Lattice QCD analysis for relation between quark confinement and chiral symmetry breaking

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in collaboration with
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references

T. M. Doi, H. Suganuma, T. Iritani, arXiv: 1405.1289 [hep-lat].
H. Suganuma, T. M. Doi, T. Iritani, arXiv: 1404.6494 [hep-lat].
T. M. Doi, H. Suganuma, T. Iritani, PoS (Lattice 2013) (2013) 375.
H. Suganuma, T. M. Doi, T. Iritani, PoS (QCD-TNT-III) (2014) 042.

Confinement XI, September 11, 2014, St. Petersburg

Contents

- Introduction
  - Quark confinement
  - Chiral symmetry breaking
- Previous works
  - QCD phase transition at finite temperature
  - Dirac-mode expansion and projection
- Our work

  Analytical part
  - An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice

  Numerical part
  - New modified KS formalism in temporally odd-number lattice
  - Numerical analysis for each Dirac-mode contribution to the Polyakov loop

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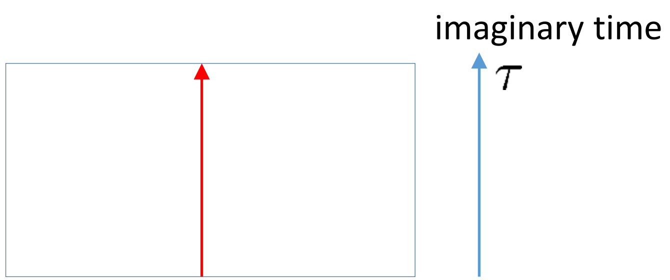
Introduction – Quark confinement

Confinement : colored state cannot be observed (quark, gluon, ･･･)
only color-singlet states can be observed (meson, baryon, ･･･)

Polyakov loop : order parameter for quark deconfinement phase transition

$$L_P(\boldsymbol{x}) = \mathrm{tr}\mathrm{T}e^{ig\int_0^\beta d\tau A_4(\boldsymbol{x},\tau)}$$ in continuum theory

$$= \mathrm{tr}\prod_{s_4=1}^{N_4} U_4(\boldsymbol{s}, s_4)$$ in lattice theory



Finite temperature :
(anti) periodic boundary condition for time direction

$$\langle L_P \rangle = \frac{1}{V}\sum_{\boldsymbol{x}} \langle L_P(\boldsymbol{x}) \rangle$$:Polyakov loop

F_q :free energy of the system with a single static quark

$$= e^{-\beta F_q} \begin{cases} = 0 & (F_q = \infty,\ \text{confinement phase}) \\ \neq 0 & (F_q : \text{finite},\ \text{deconfinement phase}) \end{cases}$$

Introduction – Chiral Symmetry Breaking

- Chiral symmetry breaking : chiral symmetry is spontaneously broken

$$\mathrm{SU(N)_L} \times \mathrm{SU(N)_R} \underset{\text{CSB}}{\to} \mathrm{SU(N)_V}$$

for example $\mathrm{SU(2)}$

- u, d quarks get dynamical mass(constituent mass)
- Pions appear as NG bosons

- Chiral condensate : order parameter for chiral phase transition

$$\langle \bar{q}q \rangle \begin{cases} \neq 0 & \text{(chiral broken phase)} \\ = 0 & \text{(chiral restored phase)} \end{cases}$$

- Banks-Casher relation

$$\langle \bar{q}q \rangle = - \lim_{m \to 0} \lim_{V \to \infty} \pi \langle \rho(0) \rangle$$

$\hat{\not{D}}$:Dirac operator

$\hat{\not{D}} |n\rangle = i\lambda_n |n\rangle$:Dirac eigenvalue equation

$\rho(\lambda) = \frac{1}{V} \sum_n \delta(\lambda - \lambda_n)$:Dirac eigenvalue density

Contents

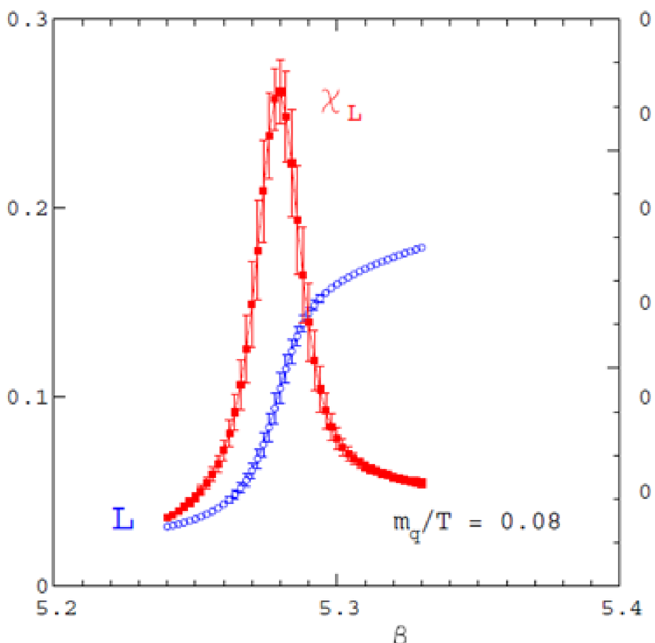
QCD phase transition at finite temperature

F. Karsch, Lect. Notes Phys. 583, 209 (2002)

$\langle L \rangle, \chi_L$: Polyakov loop and its susceptibility

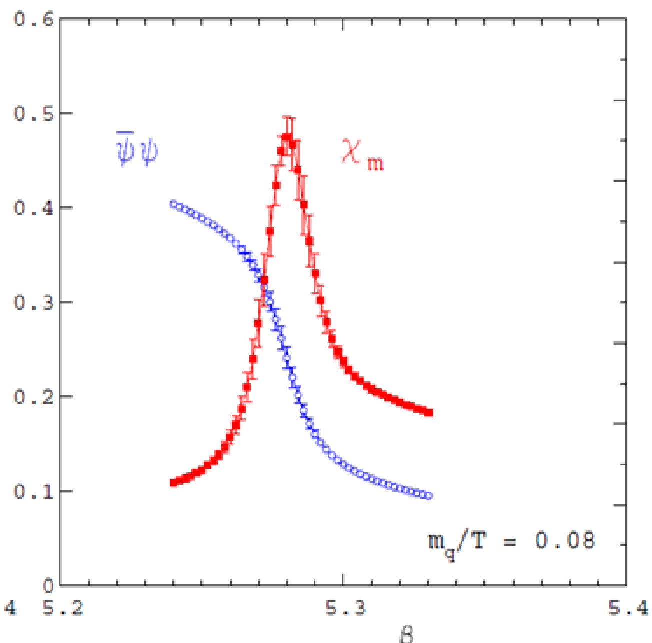
$\langle \bar{\psi}\psi \rangle, \chi_m$: chiral condensate and its susceptibility

deconfinement transition



Low T ⟷ High T

chiral transition



Low T ⟷ High T

- $\mu = 0$
- two flavor QCD with light quarks

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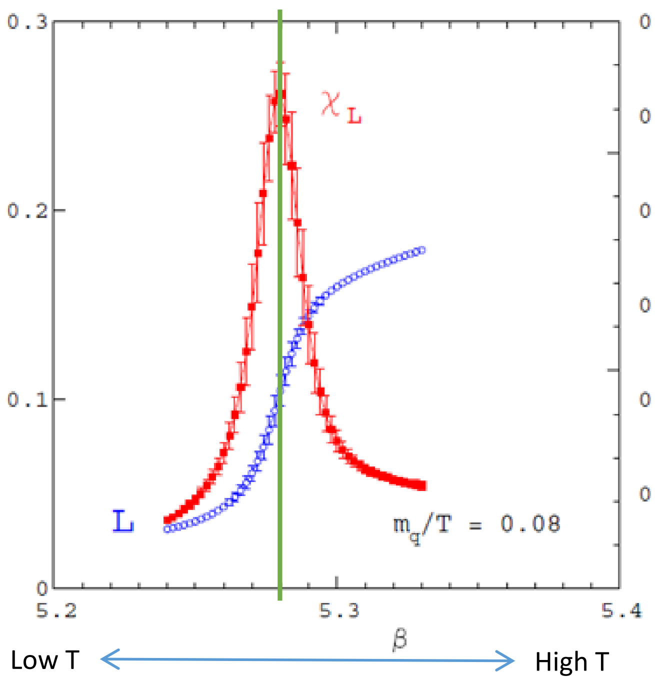
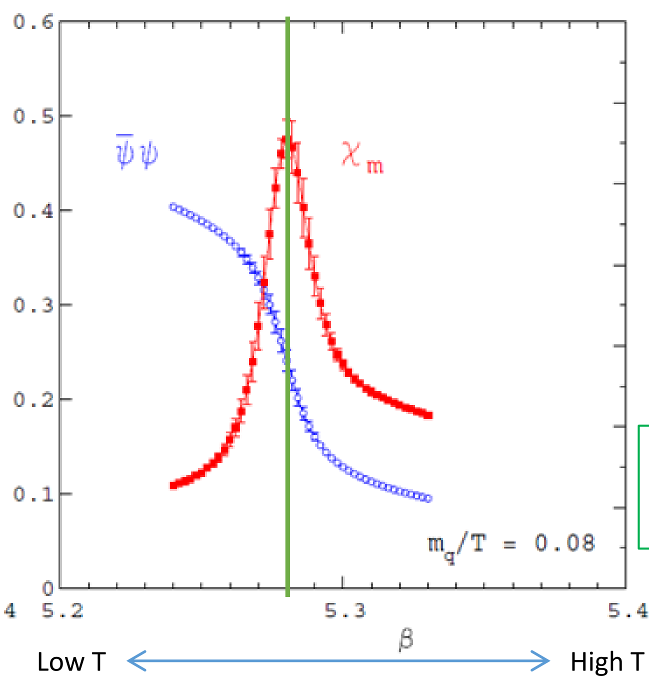


chiral transition



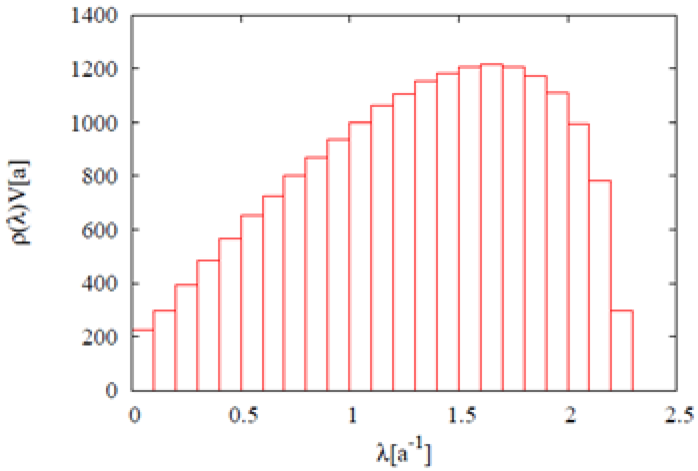
- $\mu = 0$
- two flavor QCD with light quarks

