Effective Polyakov line actions

and their solution at finite chemical potential

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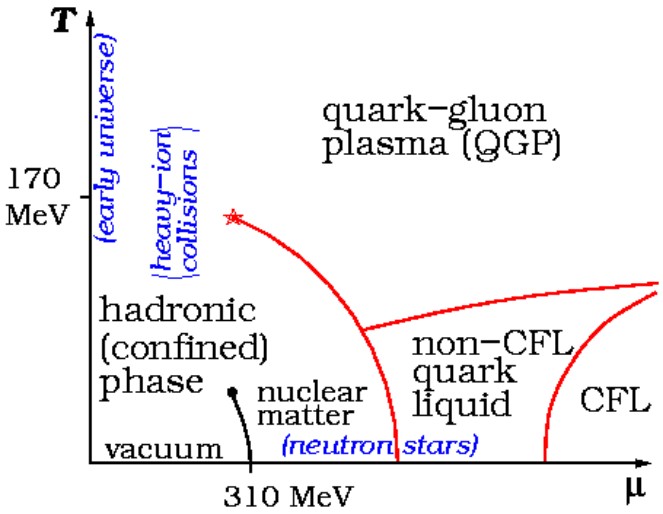
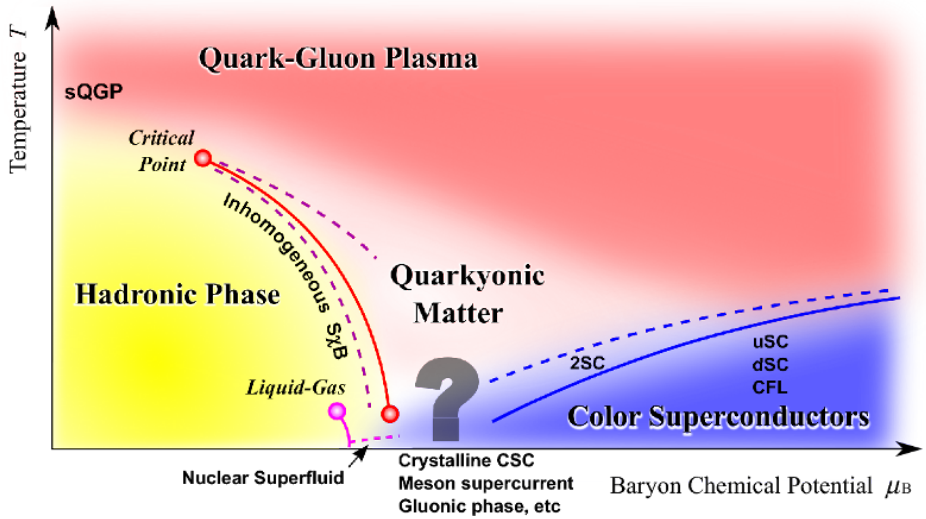
arXiv:1404.5844 with Kurt Langfeld, and arXiv:1406.4558

The Sign Problem

Sketches of the QCD phase diagram (two of many):





Except near $\mu = 0$, it is mostly conjecture.

Can the situation be improved via lattice Monte Carlo?

The Sign Problem

In order to vary particle density, introduce a chemical potential μ.
When the quark fields are integrated out, each flavor contributes a factor

$$\det(\not{D} + m + \mu\gamma_0)$$

which is complex, unless either

- $\mu = 0$;
- μ is purely imaginary;
- there are two flavors with μ of opposite sign (isospin chemical potential).

If not, then exponentiating a complex determinant $\det(M) = \exp \mathrm{Tr} \log(M)$, results in a complex action.

Straightforward importance sampling is impossible!

What to do?

There are several “direct” approaches to the sign problem in QCD, which are under development:

- **Reweighting + cumulant expansion** *(WHOT collaboration)*

  Treat $\exp[iS_I]$ as an observable, rather than as part of the Boltzman weight.

- **Stochastic quantization** *(Aarts, Seiler, Sexty, Stamatescu...)*

  Complexify the field variables and apply the Langevin equation.

- **Lefschetz thimbles** *(Cristoforetti, Di Renzo, Mukherjee, Scorzato)*

  Shift functional integration contours into the complex plane.

An “indirect” approach

First map the gauge-matter theory onto a much simpler theory – a **Polyakov line action (or “SU(3) spin”) model**.

There is still a sign problem that must be faced. I will deal with that in two ways:

Methods

1. mean field theory *(Splittorff and JG)*
2. complex Langevin equation *(Aarts and James)*

We will find that these methods sometimes agree perfectly, and sometimes not. I will discuss who is right – or who is wrong – in the latter case.

Effective Polyakov Line Action

Start with lattice gauge theory and integrate out all d.o.f. subject to the constraint that the Polyakov line holonomies are held fixed. In temporal gauge

$$e^{S_P[U_{\mathbf{x}}]} = \int DU_0(\mathbf{x},0) DU_k D\phi \left\{ \prod_{\mathbf{x}} \delta[U_{\mathbf{x}} - U_0(\mathbf{x},0)] \right\} e^{S_L}$$

At leading order in the strong coupling/hopping parameter expansion S_P has the form of an SU(3) spin model

$$\begin{aligned} S_{spin} &= J \sum_x \sum_{k=1}^{3} \left(\mathrm{Tr}[U_x] \mathrm{Tr}[U^\dagger_{x+\hat{k}}] + \text{c.c.} \right) \\ &\quad + h \sum_x \left(e^{\mu/T} \mathrm{Tr}[U_x] + e^{-\mu/T} \mathrm{Tr}[U_x] \right) \end{aligned}$$

Avoid dynamical fermion simulations for now, work instead with an SU(3) gauge-Higgs model with a fixed modulus Higgs

$$S_L = \frac{\beta}{3}\sum_p \mathrm{ReTr}[U(p)] + \frac{\kappa}{3}\sum_x \sum_{\mu=1}^{4} \mathrm{Re}\left[\Omega^\dagger(x) U_\mu(x) \Omega(x+\hat{\mu})\right]$$

If we can derive S_P at $\mu = 0$, then (in principle) we also have S_P at $\mu > 0$ by the following identity:

$$S_P^\mu[U_{\mathbf{x}}, U_{\mathbf{x}}^\dagger] = S_P^{\mu=0}\left[e^{N_t\mu} U_{\mathbf{x}}, e^{-N_t\mu} U_{\mathbf{x}}^\dagger\right]$$

which is true to all orders in the strong coupling/hopping parameter expansion.

