Advances in QCD sum – rule calculations

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The method of sum rules is 35 years old. In spite of this respectable age, the method is being permanently enriched by new ideas and new calculations and remains one of the widely used and competitive tools for determinations of QCD parameters (e.g., quark masses) and for applications of QCD to hadron properties, such as decay constants and form factors.

1. **Two-point functions, quark masses, and leptonic decay constants**
2. **Three-point sum rules, form factors, and strong couplings**
3. **Light-cone sum rules, form factors, and strong couplings**
4. **Sum rules for exotic states**

2 – point sum rules, quark masses and leptonic decay constants

- **The basic object**

T-product of 2 quark currents currents, e.g. $j_5(x) = (m_b + m)\,\bar{q}(x) i\gamma_5 b(x)$,

$$\Pi(p^2) = i \int d^4x\, e^{ipx} \left\langle \Omega \left| T\left(j_5(x) j_5^\dagger(0) \right) \right| \Omega \right\rangle$$

- **Wilsonian OPE - separation of distances:**

$$T\left(j_5(x) j_5^\dagger(0) \right) = C_0(x^2,\mu)\hat{1} + \sum_n C_n(x^2,\mu) : \hat{O}(x=0,\mu) :$$

$$\Pi(p^2) = \Pi_{\rm pert}(p^2,\mu) + \sum_n \frac{C_n}{(p^2)^n} \langle \Omega | : \hat{O}(x=0,\mu) : | \Omega \rangle$$

- **Physical QCD vacuum $|\Omega\rangle$ is complicated and differs from perturbative QCD vacuum $|0\rangle$.**

Condensates – nonzero expectation values of gauge-invariant operators over physical vacuum:

$$\boxed{\langle \Omega | : \hat{O}(0,\mu) : | \Omega \rangle \neq 0}$$

$$\langle \Omega | \bar{q} q (2\ \text{GeV} | \Omega \rangle = (271 \pm 3\ \text{MeV})^3, \qquad \langle \Omega | \alpha_s/\pi GG | \Omega \rangle = 0.012 \pm 0.006\ \text{GeV}.$$

$$\Pi(p^2) = \int \frac{ds}{s - p^2} \rho(s), \qquad \Pi_{\rm theor}(p^2) = \Pi_{\rm phen}(p^2)$$

$$\rho_{\rm theor}(s) = \left[\rho_{\rm pert}(s,\mu) + \sum_n C_n \delta^{(n)}(s) \langle \Omega | O_n(\mu) | \Omega \rangle \right], \qquad \rho_{\rm hadr}(s) = R_B \delta(s - M_B^2) + \rho_{\rm cont}(s)$$

Moment sum rules and quark masses

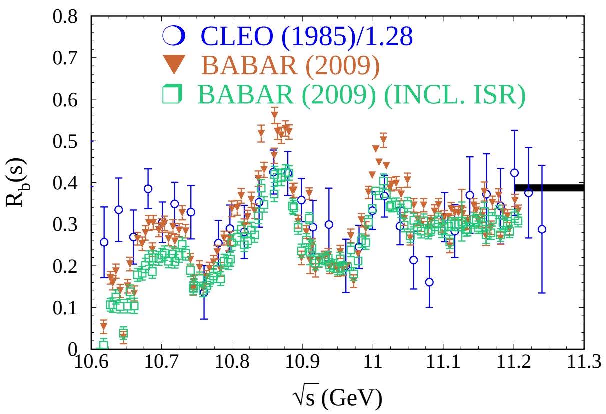
For $\bar{Q}Q$ systems, mainly moment sum rules are used

$$M_n = \int \frac{ds}{s^{n+1}} \rho_{\bar{Q}Q}(s).$$

Moments are known to $O(\alpha_s^3)$ accuracy for several n.

Moment SRs + experimental data or lattice QCD calculation of moments $\rightarrow$ *Quark masses*

$m_c(m_c) = 1.282 \pm 0.006_{stat} \pm 0.009_{syst} \pm 0.019_{pert} \pm 0.010_{\alpha} \pm 0.002_{GG}$ **GeV**
(Hoang et al, charmonium SR at order α_s^3).



$m_b(m_b) = 4.163 \pm 0.016$ **GeV (Chetyrkin et al, relativistic, i.e. low-n, moment sum rules**

$m_b(m_b) = 4.171 \pm 0.009$ **GeV (Dominguez finite energy sum rules)**

$m_b(m_b) = 4.235 \pm 0.055_{pert} \pm 0.003_{exp}$ **GeV (Hoang et al, nonrelativistic sum rules at NNLL).**

Properties of individual resonances from Borel sum rules

$$\Pi(\tau) = \int ds\, \exp(-s\tau)\rho(s) = f_B^2 M_B^4 e^{-M_B^2\tau} + \int\limits_{s_{\rm phys}}^{\infty} ds\, e^{-s\tau}\rho_{\rm hadr}(s) = \int\limits_{(m_b+m)^2}^{\infty} ds\, e^{-s\tau}\rho_{\rm pert}(s,\mu) + \Pi_{\rm power}(\tau,\mu).$$