We define critical temperature as the peak of susceptibility

$\Longrightarrow$ These two phase transitions are strongly correlated(?)

S. Gongyo, T. Iritani, H. Suganuma, PRD86 (2012) 034510
T. Iritani and H. Suganuma, PTEP, 2014 3, 033B03 (2014).

Dirac-mode expansion and projection



Dirac eigenvalue equation: $\hat{\not{D}}|n\rangle = i\lambda_n|n\rangle$

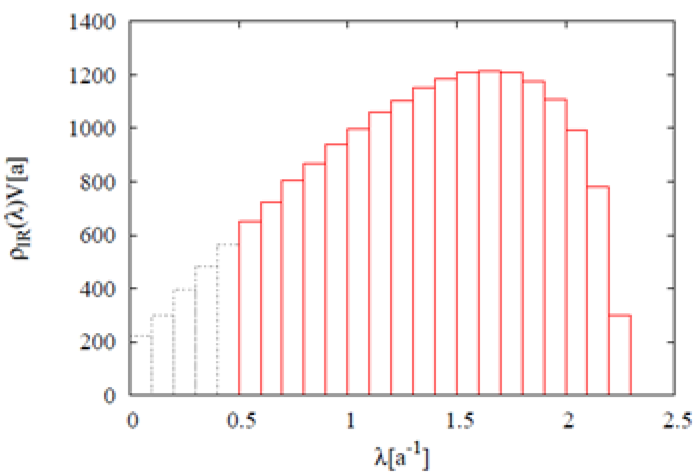
Dirac eigenmode: $|n\rangle$

Dirac eigenvalue: $i\lambda_n$

Dirac eigenvalue density: $\rho(\lambda) = \frac{1}{V}\sum_n \delta(\lambda - \lambda_n)$

Banks-Casher relation: $\langle \bar{q}q\rangle = -\lim_{m\to 0}\lim_{V\to\infty}\pi\langle\rho(0)\rangle$

removing low-lying Dirac modes(Dirac IR cut) ⟺ removing the essence of CSB



$$\langle \bar{q}q\rangle_{\mathrm{IR}} = -\frac{1}{V}\sum_{\lambda_n > \Lambda_{\mathrm{IR}}} \frac{2m}{\lambda_n^2 + m^2}$$

$$\frac{\langle \bar{q}q\rangle_{\mathrm{IR}}}{\langle \bar{q}q\rangle} \simeq 0.02 \quad (\Lambda_{\mathrm{IR}} \simeq 0.4\ \mathrm{GeV},\ m \simeq 5\ \mathrm{MeV})$$

※This formalism is manifestly gauge invariant.

Dirac-mode expansion and projection

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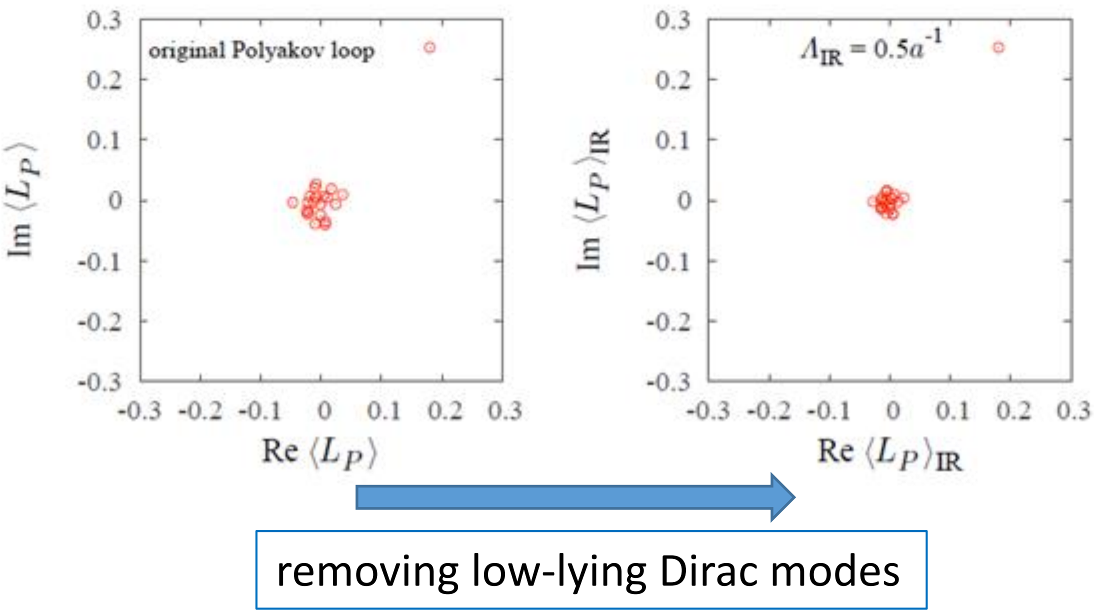

$L_{\rm P}$:Polyakov loop

$$\langle L_{\rm P} \rangle \begin{cases} = 0 & (F_q = \infty, \ \text{confinement phase}) \\ \neq 0 & (F_q : \text{finite}, \ \text{deconfinement phase}) \end{cases}$$

After removing the essence of CSB, the confinement property is kept

$\Longrightarrow$ one-to-one correspondence does not hold for confinement and chiral symmetry breaking in QCD.

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An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice

H. Suganuma, TMD, T. Iritani, arXiv: 1404.6494 [hep-lat].
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$$L_P = \frac{(2i)^{N_4-1}}{12V} \sum_n \lambda_n^{N_4-1} \langle n|\hat{U}_4|n\rangle$$ on temporally odd number lattice: N_4 is odd

(in lattice unit: $a = 1$)

notation:

- Polyakov loop : L_P
- Dirac eigenmode : $\hat{\not{D}}|n\rangle = i\lambda_n|n\rangle$
- Link variable operator : $\langle s|\hat{U}_\mu|s'\rangle = U_\mu(s)\delta_{s+\hat{\mu},s'}$
- Covariant derivative : $\hat{D}_\mu = \frac{1}{2}(\hat{U}_\mu - \hat{U}_{-\mu})$
- Dirac operator : $\hat{\not{D}} = \frac{1}{2}\sum_\mu \gamma_\mu(\hat{U}_\mu - \hat{U}_{-\mu})$

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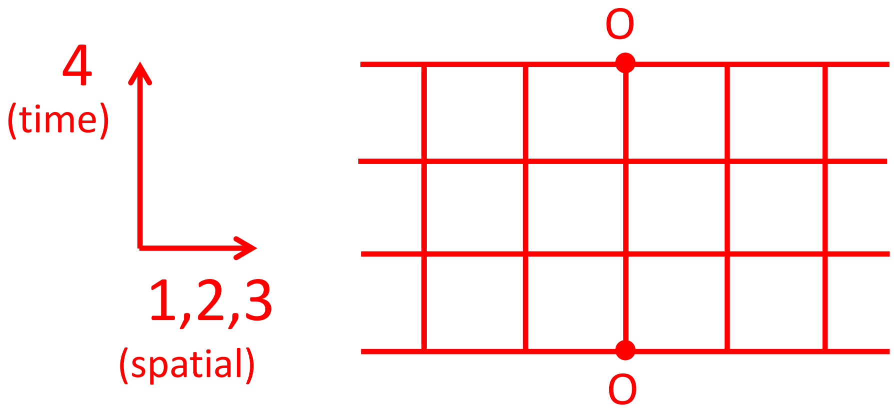
derivation

An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice



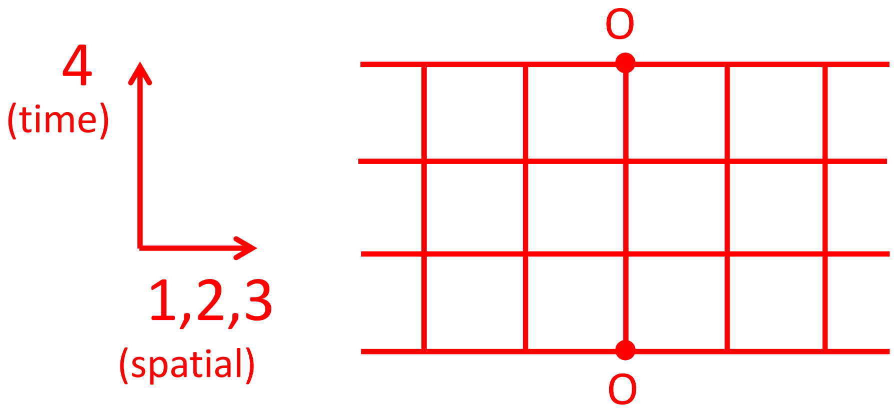
$N_4 = 3$ case

In this study, we use

- standard square lattice
- with ordinary periodic boundary condition for gluons,
- with the odd temporal length N_4

