla}_r\}$$
$$+\frac{1}{8M}\boldsymbol{\eta}\cdot\boldsymbol{\nabla}_R\times(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}) + \frac{1}{4M}\boldsymbol{\eta}\cdot\boldsymbol{\nabla}_r\times(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)})$$
$$+\mathcal{O}(1/M^3)\Big)S(t',\mathbf{R}',\mathbf{r}') \tag{29}$$

# Boost transformations in pNRQCD - 3/3

And similarly the octet field is transformed as

$$O'(t,\mathbf{R},\mathbf{r}) = \Big(1 + 2iM\boldsymbol{\eta}\cdot\mathbf{R} - \frac{i}{4M}\boldsymbol{\eta}\cdot\mathbf{D}_R - \frac{i}{4M}\{\boldsymbol{\eta}\cdot\mathbf{r}, \mathbf{D}_R\cdot\boldsymbol{\nabla}_r\}$$
$$+\frac{1}{8M}\boldsymbol{\eta}\cdot\mathbf{D}_R\times(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}) + \frac{1}{4M}\boldsymbol{\eta}\cdot\boldsymbol{\nabla}_r\times(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)})\Big)$$
$$\times O(t',\mathbf{R}',\mathbf{r}') - \frac{i}{8}k^{(0,2)}_{OOa}(r)(\boldsymbol{\eta}\cdot\mathbf{r})(\mathbf{r}\cdot[g\mathbf{E}, O(t',\mathbf{R}',\mathbf{r}')])$$
$$-\frac{i}{8}k^{(0,2)}_{OOb}(r)\mathbf{r}^2(\boldsymbol{\eta}\cdot[g\mathbf{E}, O(t',\mathbf{R}',\mathbf{r}')]) + \dots \qquad (30)$$

where the coefficients $k^{(0,2)}_{OOa}$ and $k^{(0,2)}_{OOb}$ are determined under the condition that the Lagrangian is invariant under the Lorentz transformation.

# Constraints: singlet sector (1/2)

From the singlet sector of the pNRQCD

$$\mathcal{L}_S = S^\dagger \Big( i\partial_0 + \frac{1}{2M}\left\{ c_s^{(1,-2)}, \boldsymbol{\nabla}_r^2 \right\} + \frac{c_S^{(1,0)}}{4M}\boldsymbol{\nabla}_R^2 - V_S^{(0)} - \frac{V_S^{(1)}}{M}$$
$$+ \frac{V_{P^2Sa}}{8M^2}\boldsymbol{\nabla}_R^2 + \frac{1}{2M^2}\left\{ \boldsymbol{\nabla}_r^2, V_{p^2Sb} \right\} + \frac{V_{L^2Sa}}{4M^2r^2}(\mathbf{r}\times\boldsymbol{\nabla}_R^2)^2$$
$$+ \frac{V_{L^2Sb}}{4M^2r^2}(\mathbf{r}\times\boldsymbol{\nabla}_r)^2 - \frac{V_{S_{12}S}}{M^2r^2}\Big( 3(\mathbf{r}\cdot\boldsymbol{\sigma}^{(1)})(\mathbf{r}\cdot\boldsymbol{\sigma}^{(2)}) - \mathbf{r}^2\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)} \Big)$$
$$- \frac{V_{S^2S}}{4M^2}\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)} + \frac{iV_{LSSa}}{4M^2}(\mathbf{r}\times\boldsymbol{\nabla}_R)\cdot(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)})$$
$$+ \frac{V_{LSSb}}{4M^2}(\mathbf{r}\times\boldsymbol{\nabla}_r)\cdot(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}) \Big) S, \tag{31}$$

# Constraints: singlet sector (2/2)

By imposing Lorentz invariance, $\delta\mathcal{L}_S = 0$, in which

$$\begin{aligned}\delta\mathcal{L}_S &= S^\dagger\Big(i\Big(1-c_S^{(1,0)}\Big)\boldsymbol{\eta}\cdot\boldsymbol{\nabla}_R - \frac{1}{2M}\Big(1-c_S^{(1,0)}\Big)\boldsymbol{\eta}\cdot\boldsymbol{\nabla}_R\partial_0 \\ &\quad -\frac{i}{M}\Big(V_{p^2Sa}+V_{L^2Sa}+\frac{1}{2}V_S^{(0)}\Big)\boldsymbol{\eta}\cdot\boldsymbol{\nabla}_R \\ &\quad +\frac{i}{Mr^2}\Big(V_{L^2Sa}+\frac{r}{2}\partial_r V_S^{(0)}\Big)(\boldsymbol{\eta}\cdot\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\nabla}_R) \\ &\quad +\frac{1}{2M}\Big(V_{LSSa}+\frac{1}{2r}\partial_r V_S^{(0)}\Big)\boldsymbol{\eta}\cdot(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)})\times\mathbf{r}\Big)S \end{aligned} \quad (32)$$

the Wilson coefficients are constrained

$$\begin{aligned} c_S^{(1,0)} &= 1, \quad V_{p^2Sa}+V_{L^2Sa}+\frac{1}{2}V_S^{(0)}=0, \\ V_{L^2Sa} &= -\frac{r}{2}\partial_r V_S^{(0)}, \quad V_{LSSa} = -\frac{1}{2r}\partial_r V_S^{(0)} \end{aligned} \quad (33)$$

matches with the literature [Brambilla, Gromes, Vairo, 2003].

# Summary and Outlook

Summary

- Induced representation of the non-relativistic field
- Wilson coefficients in NRQCD
- Wilson coefficients in pNRQCD (singlet sector)

Outlook

- Higher order terms in NRQCD
- Higher order terms in pNRQCD (in progress)
- Dark matter with heavy mass (Hill's group in UChicago)
- Sterile neutrinos (in preparation)
- (Group theoretic) Reason why this method is valid remains to be answered

# References


N. Brambilla, D. Gromes, and A. Vairo (2003)
Poincaré invariance constrains on NRQCD and potential NRQCD
*Phys. Lett. B* 576, 314 - 327.

J. Heinonen, R. Hill, and M. Solon (2012)
Lorentz invariance in heavy particle effective theories
*Phys. Rev. D* 86, 094020 (2012).