In practice we will also make use of imaginary chemical potentials $\mu/T = i\theta$.

How to compute S_P at $\mu = 0$?

- strong-coupling expansions *(Philipsen et al.)*
- inverse Monte Carlo *(Heinzl et al.)*
- relative weights *(this talk)*

And how do we know that we have derived S_P correctly?

One test: compare Polyakov line correlators

$$G(R) = \frac{1}{N_c^2}\left\langle \mathrm{Tr}[U_{\mathbf{x}}]\mathrm{Tr}[U_{\mathbf{y}}^\dagger]\right\rangle \quad , \quad R = |\mathbf{x}-\mathbf{y}|$$

computed for the effective action, and in the underlying lattice gauge theory.

The Gold Standard - SU(2) via Relative Weights

In previous papers we worked out S_P for pure SU(2) gauge theory:

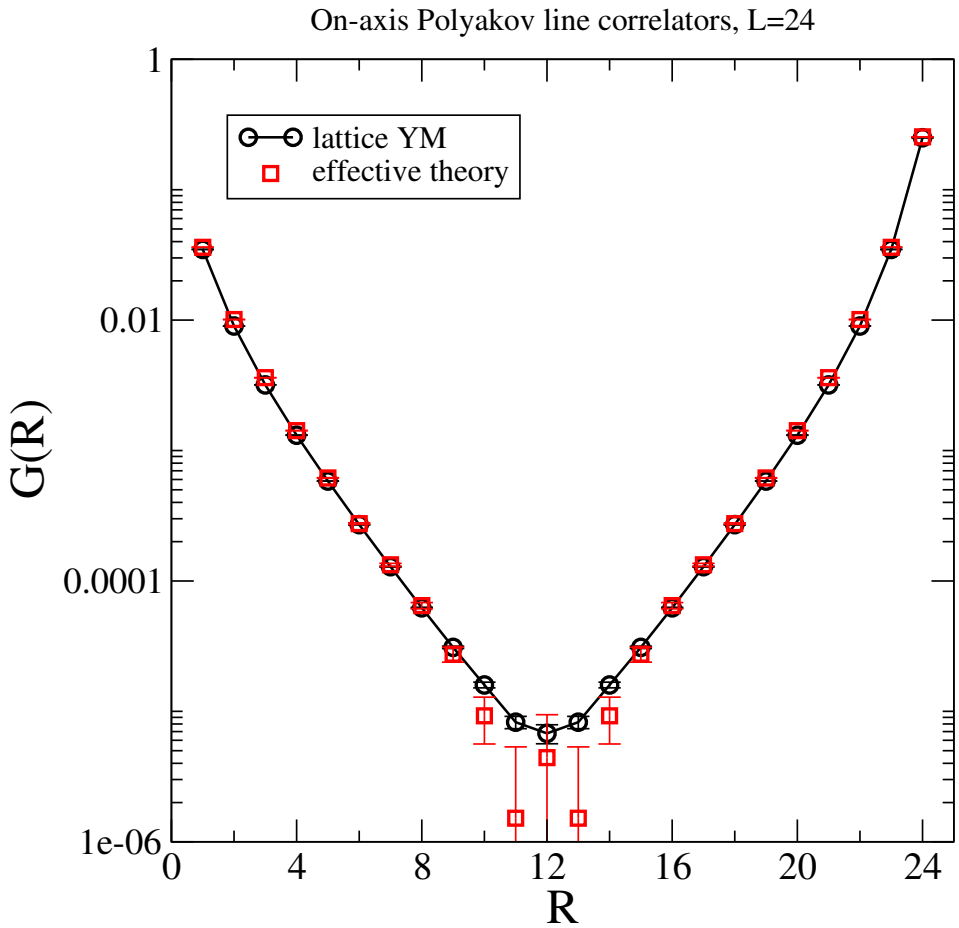
$$S_P = \sum_{\mathbf{x},\mathbf{y}} P_{\mathbf{x}} K(\mathbf{x}-\mathbf{y}) P_{\mathbf{y}}$$

where

$$P_{\mathbf{x}} = \tfrac{1}{2}\mathrm{Tr} U_{\mathbf{x}}$$

Here is the correlator comparison for

$$G(R) = \langle P_{\mathbf{x}} P_{\mathbf{y}} \rangle$$



The underlying lattice gauge theory is at $\beta = 2.2$ on a $24^3 \times 4$ lattice.