(temporally odd-number lattice)

An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice

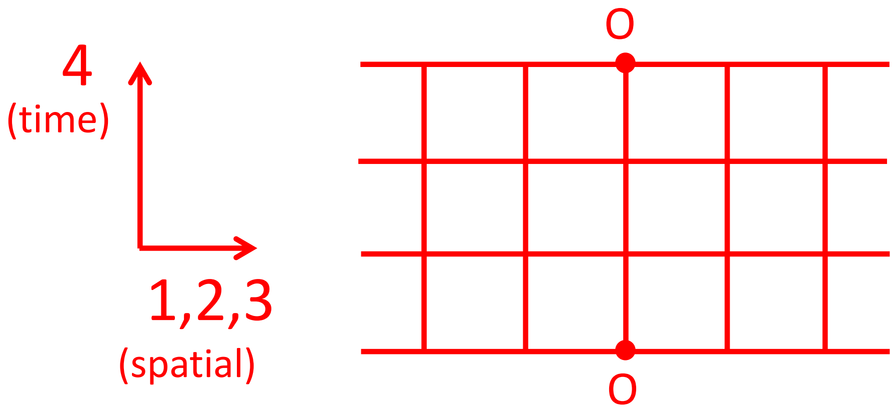


$N_4 = 3$ case

In this study, we use
- standard square lattice
- with ordinary periodic boundary condition for gluons,
- with the odd temporal length N_4
  (temporally odd-number lattice)

Note: in the continuum limit of $a \to 0$, $N_4 \to \infty$,
any number of large N_4 gives the same result.
Then, it is no problem to use the odd-number lattice.

An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice



$N_4 = 3$ case

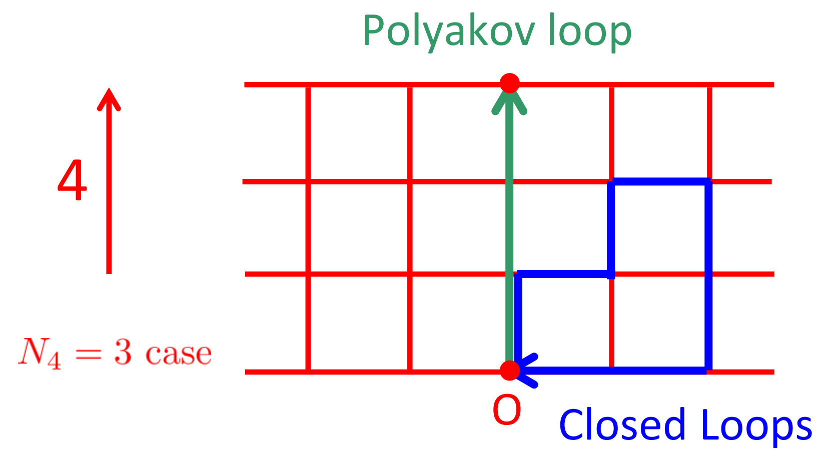
In this study, we use
- standard square lattice
- with ordinary periodic boundary condition for gluons,
- with the odd temporal length N_4

  (temporally odd-number lattice)

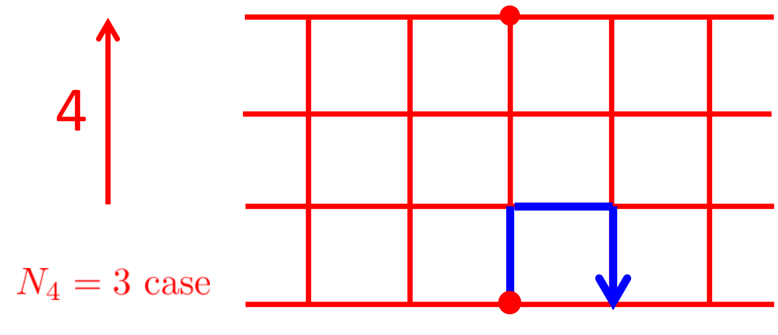
For the simple notation,
we take the lattice unit a=1 hereafter.

An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice

In general, only gauge-invariant quantities such as Closed Loops and the Polyakov loop survive in QCD. (Elitzur's Theorem)



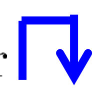
All the non-closed lines are gauge-variant and their expectation values are zero.



e.g.

$$\mathrm{Tr}\hat{U}_4\hat{U}_1\hat{U}_{-4} = \sum_s \mathrm{tr}\{U_4(s)U_1(s+\hat{4})U_4^\dagger(s+\hat{1})\}\underline{\delta_{s,s+\hat{1}}} = 0$$

gauge-variant

(Tr  =0)

$$\langle s|\hat{U}_\mu|s'\rangle = U_\mu(s)\delta_{s+\hat{\mu},s'}$$

An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice

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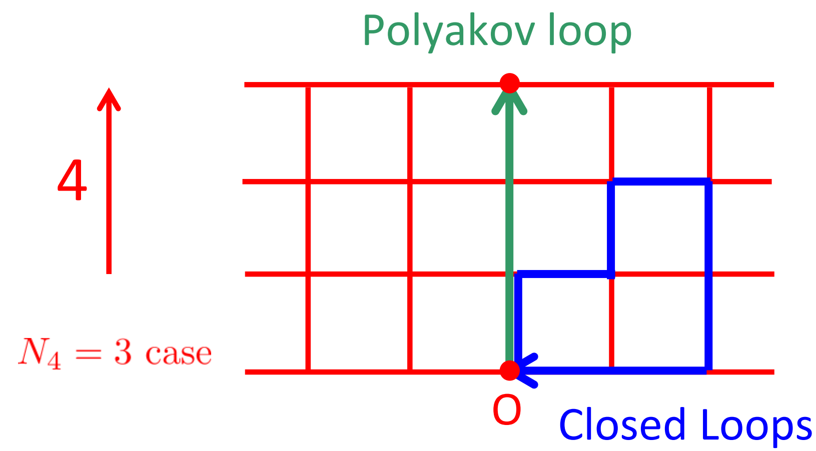

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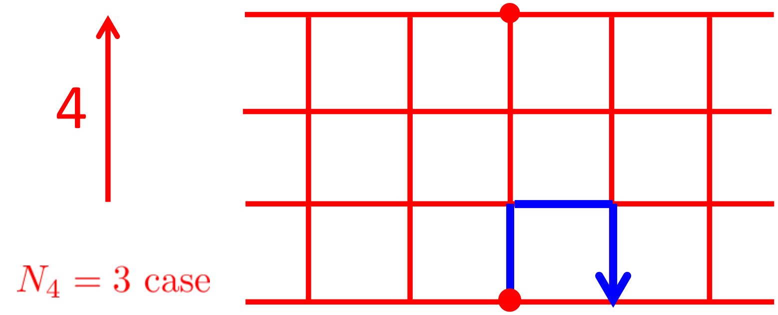

e.g.

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gauge-variant

(Tr ⌐↓ =0)

Key point

Note: any closed loop needs even-number link-variables on the square lattice.

An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice

We consider the functional trace I on the temporally odd-number lattice:

$$I \equiv \mathrm{Tr}_{c,\gamma}(\hat{U}_4 \hat{\not{D}}^{N_4-1}) \quad (N_4 : \mathrm{odd})$$

definition:

$\langle s|\hat{U}_\mu|s'\rangle = U_\mu(s)\delta_{s+\hat{\mu},s'}$

$\mathrm{Tr}_{c,\gamma} \equiv \Sigma_s \mathrm{tr}_c \mathrm{tr}_\gamma$

site & color & spinor

Dirac operator : $\hat{\not{D}} = \frac{1}{2}\sum_\mu \gamma_\mu(\hat{U}_\mu - \hat{U}_{-\mu})$

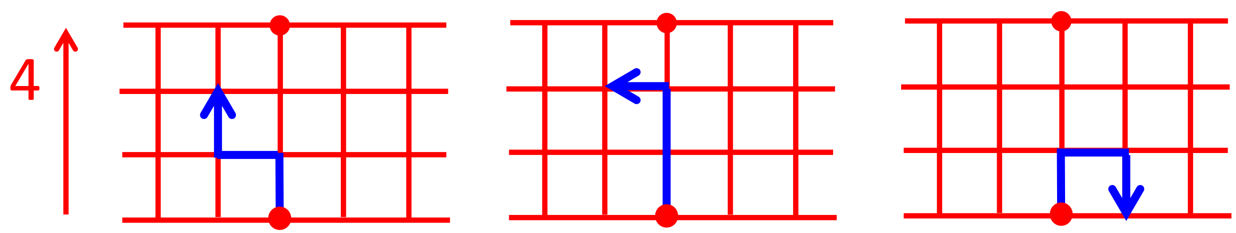
$\hat{U}_4 \hat{\not{D}}^{N_4-1}$ is expressed as a sum of products of N_4 link-variable operators because the Dirac operator $\hat{\not{D}}$ includes one link-variable operator in each direction $\hat{\mu}$.

$I \equiv \mathrm{Tr}_{c,\gamma}(\hat{U}_4 \hat{\not{D}}^{N_4-1})$ includes many trajectories on the square lattice.

