How center vortices break chiral symmetry

in cooperation with
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A scenario

Chiral symmetry breaking (χSB) via instantons

- single instanton or anti-instanton
  topological charge $Q = \pm 1$
  zero mode of fermionic determinant
- many instantons
  interact
  zero modes $\rightarrow$ near-zero modes
- Banks-Casher relation
  density of near-zero modes determines chiral condensate $\bar{\psi}\psi$
- problems
  1. instantons are minima of action
     action of QCD-vacuum is not minimal
     action $\leftrightarrow$ entropy competition
  2. instantons don't exist in 3+1D, only in 4D
     on the lattice: instantons produced via cooling

instanton liquid model gives a nice mechanism how zero modes by interaction get near-zero modes.

Another scenario

Chiral symmetry breaking (χSB) via vortices

- vortices carry topological charge
  via intersections $Q = \pm 1/2$
  via writhing points $Q = \pm 1/4$
  via SU(2) color structure $Q = \pm 1$
- topological charge $Q = \pm 1$
  $\leftrightarrow$ zero mode of fermionic determinant
- interaction of vortices
  $\leftrightarrow$ near-zero modes
- Banks-Casher relation
  density of near-zero modes determines chiral condensate $\bar{\psi}\psi$

Banks-Casher relation

Chiral symmetry breaking (χSB) $\longrightarrow$

$\longrightarrow$ Low-lying eigenmodes of Dirac operator

Dirac equation: $D[A]\,\psi_n = \mathrm{i}\lambda_n\psi_n$,
$\{\gamma_5, \gamma_\mu\} = 0, \quad D[A]\,\gamma_5\psi_n = -\mathrm{i}\lambda_n\gamma_5\psi_n$
Non-zero eigenvalues appear in imaginary pairs $\pm\mathrm{i}\lambda_n$.

$$\langle\bar{\psi}\psi\rangle = -\lim_{m\to 0}\lim_{V\to\infty}\left\langle \frac{1}{V}\sum_n \frac{1}{m+\mathrm{i}\lambda_n}\right\rangle =$$

$$= -\lim_{m\to 0}\lim_{V\to\infty}\left\langle \frac{1}{V}\int \mathrm{d}\lambda\,\rho_V(\lambda)\frac{1}{2}\left(\frac{1}{m+\mathrm{i}\lambda}+\frac{1}{m-\mathrm{i}\lambda}\right)\right\rangle$$

$$-\lim_{m\to 0}\frac{m}{m^2+\lambda^2} = \lim_{m\to 0}\frac{\mathrm{d}}{\mathrm{d}\lambda}\arctan\frac{m}{\lambda} \quad\longrightarrow\quad \pi\delta(0)$$

Chiral condensate $\Longrightarrow$ Density of Near-Zero-modes

$$\langle\bar{\psi}\psi\rangle = \frac{\pi\rho_V(0)}{V}$$

➜ *Banks, Casher, 1980*

Atiyah-Singer index theorem

- zero-modes of fermionic matrix: $D[A]\psi(x) = 0$
- ψ has definite chirality:

$$\psi_{\substack{R\\L}} = \frac{1}{2}(1 \pm \gamma_5)\psi, \quad \Rightarrow \quad \gamma_5 \psi_{\substack{R\\L}} = \pm \psi_{\substack{R\\L}}$$

- Index theorem (wilson, overlap fermions):

$$n_-, n_+: \text{ number of left-/right-handed zeromodes}$$

$$\mathrm{ind} D[A] = n_- - n_+ = Q[A]$$

- (Asqtad) staggered fermions:

$$\mathrm{ind}\ D[A] = 2Q[A] \text{ (SU(2), double degeneracy)}$$

- Adjoint overlap fermions:

$$\mathrm{ind}\ D[A] = 2NQ[A] = 4Q[A] \text{ (real representation)}$$

➜ *Neuberger, Fukaya (1999)*

Topological charge Q

- QCD-vacua characterised by winding number n_w
- scalar (gauge) function: $g(x) = \mathrm{e}^{-\mathrm{i}\vec{\alpha}(x)\vec{\sigma}} \in \mathrm{SU}(2) \simeq \mathrm{S}^3$
- $\mathrm{R}^3 \to \mathrm{S}^3$: $n_w = -\frac{1}{4\pi^2} \int_{\mathrm{S}^3} \mathrm{d}^3x \mathrm{Sp}(\partial_i g g^\dagger \partial_j g g^\dagger \partial_k g g^\dagger)$
- vector field: $\mathrm{i}\partial_\mu g g^\dagger = \mathcal{A}_\mu = \frac{\vec{\sigma}}{2}\vec{A}_\mu$
- $F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - \varepsilon_{ijk} A^j_\mu A^k_\nu = 0$
- Topological charge: $Q = \frac{1}{32\pi^2} \int \mathrm{d}^4x \, F^i_{\mu\nu} \, \tilde{F}^i_{\mu\nu} = \frac{1}{4\pi^2} \vec{E}\vec{B}$
- Lattice: $F_{\mu\nu} = \frac{1}{2\mathrm{i}}(U_{\mu\nu} - U^\dagger_{\mu\nu})$
- admissibility condition: $\mathrm{tr}(\mathbb{1} - U_{\mu\nu}) < 0.0011$
  uniquness of topological charge on the lattice: ➜ Lüscher (1998)

Center vortices

closed quantised magnetic flux tubes

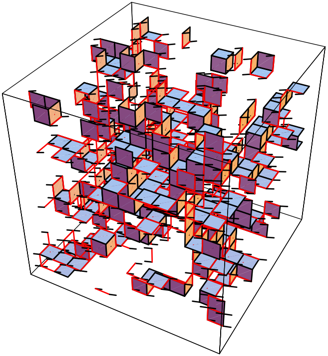
recall:

- vortices explain confinement ➜ Greensite (2011), Springer
- identifying vortices via center projection → P-vortices

Monte-Carlo:

P-vortices:

- closed surfaces
- random surfaces
- in 4D and in 3+1D



3-dimensional cut through dual of 12^4-lattice

prototypes of vortices: to understand mechanism of χSB

plane vortices

colorful spherical vortices

Center vortices and topological charge

We conclude that in a non-Abelian gauge theory spontaneous chiral symmetry breaking is intimately linked to gauge configurations having topological charge. So one may ask whether center vortices can give rise to topological charge.

Recall that the topological charge density is defined as

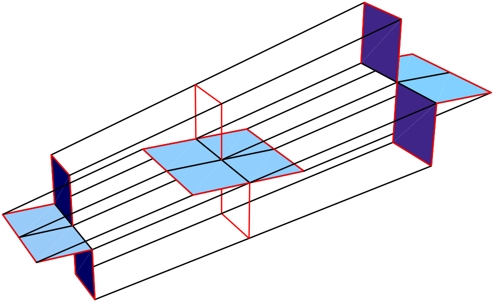
$$q(x) = \frac{1}{16\pi^2}\mathrm{Tr}\left(F_{\mu\nu}\tilde{F}_{\mu\nu}\right) = \frac{1}{4\pi^2}\vec{E}\cdot\vec{B}\,, \qquad \tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}\,.$$

So we need flux in all four directions. A vortex has flux perpendicular to its world sheet. Hence to generate topological charge we need two orthogonal vortices intersecting, or one vortex “writhing,” *i.e.*, twisting around itself.

P-vortices need an orientation
regions of different orientation are separated by monopole lines

➜ *Engelhardt, Reinhardt (2000)*

Generating topological charge



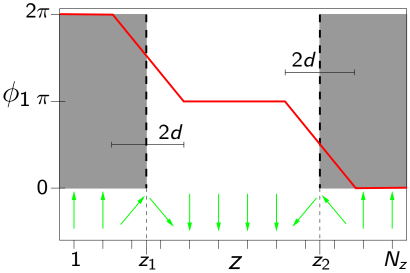
Intersections and writhing points contribute to the topological charge of a P-vortex surface

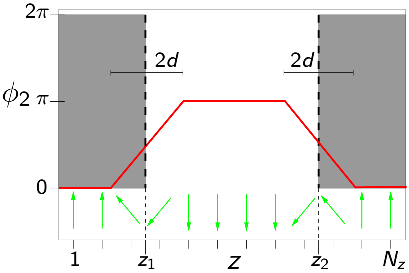
- intersections $Q = \pm\frac{1}{2}$
- writhing points $Q = \pm\frac{1}{4}$

H. Reinhardt, NPB628 (2002) 133 [hep-th/0112215], hep-th/0204194

Plane vortices

We can create a pair of plane vortices, or a vortex-antivortex pair, in the xy plane by setting the time like links in one time slice t_0 to $U_4(t_0, z) = \exp\{i\sigma_3\phi(z)\}$ with ϕ as