The Relative Weights Method

Let S'_L be the lattice action in temporal gauge with $U_0(\mathbf{x}, 0)$ fixed to $U'_{\mathbf{x}}$. It is not so easy to compute

$$\exp\Big[S_P[U'_{\mathbf{x}}]\Big] = \int DU_k D\phi \; e^{S'_L}$$

directly. But the ratio ("relative weights")

$$e^{\Delta S_P} = \frac{\exp[S_P[U'_{\mathbf{x}}]]}{\exp[S_P[U''_{\mathbf{x}}]]}$$

is easily computed as an expectation value

$$\begin{aligned} \exp[\Delta S_P] &= \frac{\int DU_k D\phi \; e^{S'_L}}{\int DU_k D\phi \; e^{S''_L}} \\ &= \frac{\int DU_k D\phi \; \exp[S'_L - S''_L] e^{S''_L}}{\int DU_k D\phi \; e^{S''_L}} \\ &= \Big\langle \exp[S'_L - S''_L] \Big\rangle'' \end{aligned}$$

where $\langle \ldots \rangle''$ means the VEV in the Boltzman weight $\propto e^{S''_L}$.

Suppose $U_{\mathbf{x}}(\lambda)$ is some path through configuration space parametrized by λ, and suppose $U'_{\mathbf{x}}$ and $U''_{\mathbf{x}}$ differ by a small change in that parameter, i.e.

$$U'_{\mathbf{x}} = U_{\mathbf{x}}(\lambda_0 + \tfrac{1}{2}\Delta\lambda) \quad , \quad U''_{\mathbf{x}} = U_{\mathbf{x}}(\lambda_0 - \tfrac{1}{2}\Delta\lambda)$$

Then the relative weights method gives us the derivative of the true effective action S_P along the path:

$$\left(\frac{dS_P}{d\lambda}\right)_{\lambda=\lambda_0} \approx \frac{\Delta S}{\Delta\lambda}$$

The question is: which derivatives will help us to determine S_P itself?

Fourier components of P_x

$$P_{\mathbf{x}} \equiv \frac{1}{N_c} \mathrm{Tr} U_{\mathbf{x}} = \sum_{\mathbf{k}} a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$$

We first set a particular momentum mode $a_{\mathbf{k}}$ to zero. Call the resulting configuration $\widetilde{P}_x$. Then define ($f \approx 1$)

$$\begin{aligned} P''_{\mathbf{x}} &= \left(\alpha - \frac{1}{2}\Delta\alpha\right) e^{i\mathbf{k}\cdot\mathbf{x}} + f\widetilde{P}_x \\ P'_{\mathbf{x}} &= \left(\alpha + \frac{1}{2}\Delta\alpha\right) e^{i\mathbf{k}\cdot\mathbf{x}} + f\widetilde{P}_x \end{aligned}$$

which uniquely determine (in SU(2) and SU(3)) the eigenvalues of the corresponding holonomies $U'_{\mathbf{x}}, U''_{\mathbf{x}}$.

S_P has a remnant local symmetry $U_{\mathbf{x}} \to g_{\mathbf{x}} U_{\mathbf{x}} g_{\mathbf{x}}^\dagger$, so the holonomies $U'_{\mathbf{x}}, U''_{\mathbf{x}}$ can be taken to be diagonal. We then compute

$$\frac{1}{L^3}\left(\frac{\partial S_P}{\partial a^R_{\mathbf{k}}}\right)_{a_{\mathbf{k}}=\alpha}$$

by the relative weights simulation ($a^R_{\mathbf{k}}$ is the real part of $a_{\mathbf{k}}$).

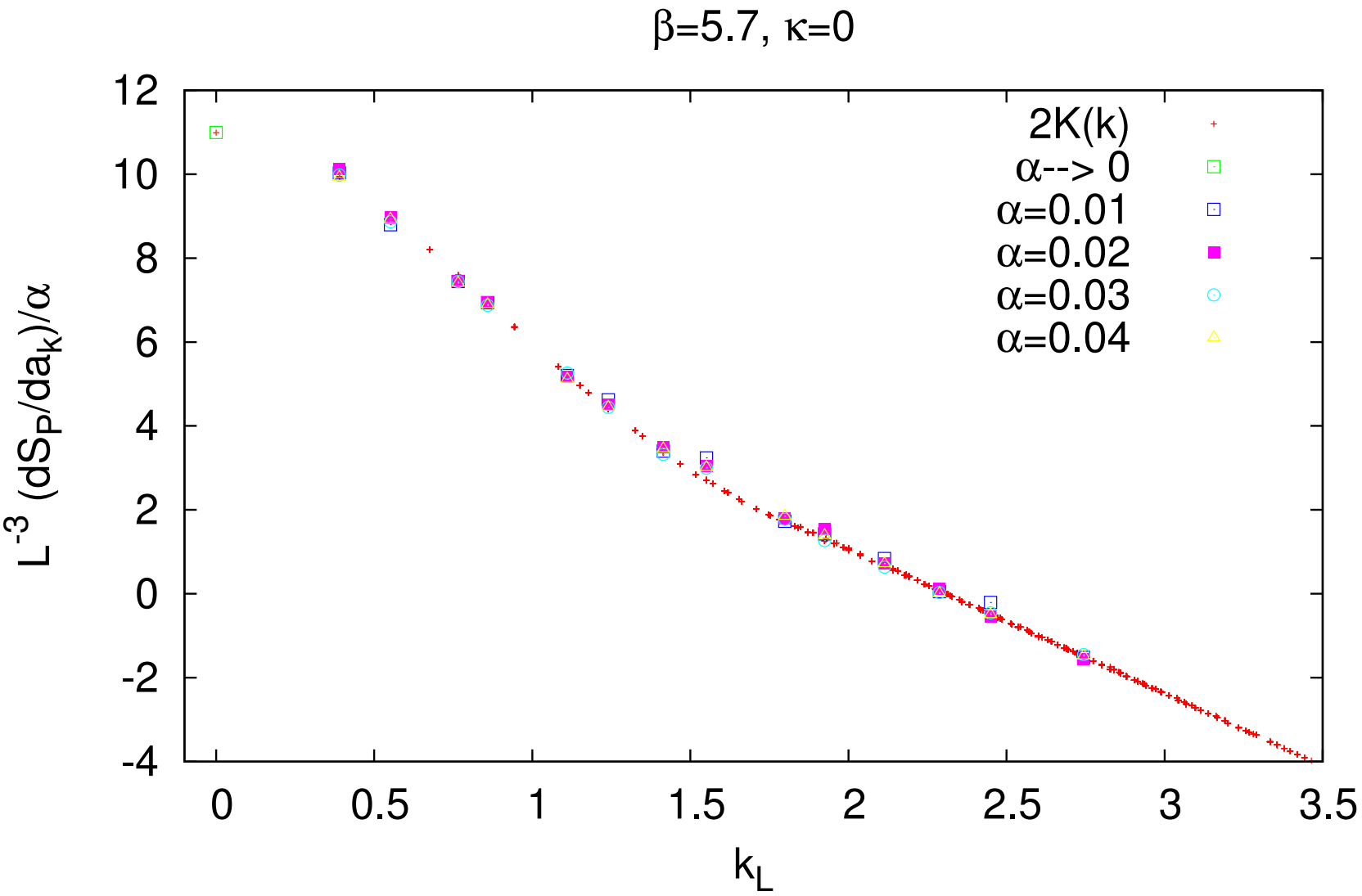
For a pure gauge theory, the part of S_P bilinear in P_x is constrained to have the form