$N_4 = 3$ case

length of trajectories: $N_4 = 3$

odd !!



An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice

We consider the functional trace I on the temporally odd-number lattice:

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definition:

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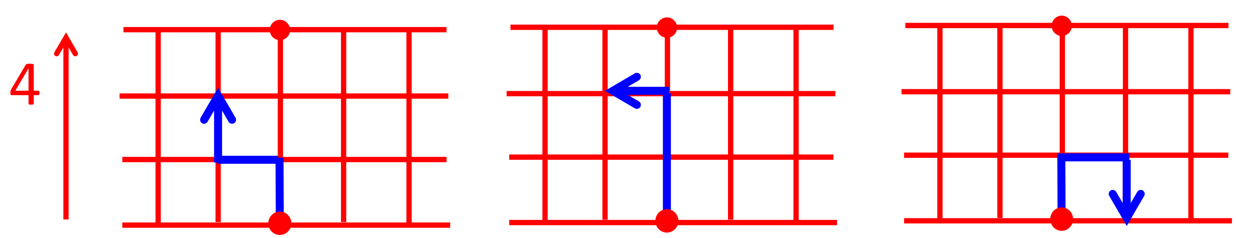


Key point

Note: any closed loop needs even-number link-variables on the square lattice.

An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice

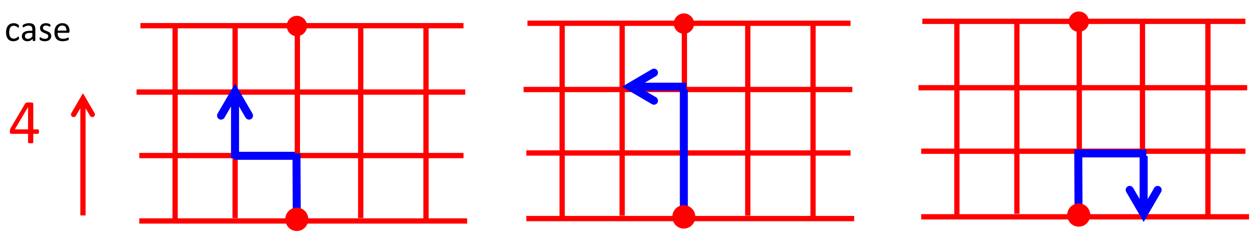
$$I \equiv \mathrm{Tr}_{c,\gamma}(\hat{U}_4 \hat{\not{D}}^{N_4-1}) \qquad (N_4 : \mathrm{odd})$$

Dirac operator : $\hat{\not{D}} = \frac{1}{2}\sum_\mu \gamma_\mu(\hat{U}_\mu - \hat{U}_{-\mu})$

In this functional trace $I \equiv \mathrm{Tr}_{c,\gamma}(\hat{U}_4 \hat{\not{D}}^{N_4-1})$,
it is impossible to form a closed loop on the square lattice,
because the length of the trajectories, N_4, is odd.

Almost all trajectories are gauge-variant & give no contribution.

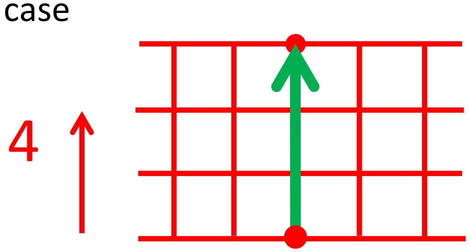
$N_4 = 3$ case



gauge variant
(no contribution)

Only the exception is the Polyakov loop.

$N_4 = 3$ case



gauge invariant !!

I is proportional to the Polyakov loop.

$$I \propto L_P$$

L_P : Polyakov loop

An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice

$$I = \mathrm{Tr}_{c,\gamma}(\hat{U}_4 \hat{\not{D}}^{N_4-1}) \qquad (\mathrm{Tr}_{c,\gamma} \equiv \Sigma_s \mathrm{tr}_c \mathrm{tr}_\gamma)$$

$$= \mathrm{Tr}_{c,\gamma}\{\hat{U}_4(\gamma_4 \hat{D}_4)^{N_4-1}\}$$ (∵ only gauge-invariant quantities survive)

$$= 4\mathrm{Tr}_c(\hat{U}_4 \hat{D}_4^{N_4-1})$$ (∵ $N_4 - 1$: even, $\gamma_4^2 = 1$ and $\mathrm{tr}_\gamma 1 = 4$)

$$= \frac{4}{2^{N_4-1}} \mathrm{Tr}_c\{\hat{U}_4(\hat{U}_4 - \hat{U}_{-4})^{N_4-1}\}$$

$$= \frac{4}{2^{N_4-1}} \mathrm{Tr}_c\{\hat{U}_4^{N_4}\}$$ (∵ only gauge-invariant quantities survive)

$$= \frac{12V}{2^{N_4-1}} L_P$$ (∵ $L_P = \frac{1}{3V}\mathrm{Tr}_c\{\hat{U}_4^{N_4}\}$: Polyakov loop)

($V = N_1 N_2 N_3 N_4$: lattice volume)

Thus, $I \equiv \mathrm{Tr}_{c,\gamma}(\hat{U}_4 \hat{\not{D}}^{N_4-1})$ is proportional to the Polyakov loop.

$$I = \mathrm{Tr}_{c,\gamma}(\hat{U}_4 \hat{\not{D}}^{N_4-1}) = \frac{12V}{(2a)^{N_4-1}} L_P$$

An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice

On the one hand,

$$I = \frac{12V}{2^{N_4-1}} L_P \quad \cdots ①$$

On the other hand, take the Dirac modes as the basis for functional trace

$$\begin{aligned} I &= \mathrm{Tr}_{c,\gamma}(\hat{U}_4 \hat{\not{D}}^{N_4-1}) \\ &= \sum_n \langle n|\hat{U}_4 \hat{\not{D}}^{N_4-1}|n\rangle \\ &= i^{N_4-1} \sum_n \lambda_n^{N_4-1} \langle n|\hat{U}_4|n\rangle \quad \cdots ② \end{aligned}$$

Dirac eigenmode

$$\hat{\not{D}}|n\rangle = i\lambda_n|n\rangle$$

$$\sum_n |n\rangle\langle n| = 1$$

from ①、② →

$$L_P = \frac{(2i)^{N_4-1}}{12V} \sum_n \lambda_n^{N_4-1} \langle n|\hat{U}_4|n\rangle$$

Note 1: this relation holds gauge-independently. (No gauge-fixing)

Note 2: this relation does not depend on lattice fermion for sea quarks.

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H. Suganuma, TMD, T. Iritani, arXiv: 1404.6494 [hep-lat].
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(in lattice unit: $a = 1$)

notation:

- Polyakov loop : L_P
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- Link variable operator : $\langle s|\hat{U}_\mu|s'\rangle = U_\mu(s)\delta_{s+\hat{\mu},s'}$

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- Link variable operator : $\langle s|\hat{U}_\mu|s'\rangle = U_\mu(s)\delta_{s+\hat{\mu},s'}$

- Low-lying Dirac-modes are important for CSB (Banks-Casher relation) $(\lambda_n \sim 0)$
- Low-lying Dirac-modes have little contribution to Polyakov loop

↓

The relation between Confinement and CSB is not one-to-one correspondence in QCD.

This conclusion agrees with the previous work by Gongyo, Iritani, Suganuma.
S. Gongyo, T. Iritani, H. Suganuma, PRD86 (2012) 034510
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- Low-lying Dirac-modes are important for CSB (Banks-Casher relation) $(\lambda_n \sim 0)$
- Low-lying Dirac-modes have little contribution to Polyakov loop

The relation between Confinement and CSB is **not one-to-one correspondence in QCD.**

In fact, from similar analysis,
we can derive the similar relation between Wilson loop and Dirac mode.
Therefore, low-lying Dirac-modes have little contribution
to the string tension σ, or the confining force.

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Numerical analysis for each Dirac-mode contribution to the Polyakov loop

Numerical confirmation of this relation is important.

$$L_P = \frac{(2i)^{N_4-1}}{12V} \sum_n \lambda_n^{N_4-1} \langle n|\hat{U}_4|n\rangle \quad \cdots \text{(A)}$$

$$L_P = \frac{(2i)^{N_4-1}}{12V} \sum_n \lambda_n^{N_4-1} \sum_s \psi_n^\dagger(s) U_4(s) \psi_n(s+\hat{4})$$

$\langle L_P\rangle,\ U_4(s)$: easily obtained

*This formalism is gauge invariant.

$\lambda_n,\ \psi_n^\dagger(s),\ \psi_n(s+\hat{4})$:are determined from $\hat{\not{D}}|n\rangle = i\lambda_n|n\rangle$

notation and coordinate representation

$$\langle s|\hat{U}_4|s'\rangle = U_4(s)\delta_{s+\hat{4},s'}$$

$$\langle s'|n\rangle = \psi_n(s')$$

$$\langle n|s\rangle = \psi_n^\dagger(s)$$

$$\hat{D}_\mu = \frac{1}{2}(\hat{U}_\mu - \hat{U}_{-\mu})$$

$$1 = \sum_s |s\rangle\langle s| \qquad |s\rangle \text{ : site}$$

explicit form of the Dirac eigenvalue equation

$$\sum_{s',j,\beta} \not{D}_{ss'}^{ij,\alpha\beta} \psi_n(s')^{j,\beta} = i\lambda_n \psi_n(s)^{i,\alpha}$$

s, s' : site
i, j : color
α, β : spinor

where $\not{D}_{ss'}^{ij,\alpha\beta} = \frac{1}{2}\sum_{\mu=1}^{4} \gamma_\mu^{\alpha\beta}\left[U_\mu(s)^{ij}\delta_{s+\hat{\mu},s'} - U_{-\mu}(s)^{ij}\delta_{s-\hat{\mu},s'}\right]$

$$U_{-\mu}(s) \equiv U_\mu(s-\hat{\mu})^\dagger$$

New Modified Kogut-Susskind(KS) Formalism on Temporally Odd Number Lattice

TMD, H. Suganuma, T. Iritani, arXiv: 1405.1289 [hep-lat].
TMD, H. Suganuma, T. Iritani, PoS (Lattice 2013) (2013) 375.

