Solutions to QCD 't Hooft Equation in terms of Airy functions

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Abstract

We consider the 't Hooft equation in the limit of large fermion masses. We provide a proof that in this limit the 't Hooft equation becomes the Schrodinger equation for a particle in a triangular potential well. The eigenfuntions are Airy functions with appropriate symmetry. The spectrum is proportional to zeros of Airy function and its derivative. Numerical study confirms our results.

Introduction to 't Hooft equation

The equation was proposed by 't Hooft in 1974 in his article "A two-dimensional model for mesons". He considered a simple 1+1 dimensional QCD model with local gauge group *U(N)*. The parameter *N* is so large that the quark–antiquark approximation is valid.

The Lagrangian is

$$\mathcal{L} = \frac{1}{4} G_{\mu\nu i}{}^{j} G_{\mu\nu j}{}^{i} - \bar{q}^{ai}(\gamma_\mu D_\mu + m_{(a)}) q^a{}_i$$

where

$$G_{\mu\nu i}{}^{j} = \partial_\mu A_i{}^j{}_\nu - \partial_\nu A_i{}^j{}_\mu + g[A_\mu, A_\nu]_i{}^j$$
$$D_\mu q^a{}_i = \partial_\mu q^a{}_i + g A_i{}^j{}_\mu q^a{}_j$$
$$A_i{}^j{}_\mu(x) = -A^{*i}_{j\ \mu}$$
$$q^1 = p;\ q^2 = n;\ q^3 = \lambda$$

The Lorentz indices μ, ν can take the two values 0 and 1. It is convenient to use light front coordinates:

$$x^{\pm} = \frac{1}{\sqrt{2}}(x^1 \pm x^0)$$

The model becomes particularly simple if we impose the light-front gauge condition:

$$A_- = A^+ = 0$$

In that gauge we have

$$G_{+-} = -\partial_- A_+$$

and

$$\mathcal{L} = -\frac{1}{2}\mathrm{tr}(\partial_- A_+)^2 - \bar{q}^a(\gamma\partial + m_{(a)} + g\gamma_- A_+) q^a$$

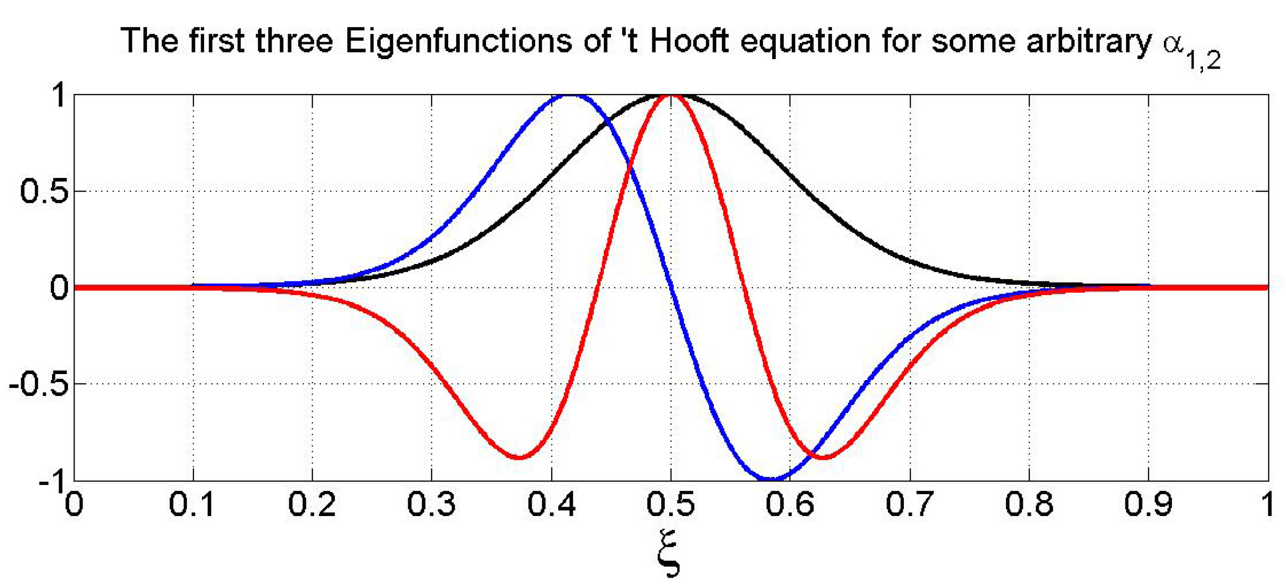
We assume quark and antiquark have momenta *p* and *r – p* and masses m_1 and m_2 correspondingly. All this leads to the eigenvalue equation for the mass squared:

$$\mu^2 \varphi(\xi) = \left(\frac{\alpha_1}{\xi} + \frac{\alpha_2}{1-\xi}\right)\varphi(\xi) - \mathrm{P}\int_0^1 \frac{\varphi(\xi')}{(\xi' - \xi)^2} d\xi'$$

where

$$\xi = \frac{p_-}{r_-}$$
$$\alpha_{1,2} = \frac{\pi}{g^2} m_{1,2}^2 - 1,$$

$\varphi(\xi)$ is the eigenfunction of the quark–antiquark system and the integral is understood in the principal value sense.



Schrödinger and Airy equations

Lets recall the Schrödinger equation for a particle in a triangular potential well (i.e. with a potential |*x*|):

$$-\frac{1}{k}\varphi''(x) + (\lambda|x| + E_0)\varphi(x) = E\varphi(x)$$

From the symmetry of the potential we have $|\varphi|^2(x) = |\varphi|^2(-x)$ and therefore the boundary conditions are:

$$\varphi(0) = 0, \quad \text{or} \quad \varphi'(0) = 0$$

So we can solve the problem for *x* ≥ 0.

Lets make a change of variables

$$x \to (\lambda k)^{-\frac{1}{3}} y + \frac{E - E_0}{\lambda}$$

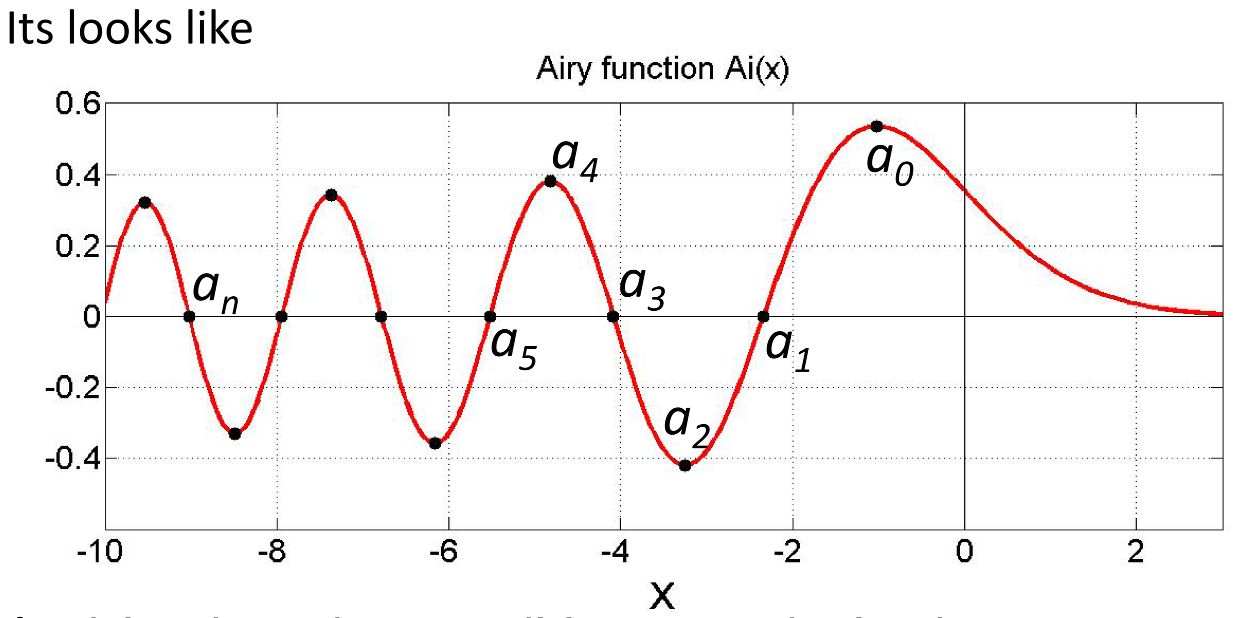
and get

$$-\varphi''(x) + y\varphi(y) = 0$$

This is Airy differential equation. Its finite solution is an Airy function:

$$\varphi(y) = \mathrm{Ai}(y) = \mathrm{Ai}\left[(\lambda k)^{\frac{1}{3}}\left(x - \frac{E - E_0}{\lambda}\right)\right]$$

Its looks like

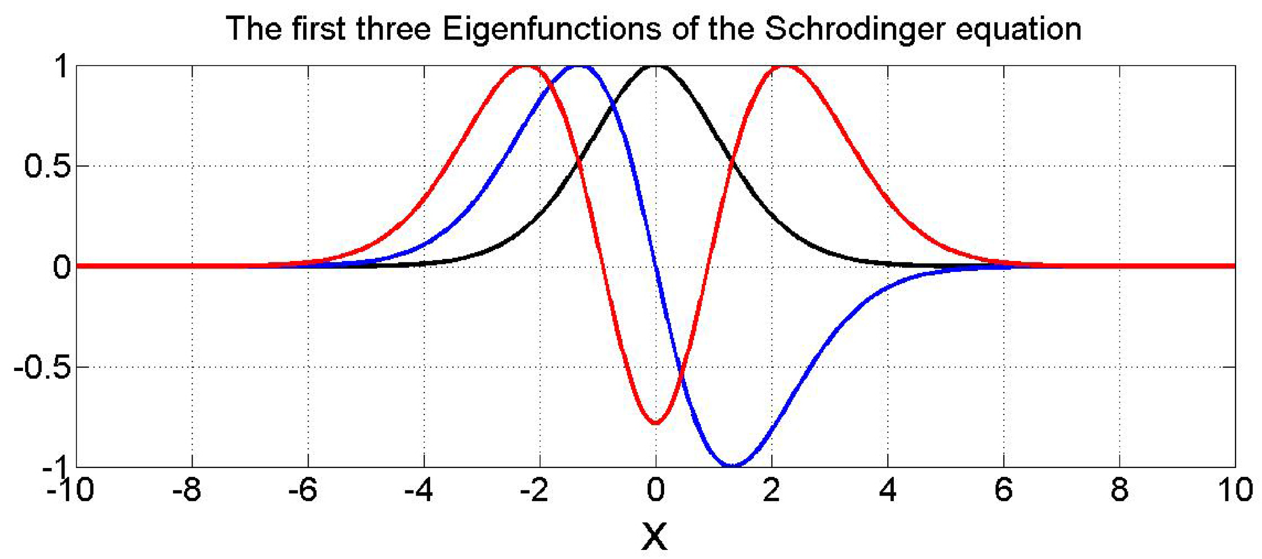


Applying boundary conditions we obtain the spectrum and eigenfunctions of the Schrödinger equation:

$$E_n = -\left(\frac{\lambda^2}{k}\right)^{\frac{1}{3}} a_n + E_0$$

$$\varphi_n(x) = \begin{cases} \mathrm{Ai}\left[(\lambda k)^{\frac{1}{3}}\left(x - \frac{E - E_0}{\lambda}\right)\right] = \mathrm{Ai}\left[(\lambda k)^{\frac{1}{3}} x + a_n\right] & \text{if } x \geq 0 \\ (-1)^n \varphi(-x) & \text{if } x < 0 \end{cases}$$

where a_n is the n^{th} zero of Airy function or its derivative.