$$S_P = \sum_{\mathbf{xy}} P_{\mathbf{x}} P^\dagger_{\mathbf{y}} K(\mathbf{x}-\mathbf{y})$$

Then, going over to Fourier modes

$$\frac{1}{\alpha}\frac{1}{L^3}\left(\frac{\partial S_P}{\partial a^R_{\mathbf{k}}}\right)_{a_{\mathbf{k}}=\alpha} = 2\widetilde{K}(\mathbf{k})$$

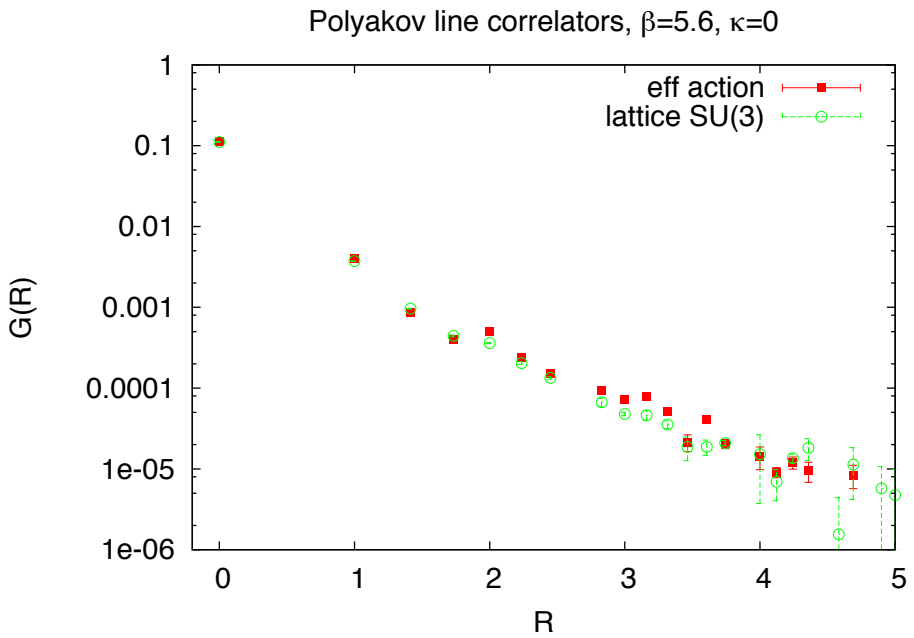
The red points are the Fourier transform of $K(\mathbf{x} - \mathbf{y})$, which gives us the effective action S_P



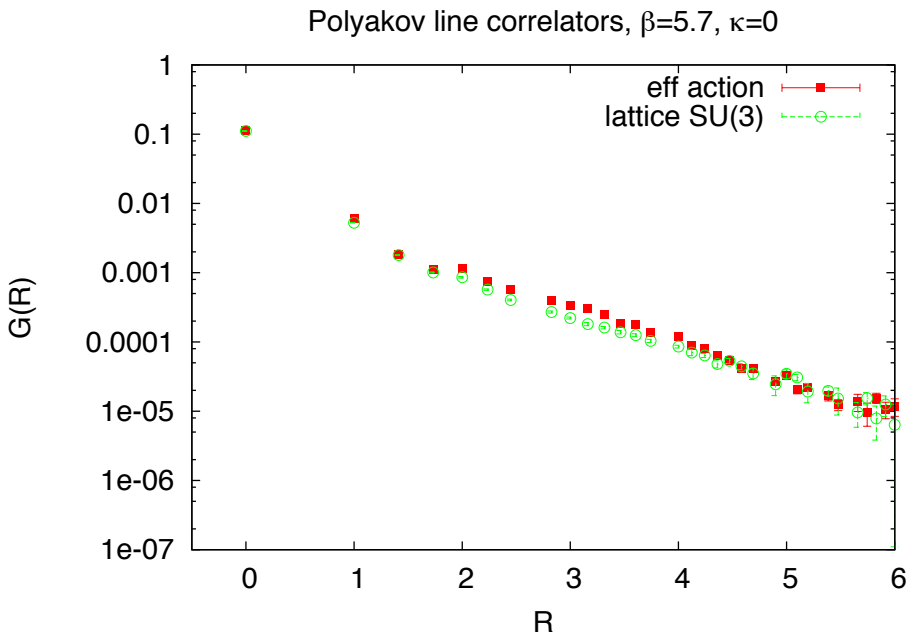
Correlator comparisons at $\beta = 5.6, 5.7$

$$S_P = \sum_{\mathbf{xy}} P_{\mathbf{x}} P^{\dagger}_{\mathbf{y}} K(\mathbf{x} - \mathbf{y})$$