N_1, N_2, N_3 : even
N_4 : odd ← "temporally odd-number lattice"

$$M(s) \equiv \gamma_1^{s_1}\gamma_2^{s_2}\gamma_3^{s_3}\gamma_4^{s_1+s_2+s_3}$$

$$\Longrightarrow M^\dagger(s)\gamma_\mu M(s\pm\hat{\mu}) = \eta_\mu(s)\gamma_4$$

We use Dirac representation(γ_4 is diagonalized)

$\not{D}$ is spin diagonalized

$$\Longrightarrow M^\dagger \not{D} M \equiv \sum_\mu M^\dagger(s)\gamma_\mu D_\mu M(s+\hat{\mu}) = \begin{pmatrix} \eta\cdot D & 0 & 0 & 0 \\ 0 & \eta\cdot D & 0 & 0 \\ 0 & 0 & -\eta\cdot D & 0 \\ 0 & 0 & 0 & -\eta\cdot D \end{pmatrix}$$

where $(\eta\cdot D)^{ij}_{ss'} = (\eta_\mu D_\mu)^{ij}_{ss'} = \frac{1}{2}\sum_{\mu=1}^{4}\eta_\mu(s)\left[U_\mu(s)^{ij}\delta_{s+\hat{\mu},s'} - U_{-\mu}(s)^{ij}\delta_{s-\hat{\mu},s'}\right]$

case of even lattice

N_1, N_2, N_3, N_4 : even

$$T(s) \equiv \gamma_1^{s_1}\gamma_2^{s_2}\gamma_3^{s_3}\gamma_4^{s_4}$$

$$\Longrightarrow T^\dagger(s)\gamma_\mu T(s\pm\hat{\mu}) = \eta_\mu(s)\mathbf{1}_{\rm spinor}$$

$$\Longrightarrow T^\dagger \not{D} T = \begin{pmatrix} \eta\cdot D & 0 & 0 & 0 \\ 0 & \eta\cdot D & 0 & 0 \\ 0 & 0 & \eta\cdot D & 0 \\ 0 & 0 & 0 & \eta\cdot D \end{pmatrix}$$

staggered phase: $\eta_\mu(s)$

$$\eta_\mu(s) = \begin{cases} 1 & (\mu=1) \\ (-1)^{s_1} & (\mu=2) \\ (-1)^{s_1+s_2} & (\mu=3) \\ (-1)^{s_1+s_2+s_3} & (\mu=4) \end{cases}$$

Numerical analysis for each Dirac-mode contribution to the Polyakov loop

TMD, H. Suganuma, T. Iritani, arXiv: 1405.1289 [hep-lat].
TMD, H. Suganuma, T. Iritani, PoS (Lattice 2013) (2013) 375.

$$L_P = \frac{(2i)^{N_4-1}}{12V}\sum_n \lambda_n^{N_4-1}\langle n|\hat{U}_4|n\rangle \quad \cdots\text{(A)}$$

Dirac eigenmode $|n\rangle$

$$\not{D}|n\rangle = i\lambda_n|n\rangle$$

⇓

$$L_P = \frac{(2i)^{N_4-1}}{3V}\sum_n \lambda_n^{N_4-1}(n|\hat{U}_4|n) \quad \cdots\text{(A)'}$$

KS Dirac eigenmode $|n)$

$$\eta\cdot D|n) = i\lambda_n|n)$$

(A)⇔(A)' relation (A)' is equivalent to (A)

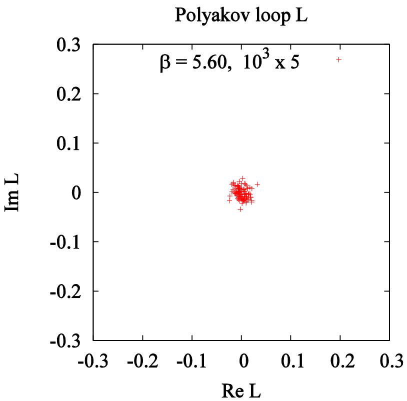
lattice setup

- quenched SU(3) lattice QCD
- standard plaquette action
- gauge coupling: $\beta = \frac{2N_c}{g^2} = 5.6\ ,\ 6.0$ ⇔ lattice spacing : $a \simeq 0.25\ ,\ 0.10$ fm
- lattice size: $N_{\rm space}^3 \times N_4 = 10^3 \times \underline{5}$ (odd)
- periodic boundary condition for link-variables

λ_n V.S. $(n|\hat{U}_4|n)$, $\lambda_n^{N_4-1}(n|\hat{U}_4|n)$

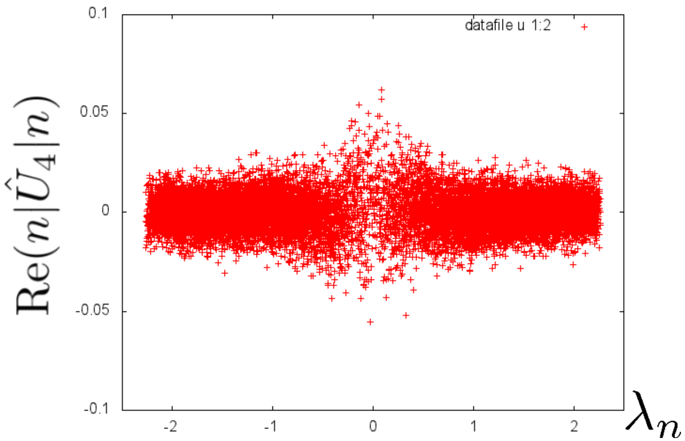
$$L_P = \frac{(2i)^{N_4-1}}{3V} \sum_n \lambda_n^{N_4-1}(n|\hat{U}_4|n)$$

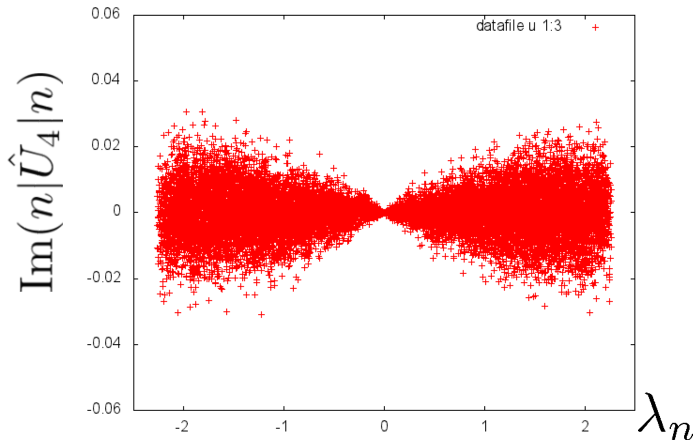
$\beta = 5.6$
lattice size : $10^3 \times 5$

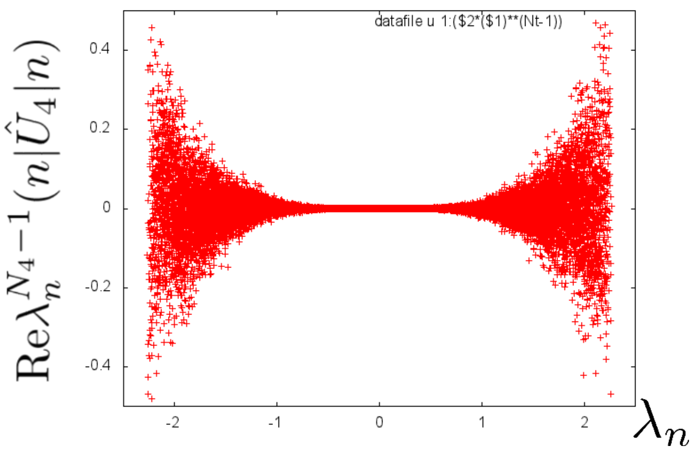


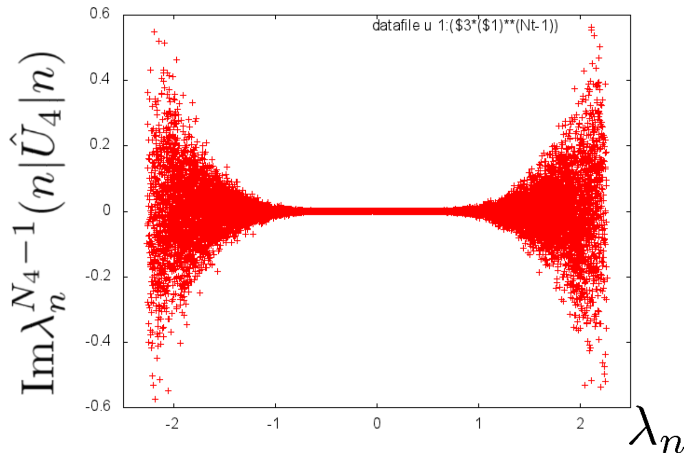
$\langle L_P \rangle = 0$
(confined phase)