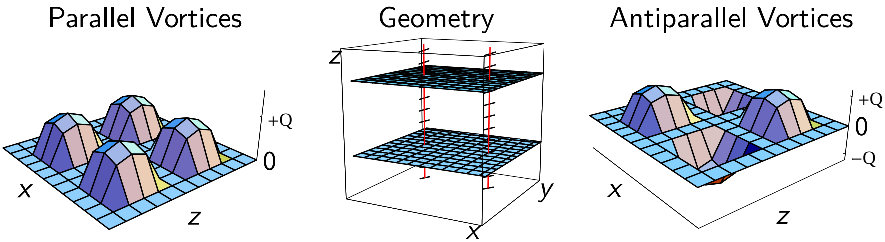


All other links are set to unity, $U_\mu = \mathbb{1}$. Due to periodic boundary conditions, plane vortices come in parallel or antiparallel pairs.

The corresponding P-vortices are located at z_1 and z_2.

Intersecting plane vortices

Intersecting two orthogonal pairs of plane vortices we can generate topology. A xy vortex generates a chromo-electric field, E_z, and a zt vortex a chromo-magnetic field, B_z. Each intersection point contributes $Q = \pm 1/2$ to the total topological charge.



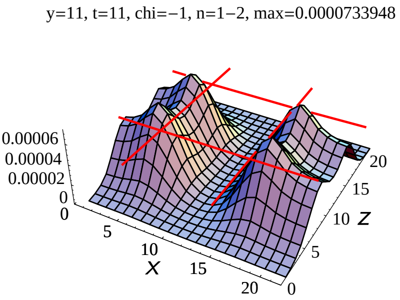
So we can get $Q = 2$ with parallel intersecting vortices and $Q = 0$ with antiparallel intersecting vortices.

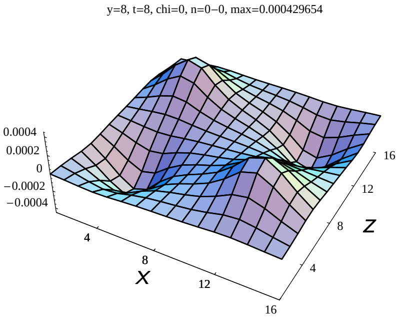
Intersecting plane vortices

We define the scalar density as $\rho = \psi^\dagger \psi$ and the chiral density as $\rho_5 = \psi^\dagger \gamma_5 \psi$, with ψ an overlap Dirac operator eigenmode.

Left: scalar density of the two zero modes for the $Q = 2$ intersecting vortex pairs on a 22^4 lattice.

Right: chiral density of the near-zero modes of the $Q = 0$ intersecting vortex pairs on a 16^4 lattice.





Planar vortices are “special”

recall: vortices are closed surfaces in 4D

planar vortices:

- are closed via periodic boundary conditions
- they belong to a single U(1) subgroup, $1 \stackrel{\sigma_3}{\to} -1$

but

- closed surfaces may have topology of S^2
- transition $1 \to -1$ may belong to varying subgroups
  3 space directions $\leftrightarrow$ 3 color degrees of freedom in SU(2)

$\Rightarrow$ colorful spherical vortex

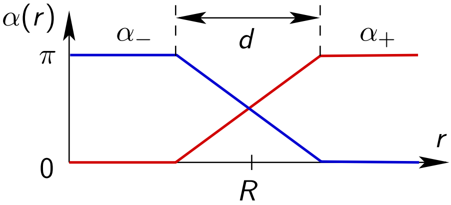
- closed surface forms a sphere
- transition

  space direction $0 \to \vec{R} = \vec{e}_R / R$ correlates with

  colour direction $1 \stackrel{\vec{\sigma}\vec{e}_r}{\to} -1$

Colorful spherical vortex

radius R
thickness d
$g(\vec{r}) = \exp\{\mathrm{i}\alpha(r)\,\vec{e}_r\cdot\vec{\sigma}\}$



$$U_\mu(x) = \begin{cases} g(|\vec{r}-\vec{r}_0|) & t=1, \mu=4 \\ \mathbb{1} & \text{elsewhere} \end{cases}$$

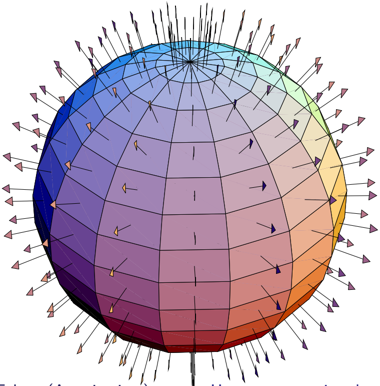
$$\left.\begin{array}{l} g(\vec{r}) \\ \text{time-like Wilson lines} \end{array}\right\} \text{ cover } S^3 \quad \Rightarrow \quad R^3 \mapsto S^3$$