Simulate the effective theory in the usual way, and compare the Polyakov line correlators in the effective theory with the correlators in the underlying pure gauge theory



$\beta = 5.6$



$\beta = 5.7$

S_P for the SU(3) gauge-Higgs model

Including linear and bilinear center symmetry-breaking terms, it can be shown that at finite chemical potential

$$\begin{aligned} S_P &= \sum_{\mathbf{xy}} P_{\mathbf{x}} P^\dagger_{\mathbf{y}} K(\mathbf{x}-\mathbf{y}) + \sum_{\mathbf{xy}} (P_{\mathbf{x}} P_{\mathbf{y}} Q(\mathbf{x}-\mathbf{y},\mu) + P^\dagger_{\mathbf{x}} P^\dagger_{\mathbf{y}} Q(\mathbf{x}-\mathbf{y};-\mu)) \\ &\quad + \sum_{\mathbf{x}} \left\{ (d_1 e^{\mu/T} - d_2 e^{-2\mu/T}) P_{\mathbf{x}} + (d_1 e^{-\mu/T} - d_2 e^{2\mu/T}) P^\dagger_{\mathbf{x}} \right\} \end{aligned}$$

where

$$Q(\mathbf{x}-\mathbf{y};\mu) = Q^{(1)}(\mathbf{x}-\mathbf{y}) e^{-\mu/T} + Q^{(2)}(\mathbf{x}-\mathbf{y}) e^{2\mu/T} + Q^{(4)}(\mathbf{x}-\mathbf{y}) e^{-4\mu/T}$$

The problem is to determine $K(\mathbf{x}-\mathbf{y}), d_1, d_2, Q(\mathbf{x}-\mathbf{y};\mu)$.

Use of the imaginary chemical potential $\mu/T = i\theta$

In terms of Fourier components

$$\frac{1}{L^3} S_P = \sum_{\mathbf{k}} a_{\mathbf{k}} a^*_{\mathbf{k}} \widetilde{K}(k_L) + a_0 \left(d_1 e^{i\theta} - d_2 e^{-2i\theta} \right) + a^*_0 \left(d_1 e^{-i\theta} - d_2 e^{2i\theta} \right) + \sum_{\mathbf{k}} \left(a_{\mathbf{k}} a_{-\mathbf{k}} \widetilde{Q}(k_L, \theta) + a^*_{\mathbf{k}} a^*_{-\mathbf{k}} \widetilde{Q}(k_L, \theta) \right)$$

Then

$$\frac{1}{L^3} \left(\frac{\partial S_P}{\partial a_0^R} \right)_{a_0 = \alpha} = 2\widetilde{K}(0)\alpha + 2d_1 \cos(\theta) - (2d_2 - 4\widetilde{Q}(0)\alpha)\cos(2\theta)$$

Fit to

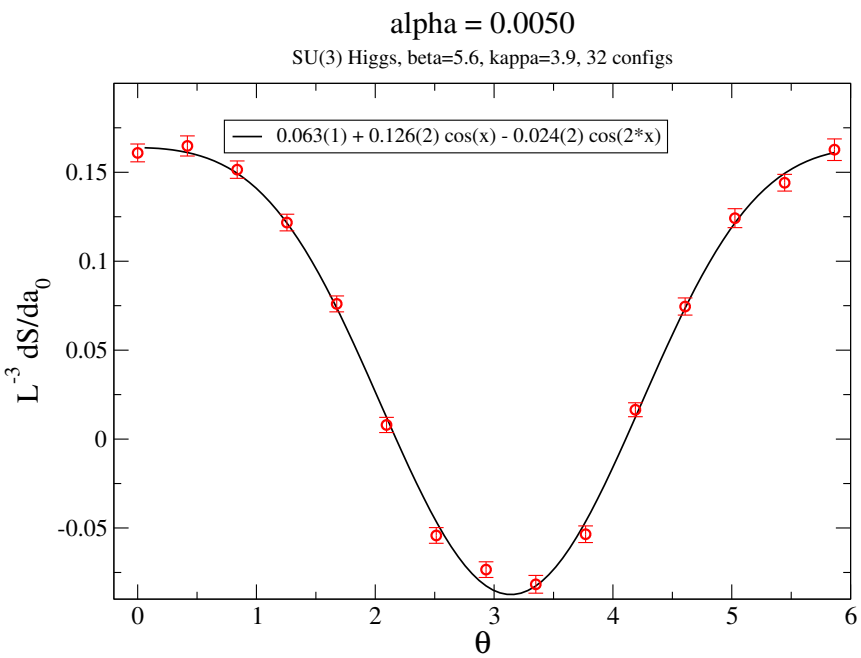
$$\frac{1}{L^3} \left(\frac{\partial S_P}{\partial a_0^R} \right)_{a_0^R = \alpha} = A(\alpha) + B(\alpha)\cos(\theta) - C(\alpha)\cos(2\theta)$$