$\not{D}|n\rangle = i\lambda_n|n\rangle$
Dirac eigenvalue: $i\lambda_n$





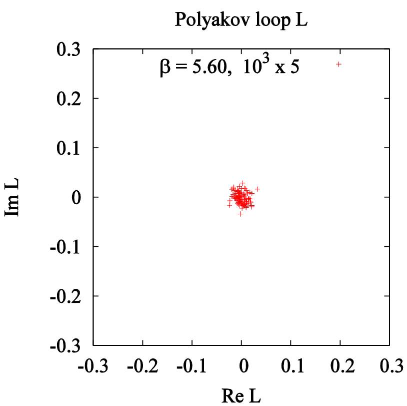




λ_n V.S. $(n|\hat{U}_4|n)$, $\lambda_n^{N_4-1}(n|\hat{U}_4|n)$

$$L_P = \frac{(2i)^{N_4-1}}{3V}\sum_n \lambda_n^{N_4-1}(n|\hat{U}_4|n)$$

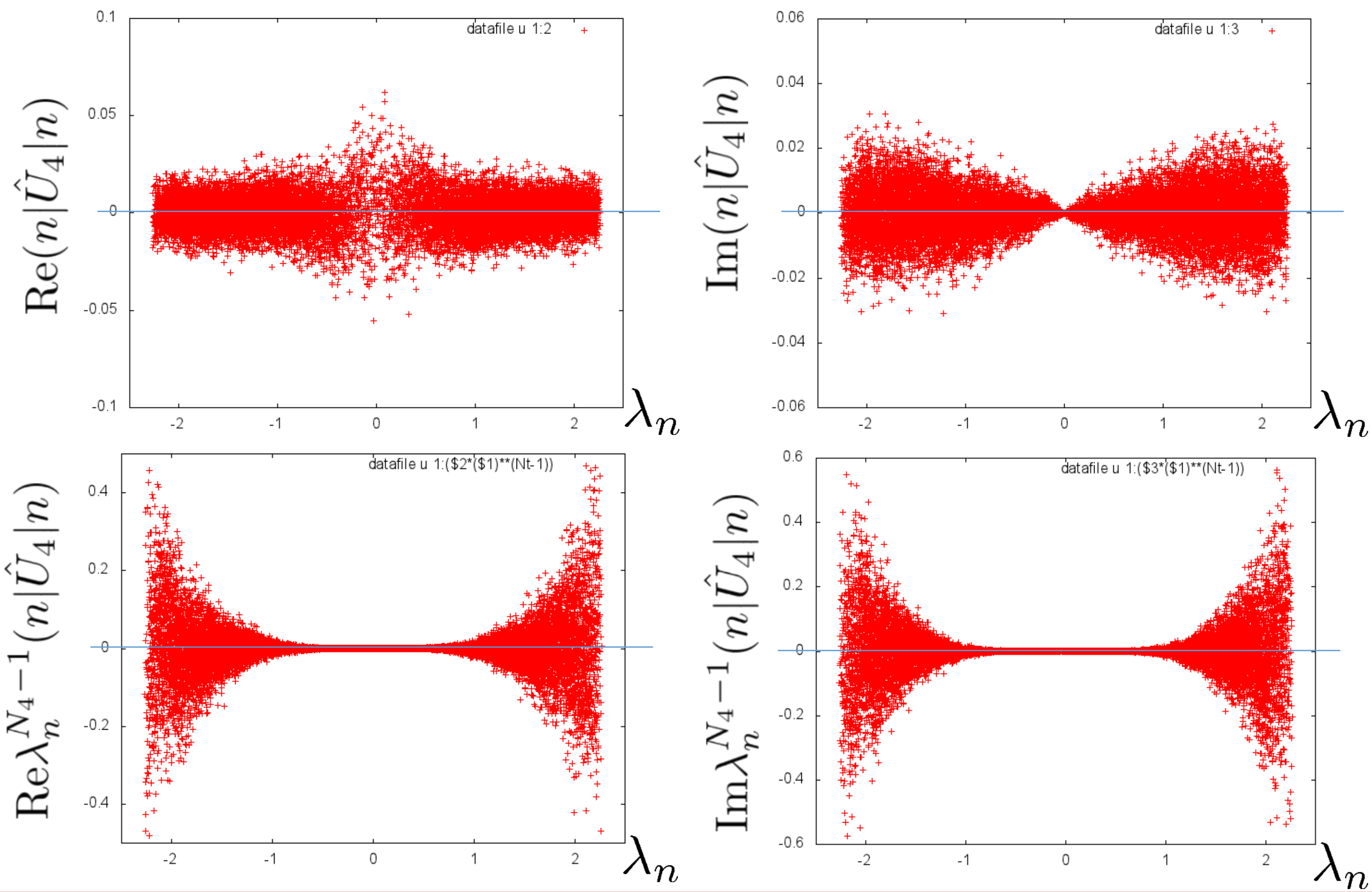
$\beta = 5.6$
lattice size : $10^3 \times 5$



$\langle L_P \rangle = 0$
(confined phase)

$$D\!\!\!/|n\rangle = i\lambda_n|n\rangle$$

Dirac eigenvalue: $i\lambda_n$



confined phase

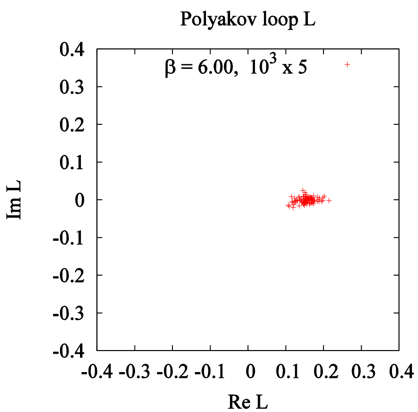
$\langle L \rangle = 0$ is due to the symmetric distribution of positive/negative value of $(n|\hat{U}_4|n)$, $\lambda_n^{N_4-1}(n|\hat{U}_4|n)$

Low-lying Dirac modes have little contribution to Polyakov loop.

λ_n V.S. $(n|\hat{U}_4|n)$, $\lambda_n^{N_4-1}(n|\hat{U}_4|n)$

$$L_P = \frac{(2i)^{N_4-1}}{3V}\sum_n \lambda_n^{N_4-1}(n|\hat{U}_4|n)$$

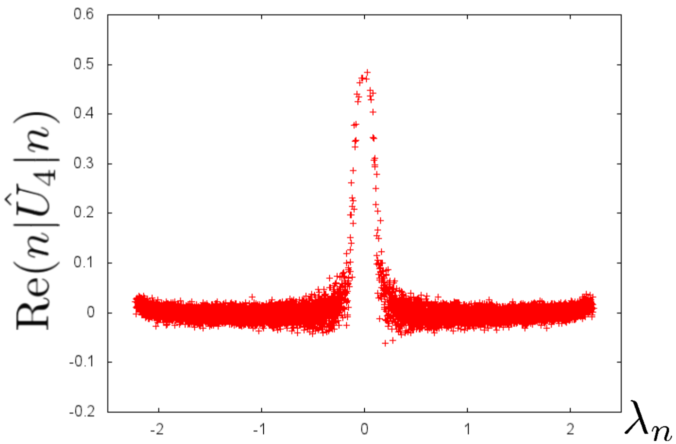
$\beta = 6.0$
lattice size : $10^3 \times 5$

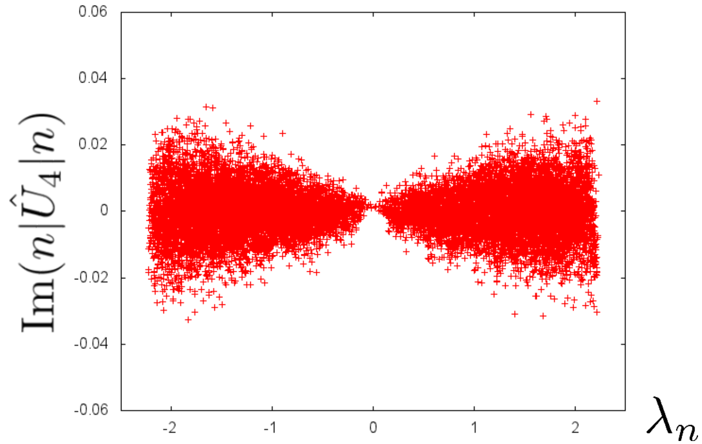


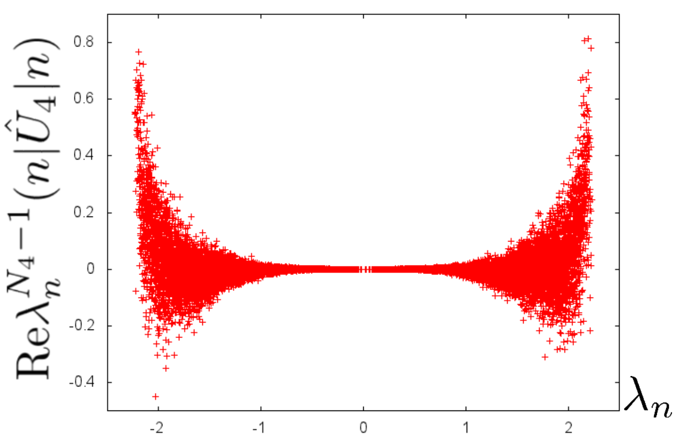
$\langle L_P \rangle \neq 0$
(deconfined phase)

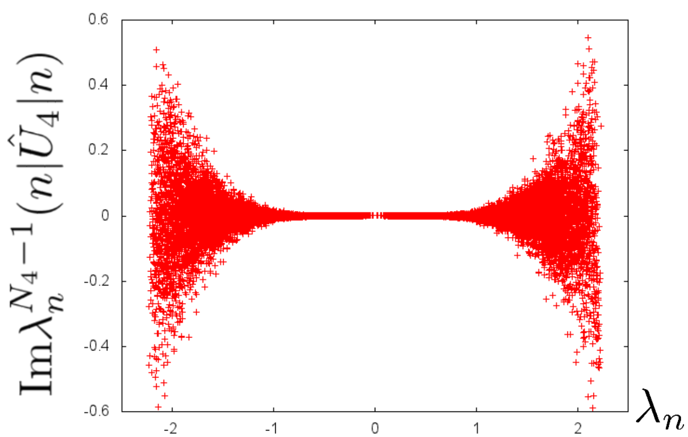
$$\not{D}|n\rangle = i\lambda_n|n\rangle$$

Dirac eigenvalue: $i\lambda_n$







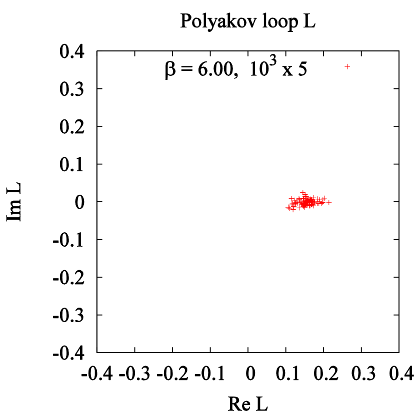


In our calculation, Polyakov loop is real, so only real part is different from it in confined phase.