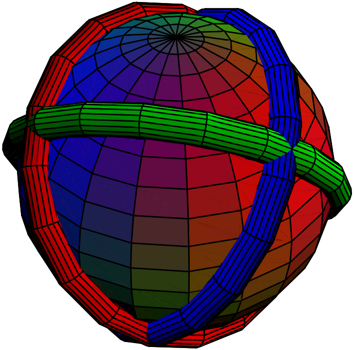
Colorful spherical vortex

The corresponding P-vortex is the sphere at radius R.

The color structure of the spherical vortex, a hedgehog configuration, is illustrated in the left plot.

The right plot illustrates the monopole lines after Abelian projection in the maximal Abelian gauge.





Continuum Form of colorful spherical vortex

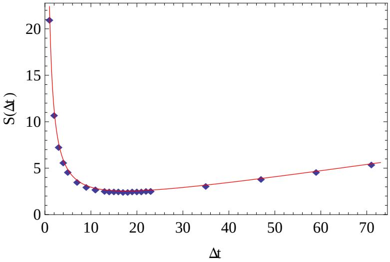
after time-dependent gauge transformation $\Omega(\vec{r}, t)$

vortex $\equiv$ vacuum - vacuum transition

$$\left.\begin{array}{l} t=1 \\ t=2 \end{array}\right\} \begin{array}{c} \text{vacuum} \\ \text{pure gauge} \end{array} \left\{\begin{array}{ll} R^3 \mapsto 1 & \text{no winding} \\ R^3 \mapsto S^3 & \text{winding} \end{array}\right.$$

smoothing possible ➜ Schweigler, 2013

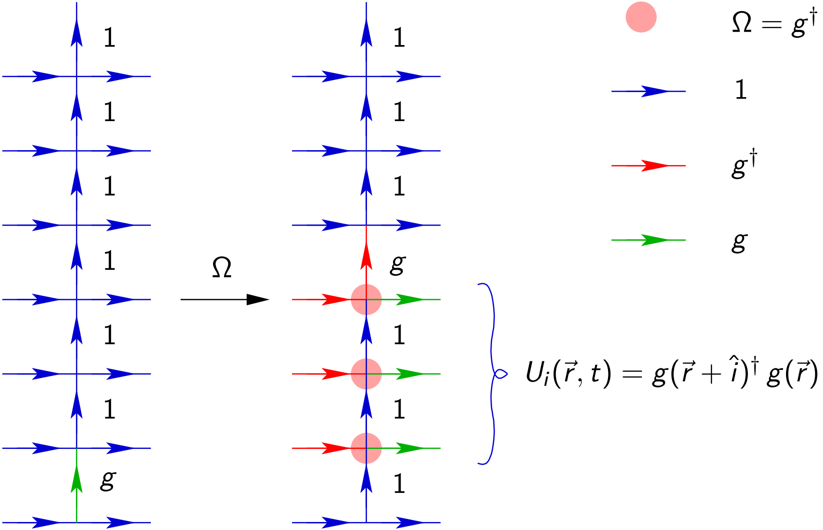
distribute to several time-slices Δt $\Rightarrow \mathcal{A}_\mu = \mathrm{i} f(t) \partial_\mu g^\dagger g$



$S_{\mathrm{min}} \approx \frac{5}{3} S_{\mathrm{Inst}}$, $S_{\mathrm{Inst}} = \frac{8\pi^2}{g^2}$

$\Delta t \to 0$	purely electric vortex
$\Delta t \to \infty$	purely magnetic vortex
Δt_{min}	electric $\approx$ magnetic contr.