Here $s_{\rm phys} = (M_{B^*} + M_P)^2$, and f_B is the decay constant defined by

$$(m_b + m)\langle 0|\bar{q} i\gamma_5 b|B\rangle = f_B M_B^2.$$

$$\Pi_{\rm power}(\tau,\mu = m_Q) = (m_Q + m)^2 e^{-m_Q^2\tau}$$
$$\times\left\{-m_Q\langle\bar{q}q\rangle\left[1 + \frac{2C_F\alpha_s}{\pi}\left(1 - \frac{m_Q^2\tau}{2}\right) - \frac{m}{2m_Q}(1 + m_Q^2\tau) + \frac{m^2}{2}m_Q^2\tau^2 + \frac{m_0^2\tau}{2}\left(1 - \frac{m_Q^2\tau}{2}\right)\right] + \frac{1}{12}\left\langle\frac{\alpha_s}{\pi}GG\right\rangle\right\}.$$

To exclude the excited-state contributions, one adopts the *duality Ansatz*: **all contributions of excited states are counterbalanced by the perturbative contribution above an** ***effective continuum threshold***, **$s_{\rm eff}(\tau)$ which differs from the physical continuum threshold.**

Applying the duality assumption yields:

$$f_B^2 M_B^4 e^{-M_B^2\tau} = \int\limits_{(m_b+m)^2}^{s_{\rm eff}(\tau)} ds\, e^{-s\tau}\rho_{\rm pert}(s,\mu) + \Pi_{\rm power}(\tau,\mu) \equiv \Pi_{\rm dual}(\tau, s_{\rm eff}(\tau)).$$

Even if the QCD inputs $\rho_{\rm pert}(s,\mu)$ and $\Pi_{\rm power}(\tau,\mu)$ are known the extraction of the decay constant requires $s_{\rm eff}(\tau)$.

Extraction of the decay constant

According to the standard procedures of QCD sum rules, one executes the following steps:

1. *The Borel window*

The working τ-window is chosen such that the OPE gives an accurate description of the exact correlator (i.e., all higher-order radiative and power corrections are under control) and at the same time the ground state gives a "sizable" contribution to the correlator.

2. *The effective continuum threshold*

The major part of hadron continuum is removed by applying the cut at $s_{\rm eff}$.

In those cases where the bound-state mass M_B is known, one can use it and improve the accuracy of f_B.

Introduce the *dual invariant mass* $M_{\rm dual}$

$$M^2_{\rm dual}(\tau) \equiv -\frac{d}{d\tau}\log \Pi_{\rm dual}(\tau, s_{\rm eff}(\tau)).$$

The deviation of $M_{\rm dual}$ from M_B measures the contamination of the dual correlator by excited states.

Starting from a trial function for $s_{\rm eff}(\tau)$ and requiring a minimum deviation of $M_{\rm dual}$ from M_B in the τ-window generates a variational solution for $s_{\rm eff}(\tau)$. We consider polynomials in τ and obtain their paramaters by minimizing the squared difference between $M^2_{\rm dual}$ and M^2_B in the τ-window:

$$\chi^2 \equiv \frac{1}{N}\sum_{i=1}^{N}\left[M^2_{\rm dual}(\tau_i) - M^2_B\right]^2 .$$

Uncertainties in the extracted decay constant

The resulting f_B is sensitive to the input values of the OPE parameters — the *OPE-related error* — and to the adopted prescription for fixing the effective continuum threshold $s_{\rm eff}(\tau)$ — the *systematic error*.

OPE – related error

Gaussian distributions for all OPE parameters but the renormalization scales; for the latter, *uniform* distribution.

Systematic error

The systematic error, related to the limited intrinsic accuracy of the method of sum rules.

The band of results obtained from linear, quadratic, and cubic trial functions for $s_{\rm eff}(\tau)$, optimized by minimizing the deviation of the dual mass from the true mass may be regarded as a realistic estimate for the systematic uncertainty of the decay constant.

$$f_B^2 M_B^4 e^{-M_B^2\tau} = \int\limits_{(m_b+m)^2}^{s_{\rm eff}(\tau)} ds\, e^{-s\tau}\rho_{\rm pert}(s,\mu) + \Pi_{\rm power}(\tau,\mu) \equiv \Pi_{\rm dual}(\tau, s_{\rm eff}(\tau)).$$

OPE

The best-known 3-loop calculations of the perturbative spectral density have been performed in form of an expansion in terms of the $\overline{\rm MS}$ **strong coupling** $\alpha_{\rm s}(\mu)$ **and the pole mass** M_b**:**

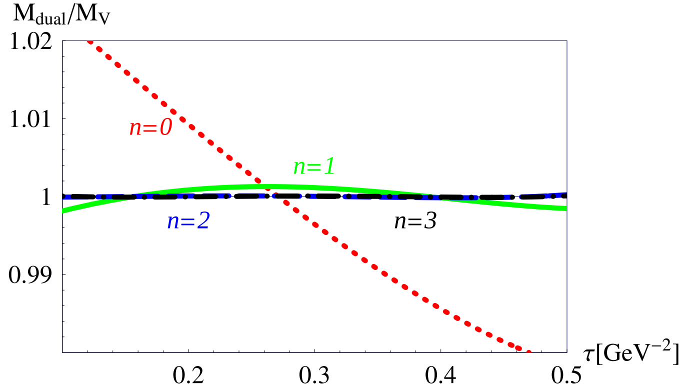
$$\rho_{\rm pert}(s,\mu) = \rho^{(0)}(s, M_b^2) + \frac{\alpha_{\rm s}(\mu)}{\pi}\rho^{(1)}(s, M_b^2) + \left(\frac{\alpha_{\rm s}(\mu)}{\pi}\right)^2 \rho^{(2)}(s, M_b^2, \mu) + \cdots .$$