λ_n V.S. $(n|\hat{U}_4|n)$, $\lambda_n^{N_4-1}(n|\hat{U}_4|n)$

$$L_P = \frac{(2i)^{N_4-1}}{3V} \sum_n \lambda_n^{N_4-1}(n|\hat{U}_4|n)$$

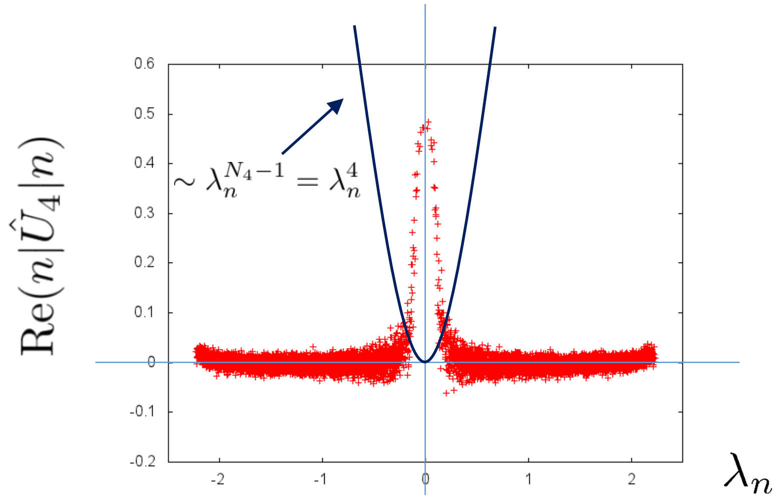
$\beta = 6.0$
lattice size : $10^3 \times 5$

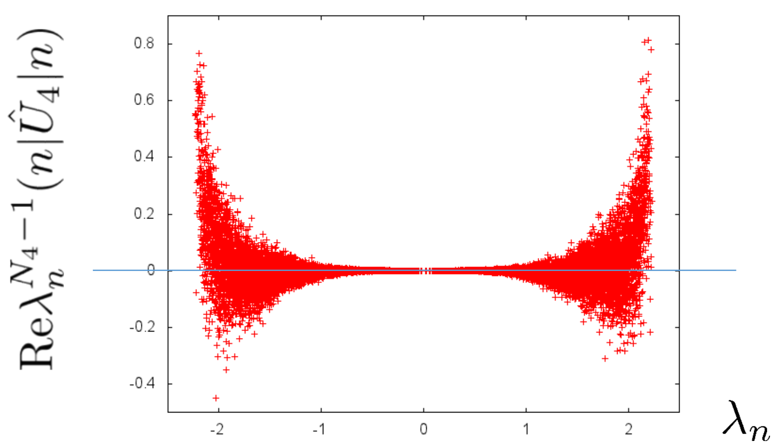


$\langle L_P \rangle \neq 0$
(deconfined phase)

$$\not{D}|n\rangle = i\lambda_n|n\rangle$$

Dirac eigenvalue: $i\lambda_n$





In low-lying Dirac modes region, $\mathrm{Re}(n|\hat{U}_4|n)$ has a large value,
but contribution of low-lying (IR) Dirac modes to Polyakov loop is very small
because of dumping factor $\lambda_n^{N_4-1}$

Summary

Analytical part

H. Suganuma, TMD, T. Iritani, arXiv: 1404.6494 [hep-lat].
H. Suganuma, TMD, T. Iritani, PoS (QCD-TNT-III) (2014) 042.

We have derived the analytical relation between Polyakov loop and Dirac eigenmodes on temporally odd-lattice lattice:

$$L_P = \frac{(2i)^{N_4-1}}{12V} \sum_n \lambda_n^{N_4-1} \langle n|\hat{U}_4|n\rangle \qquad (N_4 \text{ is odd})$$

Polyakov loop : L_P
Dirac eigenmode : $\hat{\not{D}}|n\rangle = i\lambda_n|n\rangle$
Link variable operator :
$\langle s|\hat{U}_\mu|s'\rangle = U_\mu(s)\delta_{s+\hat{\mu},s'}$

We use only
- standard square lattice
- with ordinary periodic boundary condition for link-variables,
- with the odd temporal length N_4 (temporally odd-number lattice)

conclusion:
- Low-lying Dirac modes have little contribution to the Polyakov loop
- Therefore, **The relation between confinement and chiral symmetry breaking is not one-to-one correspondence in QCD.**

Moreover, in our paper, we derived the relation between Wilson loop and Dirac modes.
From this relation, low-lying Dirac-modes have little contribution
to the string tension σ, or the confining force.

Summary

Numerical part

TMD, H. Suganuma, T. Iritani, arXiv: 1405.1289 [hep-lat].
TMD, H. Suganuma, T. Iritani, PoS (Lattice 2013) (2013) 375.

1. We numerically confirmed the relation at the quenched level.

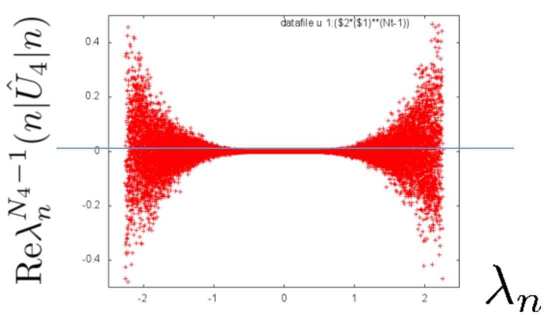
$$L_P = \frac{(2i)^{N_4-1}}{12V} \sum_n \lambda_n^{N_4-1} \langle n|\hat{U}_4|n\rangle \qquad (N_4 \text{ is odd})$$

Polyakov loop : L_P
Dirac eigenmode : $\hat{\not{D}}|n\rangle = i\lambda_n|n\rangle$
Link variable operator :
$\langle s|\hat{U}_\mu|s'\rangle = U_\mu(s)\delta_{s+\hat{\mu},s'}$

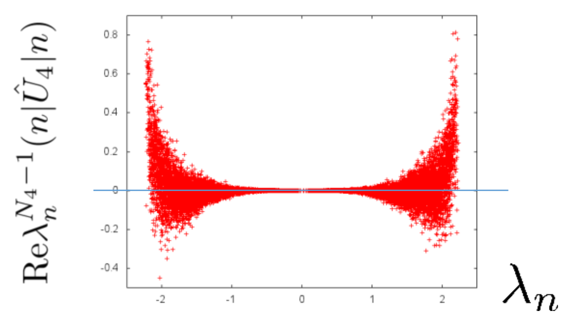
2. As the method for the numerical calculation,
we developed new Modified KS formalism applicable on temporally odd-number lattice
as well as on even lattice:

$$M(s) \equiv \gamma_1^{s_1}\gamma_2^{s_2}\gamma_3^{s_3}\gamma_4^{s_1+s_2+s_3} \quad , \quad M^\dagger \not{D} M = \begin{pmatrix} \eta\cdot D & 0 & 0 & 0 \\ 0 & \eta\cdot D & 0 & 0 \\ 0 & 0 & -\eta\cdot D & 0 \\ 0 & 0 & 0 & -\eta\cdot D \end{pmatrix}$$

3. In confined phase, $\langle L_P\rangle = 0$ is due to the positive/negative symmetry in the distribution of $\langle n|\hat{U}_4|n\rangle, \lambda_n^{N_4-1}\langle n|\hat{U}_4|n\rangle$. In deconfined phase, there is no such symmetry.



confinement phase (symmetric)



deconfinement phase (broken)

Appendix

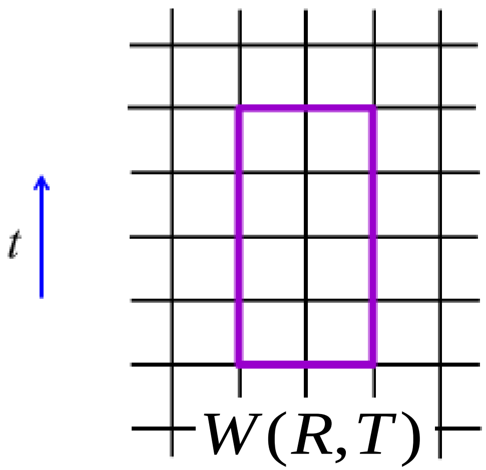
Relation between Wilson loop and Dirac mode

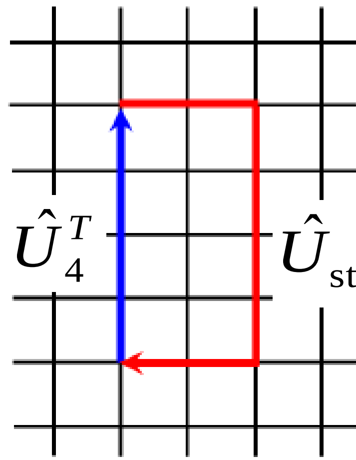
H. Suganuma, T. M. Doi, T. Iritani, arXiv: 1404.6494 [hep-lat].