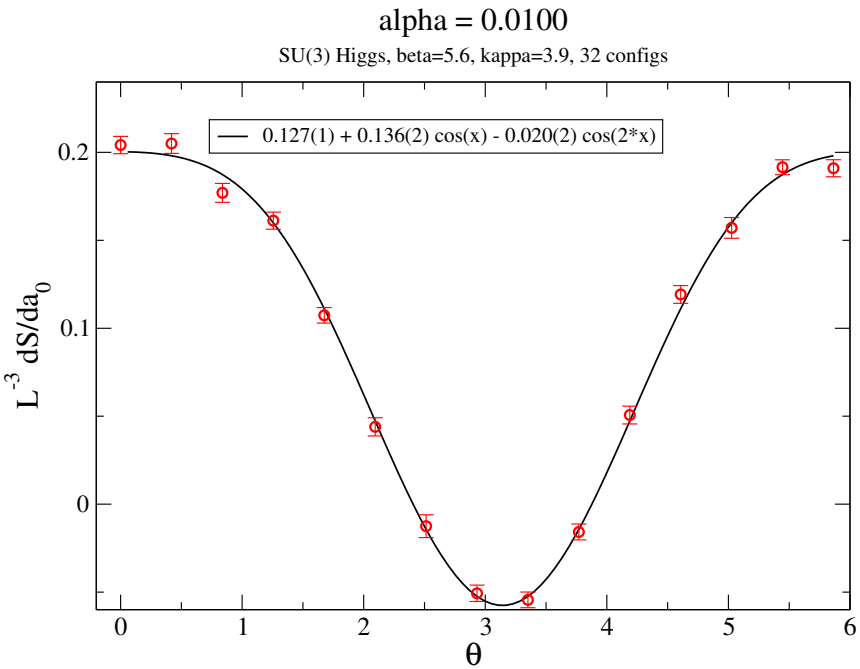
Compare the data to the fit, and we find $d_1, d_2, \widetilde{K}(0), \widetilde{Q}(0)$.

Gauge-Higgs theory at $\beta = 5.6, \kappa = 3.9$ on a $16^3 \times 6$ lattice. Calculate (lhs) and fit (rhs)

$$\frac{1}{L^3}\left(\frac{\partial S_P}{\partial a_0^R}\right)_{a_0^R=\alpha} = A(\alpha) + B(\alpha)\cos(\theta) - C(\alpha)\cos(2\theta)$$

at 15 values of θ and several α values:



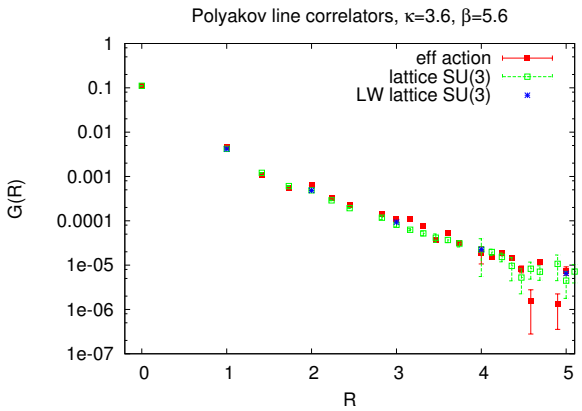


We can then extract coefficients of center symmetry-breaking terms (in this case $d_1 = 0.0585, d_2 = 0.0115$), as well as $\widetilde{K}(0)$ and $\widetilde{Q}(0)$.

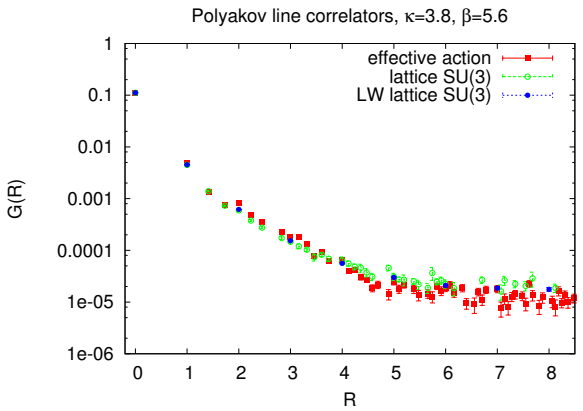
Gauge-Higgs Correlator Comparison

Effective action vs. lattice gauge theory

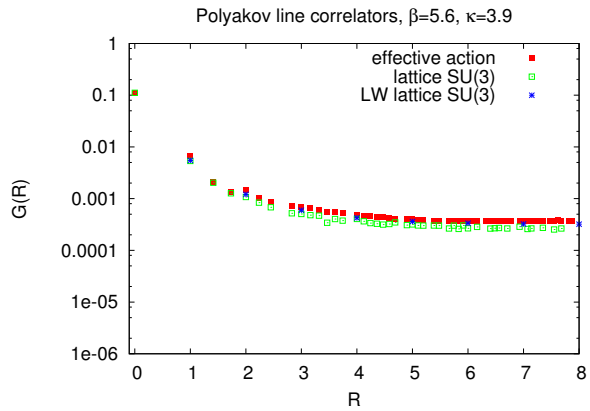
The underlying lattice gauge-Higgs theory is at $\beta = 5.6, \mu = 0$ and $\kappa = 3.6, 3.8, 3.9$ on a $16^3 \times 6$ lattice volume.



(a) $\kappa = 3.6$



(b) $\kappa = 3.8$



(c) $\kappa = 3.9$

There is still a sign problem...