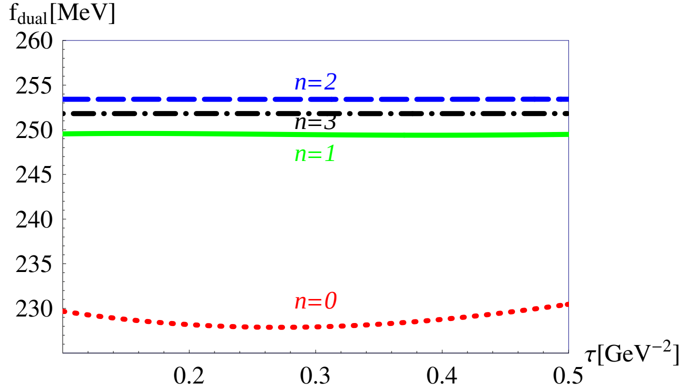
An alternative option is to reorganize the perturbative expansion in terms of the running $\overline{\rm MS}$ **mass,** $\overline{m}_b(\nu)$**, by substituting** M_b **in the spectral densities** $\rho^{(i)}(s, M_b^2)$ **via its perturbative expansion in terms of the running mass** $\overline{m}_b(\nu)$

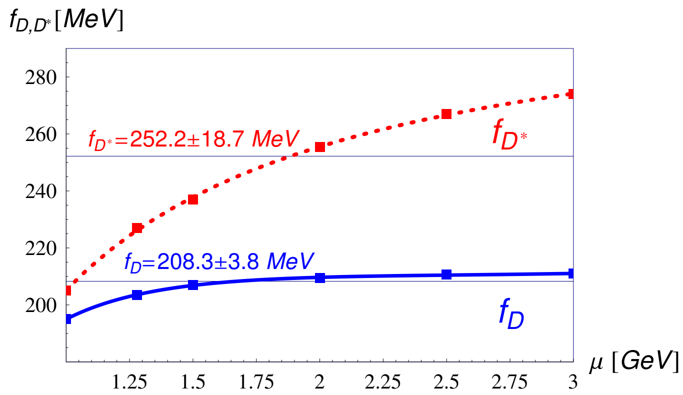
$$M_b = \overline{m}_b(\nu)\left(1 + \frac{\alpha_s(\nu)}{\pi}\, r_1 + \left(\frac{\alpha_s(\nu)}{\pi}\right)^2 r_2 + \ldots\right).$$

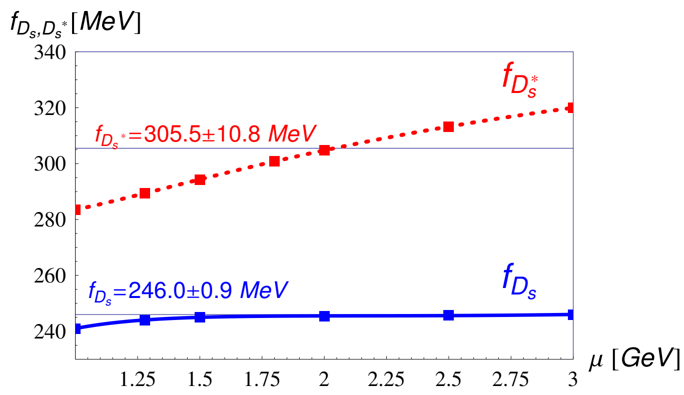
Charm sector:

$m_c(m_c) = 1.279 \pm 1.279 \pm 0.013$ **GeV.**









$f_D = (208.3 \pm 7.3_{\rm OPE} \pm 5_{\rm syst})$ MeV

$f_{D_s} = (246.0 \pm 15.7_{\rm OPE} \pm 5_{\rm syst})$ MeV **(OPE error mainly due to $\langle \bar{s}s \rangle$)**