Gauge transformation $\Omega(\vec{r}, t)$

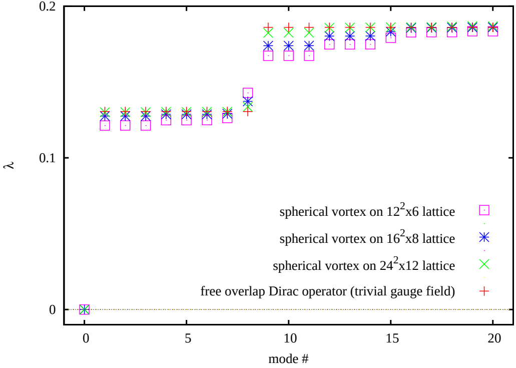


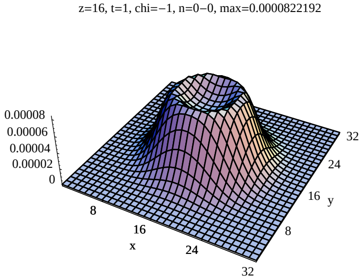
Colorful spherical vortex

Assigning to the space-like boundary constant $U_4(\mathbf{r}) = \pm 1$,
gives a map $\mathbf{r} \in S^3 \to SU(2) \ni U_4(\mathbf{r})$ characterized by a winding number

$$N = \frac{1}{24\pi^2} \int d^3x \, \epsilon_{ijk} \, \mathrm{Tr}[(\partial_i U_4 U_4^\dagger)(\partial_j U_4 U_4^\dagger)(\partial_k U_4 U_4^\dagger)],$$

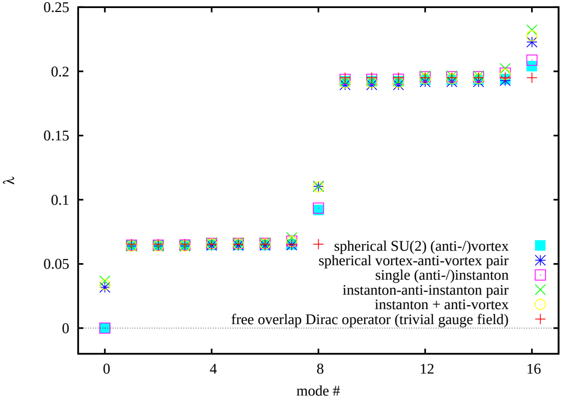
that contributes to the topology and leads to an exact zero mode.





Dirac spectra, spherical vortices and instantons

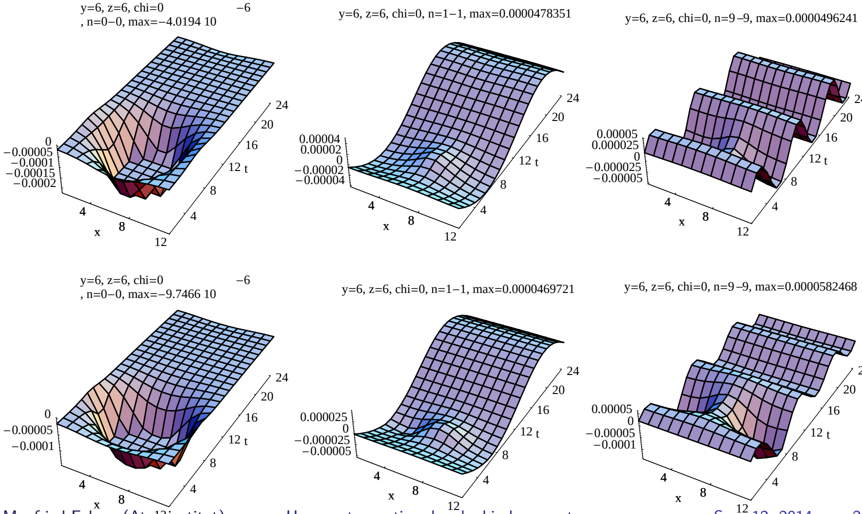
The overlap Dirac eigenvalues, and even the eigenmodes, in the background of spherical vortices are very similar to those with instantons.



With objects of opposite topological charge, the would-be zero modes interact and become near-zero modes.

Dirac spectra, spherical vortices and instantons

Chiral density for the zero mode, first and ninth modes in an instanton background (top row) and a spherical vortex background (bottom row).



Dirac spectra, spherical vortices and instantons

The similarity of the behavior of the Dirac spectrum in the background of instantons and of spherical vortices suggests that almost classical vortices, similarly to an instanton liquid model, can produce a finite spectral density near-zero and thus, by the Banks-Casher argument, lead to the spontaneous breaking of chiral symmetry.

Conclusions

- topological non-trivial color structure of vortices

  leads to topological charge of vortices
  contributes to density of near-zero modes
  explains need for orientation of P-vortices

- center vortices contribute to topological charge and near-zero modes

  via intersections
  via writhing points
  via color structure

- all objects with topological charge contribute to near-zero modes via interaction

- all topological objects contribute to $\bar{\psi}\psi$