Comparison of complex Langevin

and mean field methods

applied to effective actions at $\mu > 0$

Gauge-Higgs at $\kappa = 3.8, 3.9$

$$
\begin{aligned}
S_P &= \frac{1}{9}\sum_{xy} \mathrm{Tr}[U_{\mathbf{x}}]\mathrm{Tr}[U^{\dagger}_{\mathbf{y}}]K(\mathbf{x}-\mathbf{y}) \\
&+\frac{1}{3}\sum_{x}\Big\{(d_1 e^{\mu/T} - d_2 e^{-2\mu/T})\mathrm{Tr}[U_{\mathbf{x}}] + (d_1 e^{-\mu/T} - d_2 e^{2\mu/T})\mathrm{Tr}[U^{\dagger}_{\mathbf{x}}]\Big\}
\end{aligned}
$$

The d_2 dependent terms must originate from "double-winding "terms

$$d_2 e^{2\mu/T}\mathrm{Tr}[U^2_{\mathbf{x}}] + d_2 e^{-2\mu/T}\mathrm{Tr}[U^{\dagger 2}_{\mathbf{x}}]$$

via the SU(3) identities

$$\mathrm{Tr}[U^2_{\mathbf{x}}] = \mathrm{Tr}[U_{\mathbf{x}}]^2 - 2\mathrm{Tr}[U^{\dagger}_{\mathbf{x}}] \quad , \quad \mathrm{Tr}[U^{\dagger 2}_{\mathbf{x}}] = \mathrm{Tr}[U^{\dagger}_{\mathbf{x}}]^2 - 2\mathrm{Tr}[U_{\mathbf{x}}] \, ,$$

With that motivation, we also consider

A model with a double-winding term

$$
\begin{aligned}
S_P \;=\; & \frac{1}{9}\sum_{xy} \mathrm{Tr}[U_{\mathbf{x}}]\mathrm{Tr}[U^\dagger_{\mathbf{y}}]K(\mathbf{x}-\mathbf{y}) \\
& + \frac{1}{3}\sum_{x}\left\{ d_1 e^{\mu/T}\mathrm{Tr}[U_{\mathbf{x}}] + d_1 e^{-\mu/T}\mathrm{Tr}[U^\dagger_{\mathbf{x}}]\right\} \\
& + \frac{1}{6}\sum_{x}\left\{ d_2 e^{2\mu/T}\mathrm{Tr}[U^2_{\mathbf{x}}] + d_2 e^{-2\mu/T}\mathrm{Tr}[U^{\dagger 2}_{\mathbf{x}}]\right\}
\end{aligned}
$$

Heavy Quark model

Hopping parameter very small, chemical potential μ very large. In temporal gauge, the lattice action is simply

$$e^{S_L} = \prod_{\mathbf{x}} \det\left[1 + h e^{\mu/T} U_0(\mathbf{x}, 0)\right]^p \det\left[1 + h e^{-\mu/T} U^\dagger(\mathbf{x}, 0)\right]^p e^{S_{plaq}}$$

$p = 1$ for staggered fermions, $p = 2N_f$ for Wilson fermions. If we know the Polyakov line action for the pure gauge theory S_P^{pg}, then

$$e^{S_P} = \prod_{\mathbf{x}} \det\left[1 + h e^{\mu/T} U_{\mathbf{x}}\right]^p \det\left[1 + h e^{-\mu/T} U_{\mathbf{x}}^\dagger\right]^p e^{S_P^{pg}}$$

Complex Langevin for the effective actions

We follow the approach of *Aarts and James (2012)*.

Effective Polyakov line models depend only on the eigenvalues $\exp[i\theta_a(x)]$ of $U_{\mathbf{x}}$. In particular

$$\mathrm{Tr}[U_{\mathbf{x}}] = e^{i\theta_1(\mathbf{x})} + e^{i\theta_2(\mathbf{x})} + e^{-i(\theta_1(\mathbf{x})+\theta_2(\mathbf{x}))}$$

Treat $\theta_{1,2}(\mathbf{x})$ as the dynamical variables. Then the Haar integration measure must be incorporated into the action

$$\begin{aligned} S_P \quad \longrightarrow \quad S'_P = S_P + \sum_{\mathbf{x}} \log \Bigg[& \sin^2\left(\frac{\theta_1(\mathbf{x}) - \theta_2(\mathbf{x})}{2}\right) \\ & \times \sin^2\left(\frac{2\theta_1(\mathbf{x}) + \theta_2(\mathbf{x})}{2}\right) \sin^2\left(\frac{\theta_1(\mathbf{x}) + 2\theta_2(\mathbf{x})}{2}\right) \Bigg] \end{aligned}$$

The prescription is then to complexify the angles $\theta_{1,2}(\mathbf{x})$, and solve the complex Langevin equation.

The Møllgård-Splittorff Warning (2013)

Beware! Logarithms have branch cuts along the negative real axis. Complex Langevin can go wrong if

1. there is a logarithmic term in the action (e.g. the log of a measure or a fermion determinant), and
2. Langevin evolution frequently crosses the branch cut.

To check this, we keep track of the argument of the logarithm

$$\text{Arg} = \sin^2\left(\frac{\theta_1(\mathbf{x}') - \theta_2(\mathbf{x}')}{2}\right) \sin^2\left(\frac{2\theta_1(\mathbf{x}') + \theta_2(\mathbf{x}')}{2}\right) \sin^2\left(\frac{\theta_1(\mathbf{x}') + 2\theta_2(\mathbf{x}')}{2}\right)$$

at an arbitrarily chosen lattice site $\mathbf{x}'$.

Mean Field Theory for the effective actions

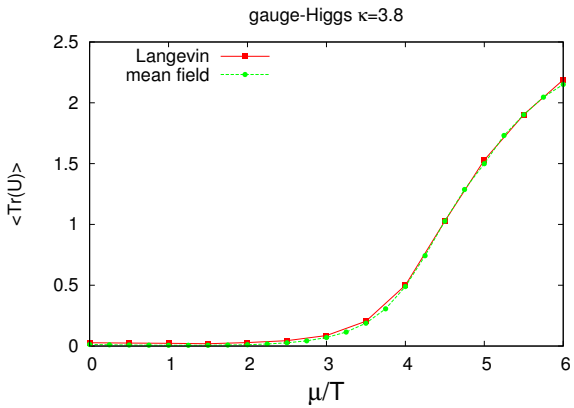
The mean field treatment of SU(3) spin models at finite μ is a minor variation of standard mean field theory at zero chemical potential.