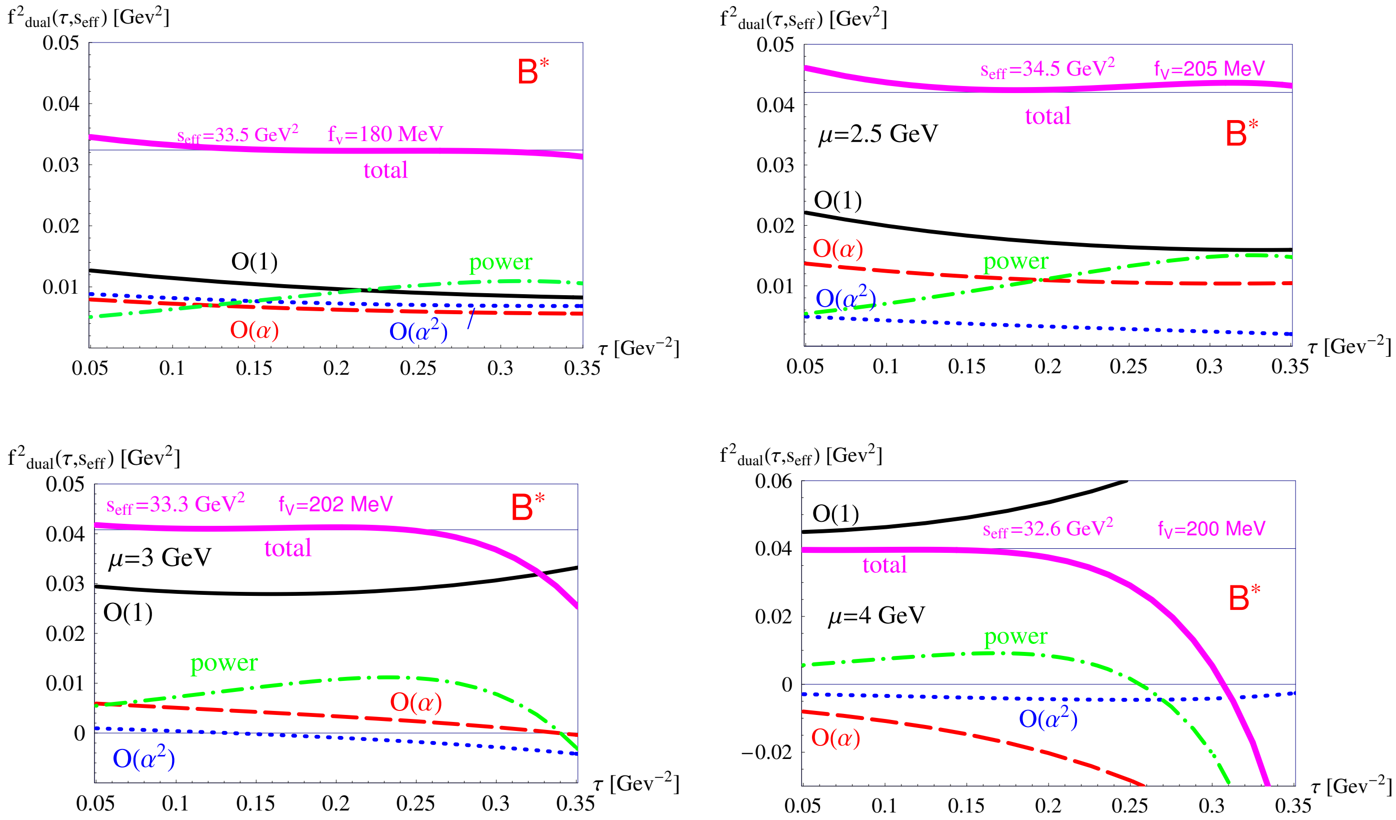
$f_{D^*} = (252.2 \pm 22.3_{\rm OPE} \pm 4_{\rm syst})$ MeV **(OPE error mainly due to $\langle \bar{s}s \rangle$ + scale-dependence)**

$f_{D_s^*} = (305.5 \pm 26.8_{\rm OPE} \pm 5_{\rm syst})$ MeV.

$f_{D^*}/f_D = 1.221 \pm 0.080_{\rm OPE} \pm 0.008_{\rm syst}$ **(lattice $f_{D^*}/f_D = 1.20 \pm 0.02$)**

<u>Beauty sector:</u>

OPE in terms of the pole and $\overline{\mathrm{MS}}$ running mass (PDG value $m_b(m_b) = 4.18$ GeV) at different scales:



1. **No perturbative hierarchy in terms of the pole mass.**
2. **$\overline{\mathrm{MS}}$ mass results depend on μ; playing with μ-choice one can acquire hierarchy.**