Shift of variable in 't Hooft equation

For our discussion it is essential to change the variable in the 't Hooft equation:

$$\rho = \frac{p_{1-} - p_{2-}}{r_-} = \frac{2p_{1-} - r_-}{r_-} = 2\xi - 1$$

Physically this means that we consider the "deviation from the mean momentum". In these terms the 't Hooft equation will be more symmetrical:

$$\mu^2\varphi(\rho) = \left(\frac{\alpha_1}{1+\rho} + \frac{\alpha_2}{1-\rho}\right) 2\varphi(\rho) - 2\mathrm{P}\int_{-1}^{1} \frac{\varphi(\rho')}{(\rho' - \rho)^2} d\rho'$$

Equivalence of equations at large *m*

Now we will show that in the limit $\alpha_{1,2} \gg 1$ the 't Hooft equation is equivalent to the Schrödinger equation for a particle in a triangular potential well. For simplicity we prove this for the case $\alpha_1 = \alpha_2 = \alpha$, and give only the final result for the general case.

If we consider the perturbation theory in $1/\alpha$, the zeroth order approximation to the 't Hooft equation is

$$\mu^2\varphi(\rho) = \frac{\alpha}{1-\rho^2} 4\varphi(\rho)$$

It has a solution

$$\mu^2 = \frac{4}{1-\rho_0^2}\alpha$$
$$\varphi(\rho) = \delta(\rho - \rho_0) \pm \delta(\rho + \rho_0)$$

Because $\alpha \gg 1$, for the lowest eigensolution we take $\rho_0 = 0$:

$$\mu^{(0)2} = 4\alpha$$
$$\varphi^{(0)}(\rho) = \delta(\rho)$$

This observation is important because it suggests that in the limit of large masses the eigenfunctions become localized in the small vicinity of zero. So we can use the expansion

$$\frac{1}{1-\rho^2}\varphi(\rho) \approx (1+\rho^2)\varphi(\rho)$$

and safely change the limits of integration:

$$\mathrm{P}\int_{-1}^{1} \frac{\varphi(\rho')}{(\rho' - \rho)^2} d\rho' \longrightarrow \mathrm{P}\int_{-\infty}^{\infty} \frac{\varphi(\rho')}{(\rho' - \rho)^2} d\rho'$$

Now our equation looks like this:

$$\mu^2\varphi(\rho) = (1+\rho^2) 4\alpha\varphi(\rho) - 2\mathrm{P}\int_{-\infty}^{\infty} \frac{\varphi(\rho')}{(\rho' - \rho)^2} d\rho'$$

The next step is to make the Fourier Transform:

$$\hat{\varphi}(s) = \int_{-\infty}^{\infty} \varphi(\rho) e^{i\rho s} d\rho$$

The variable *s* corresponds to the distance between quark and antiquark in the x^- space, i.e., $s \sim (x_{1-} - x_{2-})$.

Using the residue theorem for the integral term we finally get:

$$\mu^2\hat{\varphi}(s) = 4\alpha\hat{\varphi}(s) - 4\alpha\hat{\varphi}''(s) + 2\pi|s|\hat{\varphi}(s)$$

This is the desired Schrödinger equation with parameters:

$$k = \frac{1}{4\alpha}, \quad \lambda = 2\pi, \quad E_0 = 4\alpha$$

The more general case $\alpha_1 \neq \alpha_2$ has the similar solution. The corresponding parameters are:

$$k = \frac{4\sqrt{\alpha_1}\sqrt{\alpha_2}}{(\sqrt{\alpha_1} + \sqrt{\alpha_2})^4} \approx \frac{g^2}{\pi}\frac{4m_1m_2}{(m_1+m_2)^4}$$
$$\lambda = 2\pi$$
$$E_0 = (\sqrt{\alpha_1} + \sqrt{\alpha_2})^2 \approx \frac{\pi}{g^2}(m_1+m_2)^2$$

and the new shifted "momentum" variable is

$$\omega = \rho - \frac{\sqrt{\alpha_1} - \sqrt{\alpha_2}}{\sqrt{\alpha_1} + \sqrt{\alpha_2}} \approx \rho - \frac{m_1 - m_2}{m_1 + m_2}$$