We consider the functional trace on a arbitrary square lattice

$$J \equiv \mathrm{Tr}_{c,\gamma}(\hat{U}_{\mathrm{staple}}\, \hat{\not{D}}^{T})$$ (T : even)

instead of $I = \mathrm{Tr}_{c,\gamma}(\hat{U}_4\, \hat{\not{D}}^{N_4-1})$ for the Polyakov loop on the temporally odd-number lattice





$$\hat{U}_{\mathrm{staple}} \equiv \hat{U}_1^R \hat{U}_{-4}^T \hat{U}_1^R$$

Wilson loop : $W(R,T) \equiv \mathrm{Tr}\{\hat{U}_1^R \hat{U}_{-4}^T \hat{U}_1^R \hat{U}_4^T\} = \mathrm{Tr}\{\hat{U}_{\mathrm{staple}} \hat{U}_4^T\}$

Staple operator : $\hat{U}_{\mathrm{staple}} \equiv \hat{U}_1^R \hat{U}_{-4}^T \hat{U}_1^R$

Relation between Wilson loop and Dirac mode

$$J \equiv \mathrm{Tr}\hat{U}_{\mathrm{staple}}\hat{\not{D}}^{T}$$

$$= \mathrm{Tr}\hat{U}_{\mathrm{staple}}(\gamma_4\hat{D}_4)^{T}$$ ($\because$ only gauge-invariant quantities survive)

$$= \mathrm{Tr}\hat{U}_{\mathrm{staple}}\hat{D}_4^{T}$$ ($\because\ \gamma_4^{N_t-1}=1$, T is even)

$$= \frac{1}{2^T}\mathrm{Tr}\hat{U}_{\mathrm{staple}}(\hat{U}_4-\hat{U}_{-4})^{T}$$

$$= \frac{1}{2^T}\mathrm{Tr}\hat{U}_{\mathrm{staple}}\hat{U}_4^{T}$$ ($\because$ only gauge-invariant quantities survive)
($\because\ \mathrm{tr}_\gamma 1 = 4$, $\mathrm{Tr} = \sum_x \mathrm{tr}_c\ \mathrm{tr}_\gamma$)

$$= \frac{4}{2^T}W$$ ($\because\ W = \mathrm{Tr}\{\hat{U}_{\mathrm{staple}}\hat{U}_4^{T}\}$)

Thus, $J \equiv \mathrm{Tr}\{\hat{U}_{\mathrm{staple}}\hat{\not{D}}^{T}\}$ is proportional to Wilson loop W

On one hand, we obtain for even T

$$J \equiv \mathrm{Tr}\hat{U}_{\mathrm{staple}}\hat{\not{D}}^T = \frac{4}{2^T} W$$

On the other hand, using the complete set of the Dirac eigen-states $|n\rangle$

$$J \equiv \mathrm{Tr}\hat{U}_{\mathrm{staple}}\hat{\not{D}}^T = \sum_n \langle n|\hat{U}_{\mathrm{staple}}\hat{\not{D}}^T|n\rangle = (-)^{T/2}\sum_n \lambda_n{}^T \langle n|\hat{U}_{\mathrm{staple}}|n\rangle$$

$$\sum_n |n\rangle\langle n| = 1 \qquad\qquad \hat{\not{D}}|n\rangle = i\lambda_n|n\rangle$$

Combining them, we obtain a relation for even T:

$$W = (-)^{T/2} 2^{T-2} \sum_n \lambda_n{}^T \langle n|\hat{U}_{\mathrm{staple}}|n\rangle$$

$$\rightarrow \quad V(R) = -\lim_{T\to\infty}\frac{1}{T}\ln W = -\lim_{T\to\infty}\frac{1}{T}\ln\left|\sum_n (2\lambda_n)^T \langle n|\hat{U}_{\mathrm{staple}}|n\rangle\right|$$

$$\rightarrow \quad \sigma = -\lim_{R,T\to\infty}\frac{1}{RT}\ln W = -\lim_{R,T\to\infty}\frac{1}{RT}\ln\left|\sum_n (2\lambda_n)^T \langle n|\hat{U}_{\mathrm{staple}}|n\rangle\right|$$

$$W = (-)^{T/2} 2^{T-2} \sum_n \lambda_n{}^T \langle n | \hat{U}_{\text{staple}} | n \rangle$$

$$\sigma = -\lim_{R,T\to\infty} \frac{1}{RT} \ln W = -\lim_{R,T\to\infty} \frac{1}{RT} \ln \left| \sum_n (2\lambda_n)^T \langle n | \hat{U}_{\text{staple}} | n \rangle \right|$$

Because of the factor $\lambda_n{}^T$ in the sum,
low-lying Dirac-mode contribution is to be small
for the Wilson loop *W*, the inter-quark potential V(R) and
the string tension σ, unless the extra counter factor $1/\lambda_n{}^T$
appears from the matrix element $\langle n | \hat{U}_{\text{staple}} | n \rangle$.

Thus, the string tension σ, or the confining force,
is expected to be unchanged by the removal of
low-lying Dirac-mode contribution.

Relation between Wilson loop and Dirac mode

Setup: Arbitrary square lattice (including anisotropy lattice)

We only use the fact that
non-closed lines are gauge-variant and their expectation values are zero.

Analytical relation: connecting Wilson loop and Dirac mode

$$W(R,T) \propto \sum_n \lambda_n{}^T \langle n | \hat{U}_{\mathrm{staple}} | n \rangle$$

Wilson loop : $W(R,T) \equiv \mathrm{Tr}\{\hat{U}_1^R \hat{U}_{-4}^T \hat{U}_1^R \hat{U}_4^T\} = \mathrm{Tr}\{\hat{U}_{\mathrm{staple}} \hat{U}_4^T\}$

λ_n : Dirac eigenvalue, |n> : Dirac eigenstate

conculusion:

It is expected that the contribution from low-lying Dirac modes
to the string tension (confining force) is small because of the factor $\lambda_n{}^T$.

Reference: H.S., T.M.Doi, T. Iritani, arXiv:1404.6494 [hep-lat],
"Analytical relation between confinement and chiral symmetry breaking
in terms of the Polyakov loop and Dirac eigenmodes"

New Modified KS Formalism
Temporally Odd Number Lattice

T. M. Doi, H. Suganuma, T. Iritani, arXiv: 1405.1289 [hep-lat].
T. M. Doi, H. Suganuma, T. Iritani, PoS (Lattice 2013) (2013) 375.

N_1, N_2, N_3 : even

N_4 : odd

$$M(s) \equiv \gamma_1^{s_1}\gamma_2^{s_2}\gamma_3^{s_3}\gamma_4^{s_1+s_2+s_3}$$

explicit form of the reduced Dirac eigenvalue equation

$$\sum_{s',j}(\eta\cdot D)_{ss'}^{ij}\chi_n(s')^j = i\lambda_n\chi_n(s)^i$$

where $(\eta\cdot D)_{ss'}^{ij} = (\eta_\mu D_\mu)_{ss'}^{ij} = \frac{1}{2}\sum_{\mu=1}^{4}\eta_\mu(s)\left[U_\mu(s)^{ij}\delta_{s+\hat{\mu},s'} - U_{-\mu}(s)^{ij}\delta_{s-\hat{\mu},s'}\right]$

$\langle s|n\rangle = \psi_n(s) \equiv M(s)\chi_n(s)$ $\qquad$ $\chi_n(s) = \chi_n(s)^i$ don't have spinor index

---

This method is applicable to temporally odd number lattice.

periodic boundary condition $\Rightarrow$ $M(s+N_\mu\hat{\mu}) = M(s)$

This requirement is satisfied on odd lattice.

*Even without use of this method, the same results are obtained.

Relation between Dirac eigenmode and KS Dirac eigenmode

$$\langle L \rangle = \frac{(2i)^{N_4-1}}{12V} \sum_n \lambda_n^{N_4-1} \langle n|\hat{U}_4|n\rangle \quad \cdots \text{(A)}$$

in odd lattice

Dirac eigenmode $|n\rangle$

$$\not{D}|n\rangle = i\lambda_n|n\rangle \qquad \langle s|n\rangle = \psi_n(s) = M(s)\chi_n(s)$$

KS Dirac eigenmode $|n)$

$$\eta \cdot D|n) = i\lambda_n|n) \qquad \langle s|n) = \chi_n(s)$$

$$M^\dagger \not{D} M = \begin{pmatrix} \eta \cdot D & 0 & 0 & 0 \\ 0 & \eta \cdot D & 0 & 0 \\ 0 & 0 & -\eta \cdot D & 0 \\ 0 & 0 & 0 & -\eta \cdot D \end{pmatrix}$$

$$\Rightarrow \sum_n \lambda_n^{N_4-1} \langle n|\hat{U}_4|n\rangle = 4 \sum_n \lambda_n^{N_4-1} (n|\hat{U}_4|n)$$

$$\Rightarrow \langle L \rangle = \frac{(2i)^{N_4-1}}{3V} \sum_n \lambda_n^{N_4-1} (n|\hat{U}_4|n) \quad \cdots \text{(A)'}$$

relation (A)' is equivalent to (A)