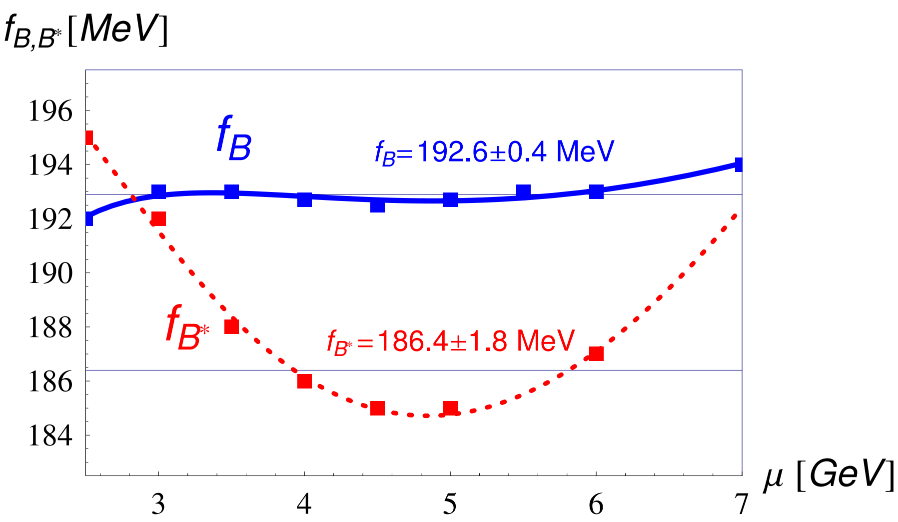
Decay constants of B and B*

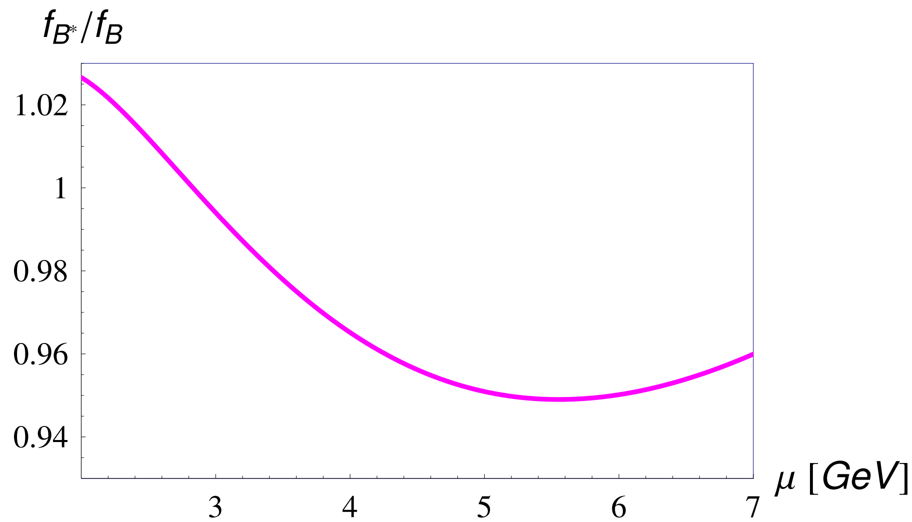
B-meson:

$$f_B^{\rm dual}(m_b, \mu = \mu^*) = 192.6\ \textbf{MeV} - 13\ \textbf{MeV}\left(\frac{m_b - 4.247\ \textbf{GeV}}{0.034\ \textbf{GeV}}\right), \quad \mu^* = 5.59\ \textbf{GeV},$$

B^*-meson:

$$f_{B^*}^{\rm dual}(m_b, \mu = \mu^*) = 186.4\ \textbf{MeV} - 10\ \textbf{MeV}\left(\frac{m_b - 4.247\ \textbf{GeV}}{0.034\ \textbf{GeV}}\right), \quad \mu^* = 5.82\ \textbf{GeV},$$





• **The extraction of hadronic properties improves by allowing** *a Borel-parameter dependence for the effective continuum threshold*, **which increases the accuracy of the duality approximation.**

• **Result obtained on the basis of** *pole-mass OPE* **are not trustable: the pole-mass OPE shows no perturbative hierarchy. Reorganizing the OPE series in terms of the running mass improves the hierarchy; however induces an explicit scale-dependence.**

• **For beauty mesons, a strong correlation between m_b and the sum-rule result for f_B was observed**

$$\frac{\delta f_B}{f_B} \approx -8\,\frac{\delta m_b}{m_b}.$$

Making use of the PDG $m_b = 4.18$ GeV leads to $f_B > 210$ MeV, in clear tention with the recent lattice QCD results for $f_B \sim 190$ MeV. Combining our sum-rule analysis with the latest results for f_B and f_{B_s} from lattice QCD yields

$$\mathbf{m_b = 4.247 \pm 0.027_{(OPE)} \pm 0.018_{(exp)} \pm 0.011_{syst}\ GeV}$$

• **For B^* unexpectedly strong μ-dependence:**

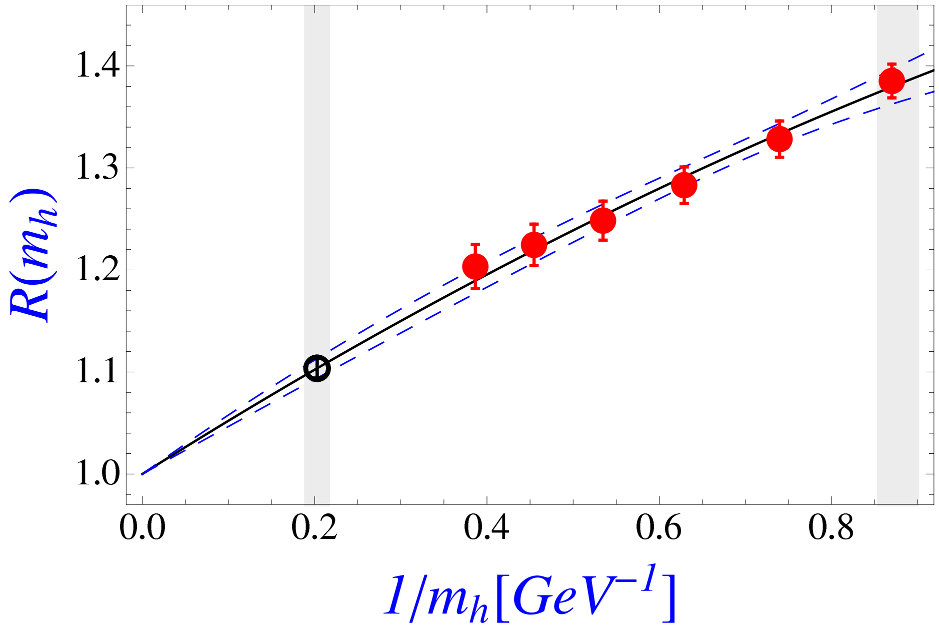
averaging over the scale range $3 < \mu[\mathrm{GeV}] < 6$ leads to

$$f_{B^*}/f_B = 0.923 \pm 0.059, \qquad f_{B_s^*}/f_{B_s} = 0.932 \pm 0.047.$$

Taking into account only low-scale results for $2.5 < \mu[\mathrm{GeV}] < 3.5$, yields $f_{B^*}/f_B = 0.994 \pm 0.01$.

$$\frac{f_{B^*}}{f_B} = \left(1 - \frac{2\alpha_s(m_b)}{3\pi}\right)[1 + \delta/m_b]\,.$$

Extrapolating lattice results



$f_{B^*}/f_B = 1.042 \pm 0.014$ **(Becirevic et al)**

3 – point sum rules, form factors, and strong coupling constants

Basic object - 3-point function

$$\Gamma(p^2, p'^2, q^2) = \int \langle 0|T(j(x)j(0)j(y))|0\rangle \exp(-ipx)\exp(-ip'y)dxdy.$$

For hadron-hadron transition - a double Borel transform $p^2 \to \tau$ and $p'^2 \to \tau'$ + again using duality property to isolate the ground-state contribution.

Successfully applied to hadron form factors at Q^2 a few GeV^2. Now interest in large Q^2.

A typical OPE has the form:

$$F_\pi(Q^2)\, f_\pi^2 = \int\limits_0^{s_{\rm eff}(Q^2,\tau)} \int\limits_0^{s_{\rm eff}(Q^2,\tau)} {\rm d}s_1\, {\rm d}s_2\, \Delta_{\rm pert}(s_1, s_2, Q^2) e^{-\frac{s_1+s_2}{2}\tau} + \frac{\left\langle \alpha_s G^2 \right\rangle}{24\pi}\tau + \frac{4\pi\,\alpha_s\,\langle \bar{q}q\rangle^2}{81}\tau^2 \left(13 + Q^2\,\tau\right) + \cdots.$$

$\boxed{\textbf{For large } Q^2 \textbf{ power corrections rise with } Q^2 \textbf{ as the consequence of using local condensates}}$

2 possibilites at large Q^2 with 3-point correlators:

- **use nonlocal condensates (Bakulev et al)**
- **work in local-duality (LD) limit $\tau = 0$ (Radyushkin). All power corrections vanish and details of non-perturbative dynamics are hidden in one quantity – the effective threshold $s_{\rm eff}(Q^2)$:**

Due to properties of the spectral functions, LD form factors obey the factorization theorems:

$$F_\pi(Q^2) \to 8\pi\alpha_s(Q^2) f_\pi^2/Q^2, \qquad F_{\pi\gamma}(Q^2) \to \sqrt{2} f_\pi/Q^2, \qquad f_\pi = 130\ \mathbf{MeV}.$$

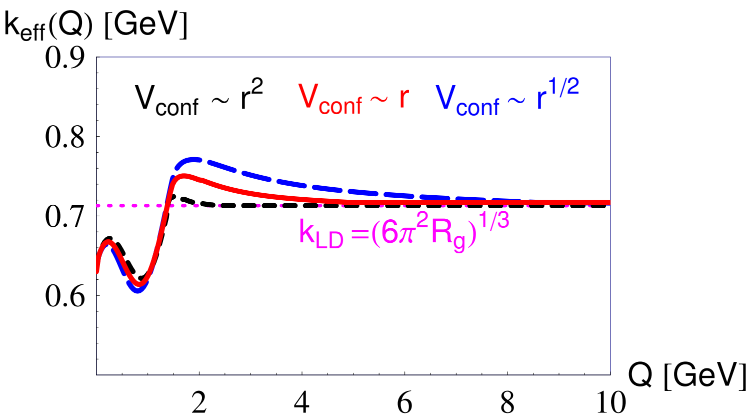
as soon as the effective thresholds satisfy:

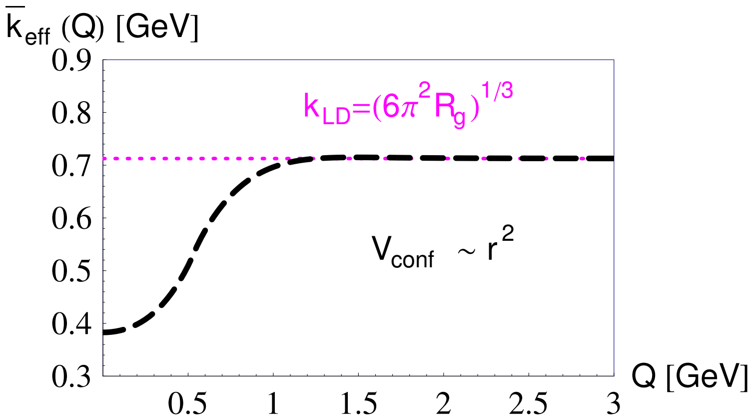
$$s_{\rm eff}(Q^2 \to \infty) = \bar{s}_{\rm eff}(Q^2 \to \infty) = 4\pi^2 f_\pi^2.$$

For finite Q^2, the effective thresholds $s_{\rm eff}(Q^2)$ and $\bar{s}_{\rm eff}(Q^2)$ depend on Q^2 and differ from each other.

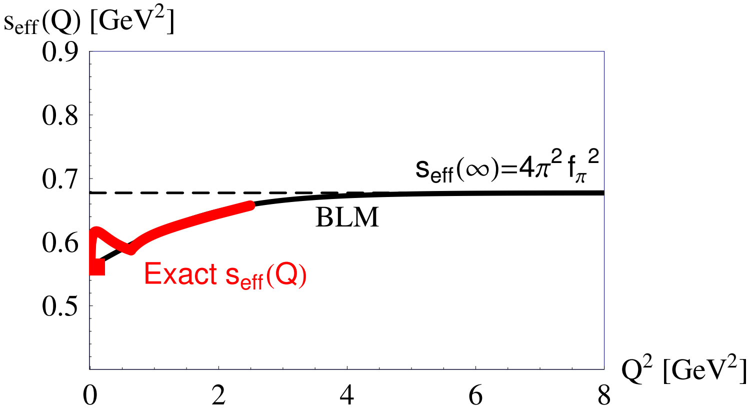
The only property of theory relevant for this property of $s_{\rm eff}$ is *factorization* of hard form factors.

Testing properties of the effective thresholds in quantum mechanics:

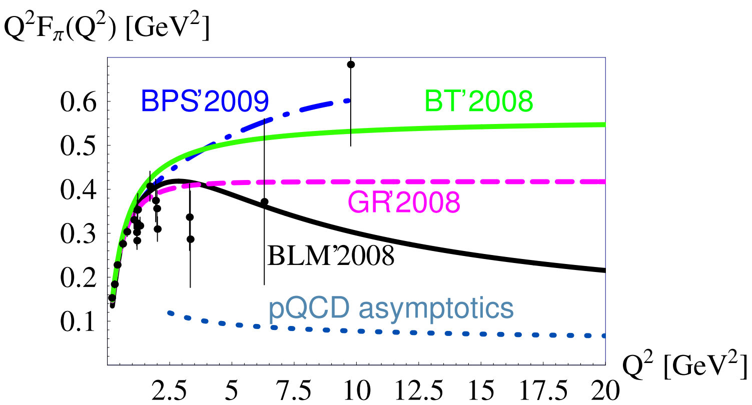


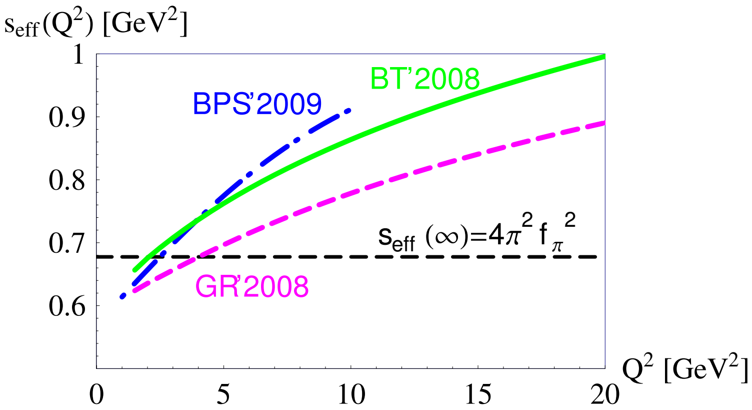


For a given result for the pion form factor, one can recalculate the *equivalent effective threshold*



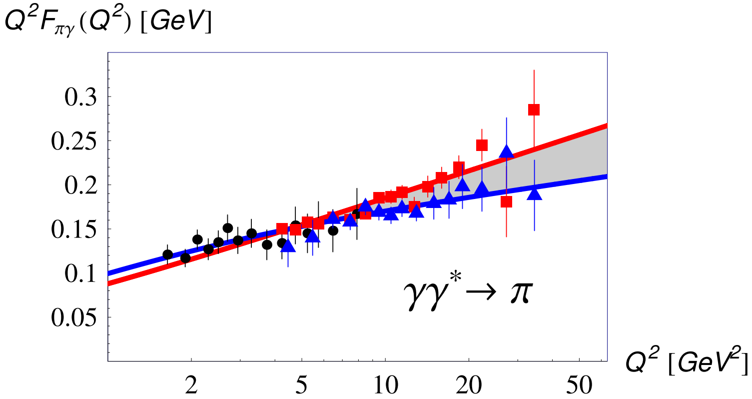
The exact $s_{\rm eff}$ extracted from the accurate data at low Q^2 *suggests* that the LD limit may be reached already at relatively low values of $Q^2 \approx 4-8$ GeV^2.

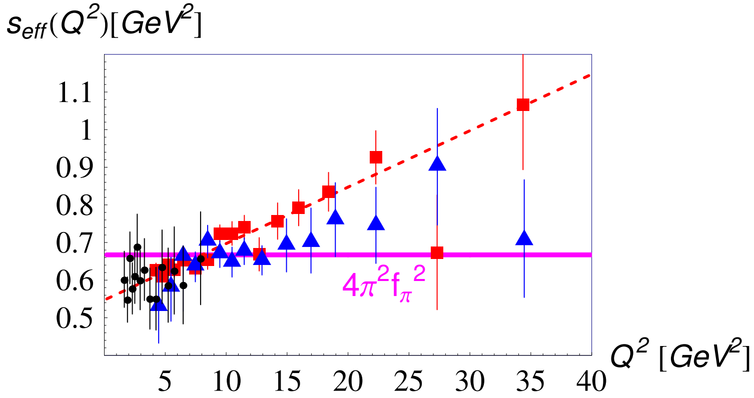




The future accurate data expected from JLab in the range up to $Q^2 = 8$ GeV^2.

For meson-photon transition form factors the LD predictions are (red squares - BaBAr, blue triangles - Belle)





The local-duality sum rule for the form factors:

- **increasingly accurate with Q in the region Q^2 = a few GeV^2**
- **requires the knowledge of $O(1)$ and $O(\alpha_s)$ double spectral densities**
- **promising for the application to heavy-to-light weak form factors**

• **Three-point sum rules have been extensively applied to <u>baryon form factors</u> (Aliev).**

(+) The influence of the choice of the baryon interpolating currents has been studied. Many new results are expected in the near future.

(-) OPE uncertainties are not well-controlled because radiative corrections are not known; control over systematics is usually insufficient.

• **<u>Strong coupling constants</u> of heavy mesons of the type $g_{DD^*\pi}$ (Nielsen, Navarra et al)**

The situation here is difficult: extrapolations of the form factors to the meson pole are necessary

$D^*D\rho$	**4.3**
$D^*D\pi$	**14**
D_s^*DK	**2.84**
$B^*B\pi$	**42**
B_s^*BK	**10.6**

More efforts are needed for obtaining reliable predictions.

Light – cone sum rules, form factors, and strong coupling constants

Basic object - 2-point function between the vaccum and one-particle state

$$T(p^2, p'^2 = M^2, q^2) = \int \langle 0|T(j(x)j(0))|p, M\rangle \exp(-iqx)dx.$$

For hadron-hadron transition - a single Borel transform $p^2 \to \tau$ **+ again using duality property to isolate the ground-state contribution.**

$$\langle 0|\bar{q}(x)q(0)|p', M\rangle|_\mu = f(x^2, xp) = \sum_n (x^2)^n \int_0^1 d\xi \exp(-ixp'\xi)\phi_n(\xi,\mu),$$

$\phi_n(\xi,\mu)$ **universal distribution amplitudes of the increasing twist (Braun et al).**

• **Main efforts: calculation of the** *baryon* **form factors, including radiative corrections.**

<u>Issues here:</u>

• **How well contributions of higher-twist DAs are suppressed compared to the leading-twist DA.**

• **Strong couplings of the type** $D^*D\pi$**; these require double Borel transforfed light-cone sum rules.**

Sum rules for exotic states

2- and 3-point functions for 4-quark interpolating currents

$j(x) = \bar{b}(x)O_1 b(x)\bar{u}(x)O_2 d(x)$

• **From the 2-point function, it is virtually impossible to study at the same time both the existence of the isolated ground state and of its coupling.**

• **For 3-point functions involving 4-quark currents there are subtleties.**

E.g., one wants to study $X \to \bar{b}b + \pi$:

$$\begin{aligned} \Gamma(p^2, p'^2, q^2) &= \int d^4x d^4y \exp(-ip'x)\exp(-iqy)\langle 0|T(\bar{b}(0)b(0)\bar{u}(0)d(0), \bar{b}(x)b(x), \bar{d}(y)u(y))|0\rangle \\ &= \frac{1}{(p^2 - M_X^2)(p'^2 - M_{bb}^2)(q^2 - M_\pi^2)} g + \cdots \end{aligned}$$

$$\Gamma(p^2, p'^2, q^2) = \Pi(p'^2)\Pi(q^2) + O(\alpha_S)$$

The LO α_s result does not depend on p^2, the invariant mass in the X-channel, at all!

The “fall-apart” decay mechanism of exotic hadrons differs from the decay mechanism of the ordinary hadrons and requires the appropriate treatment within QCD sum rules.

Summary

Many efforts in different directions within the method of QCD sum rules have been seen in the resent years.

• **QCD parameters from 2-point functions:**

Combining moment QCD sum rules with experimental/lattice data gives accurate heavy-quark masses.

• **Hadron properties from 2-point functions:**

a. **Progress in development of new algorithms of extracting ground state parameters from the OPE of the correlators (finite-energy sum rules, Borel sum rules)**

b. **Progress towards gaining control over systematic errors of the decay constants** *(it seems impossible to predict both masses and decay constants with controlled accuracy. But using the mass of the ground state as input, systematics can be controlled).*

c. **Puzzles in the b-sector:**

(i) b-quark mass 4.18 GeV used in Borel sum rules for f_B leads to tension with lattice results for f_B.

(ii) Unexpectedly strong scale-dependence of decay constants of vector mesons and of f_{B^*}/f_B. Need for 4-loop calculations?

d. **Calculation of decay constants of heavy-quarkonium states is still an open issue**

e. **Excited states?**

• **Form factors from 3-point functions:**

interesting results in a broad range of momentum transfers for elastic and transition form factors of light mesons. Seems promising to apply the same ideas to heavy-to-light transition form factors.

Need: calculations of radiative corrections to triangle diagrams are necessary.

Gain: parameter-free predictions for the form factors, increasingly accurate with increase of Q^2.

Progress in elastic and transition form factors of light baryons (however $O(1)$ calculation).

• **Form factors from light-cone sum rules:**

Baryon form factors including $O(\alpha_s)$ radiative corrections.

• **Strong couplings of the type $g_{D^*D\pi}$**
both from three-point sum rules and light-cone sum rules are still an open issue.

• **Exotic states (tetraquarks)**

Properties of exotic tetraquarks (or candidate states if their masses are measured) can be studied but require the appropriate modifications of the method.

QCD sum rules are successfully going their own way in the lattice QCD environment

The OPE parameters:

$m_d(2\ \mathbf{GeV}) = (3.42 \pm 0.09)\ \mathbf{MeV}$, $\quad m_s(2\ \mathbf{GeV}) = (93.8 \pm 2.4)\ \mathbf{MeV}$, $\quad \alpha_s(M_Z) = 0.1184 \pm 0.0007$, $\langle \bar{q}q \rangle(2\ \mathbf{GeV}) = -((271 \pm 3)\ \mathbf{MeV})^3$, $\quad \langle \bar{s}s \rangle(2\ \mathbf{GeV})/\langle \bar{q}q \rangle(2\ \mathbf{GeV}) = 0.8 \pm 0.3$, $\quad \left\langle \frac{\alpha_s}{\pi} GG \right\rangle = (0.012 \pm 0.006)\ \mathbf{GeV}^4$.