Two "magnetizations" are introduced; one for TrU and one for Tr$U^\dagger$. These are determined, as usual, by minimizing the free energy.

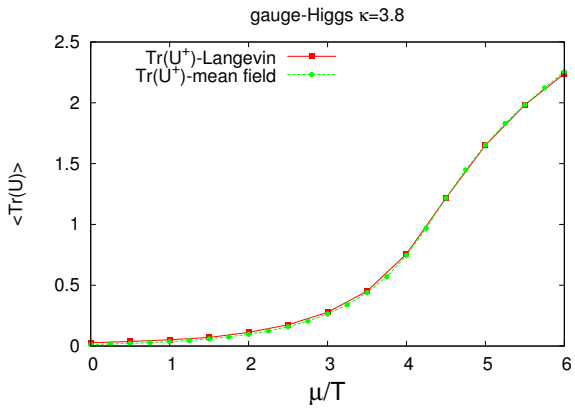
For details, see *Splittorff and JG (2012)*.

Results I - Gauge-Higgs at $\kappa = 3.8$

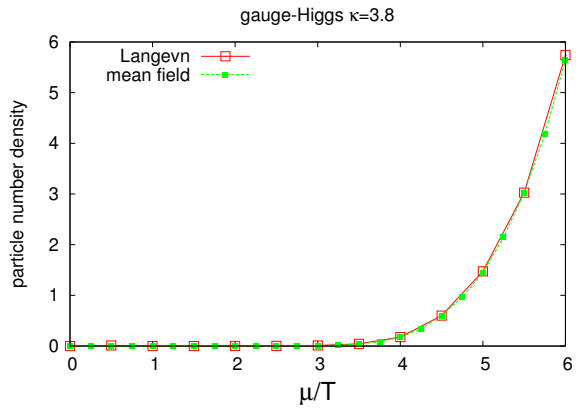
Here are the results for the Polyakov lines and the number density, derived from complex Langevin and mean field ($\beta = 5.6, 16^3 \times 6$ lattice as before):



(a) $\langle \mathrm{Tr}(U) \rangle$



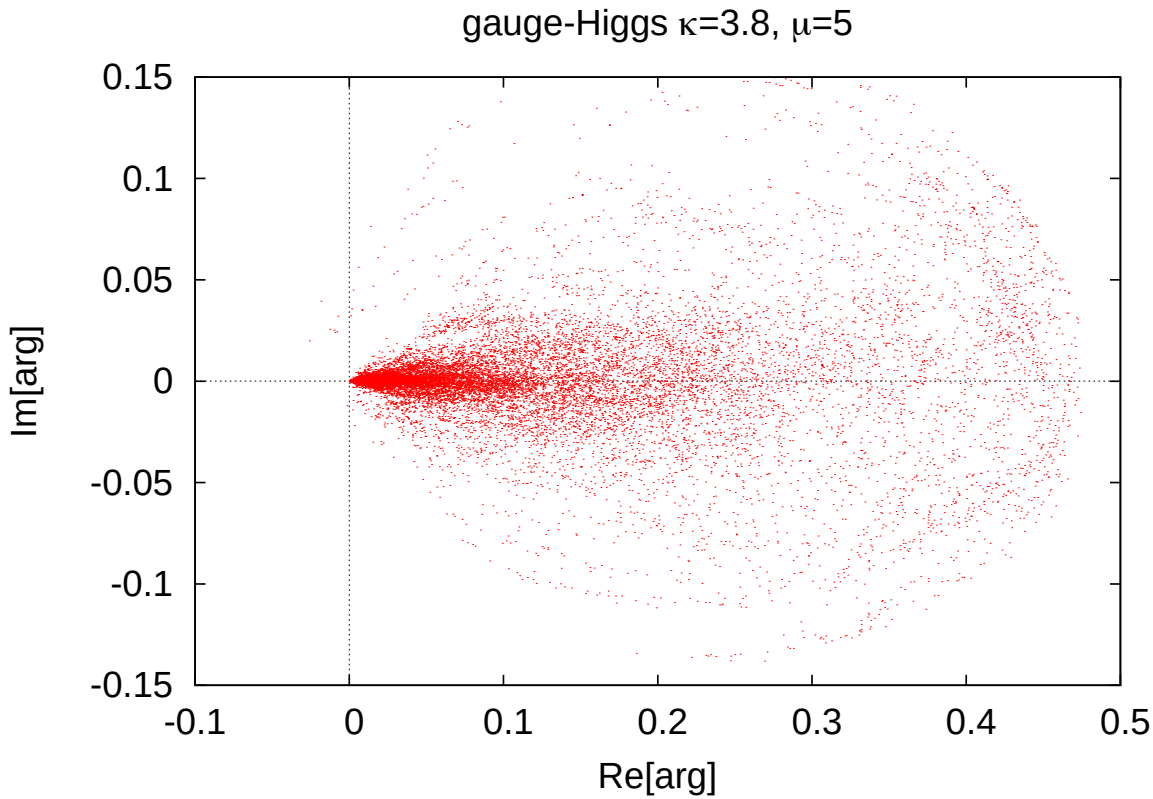
(b) $\langle \mathrm{Tr}(U^\dagger) \rangle$



(c) density

It is hard to even detect a difference between the two methods.

Here is a plot of the argument of the logarithm in the complex plane, at a fixed lattice site, at each Langevin time step for $\mu = 5$:

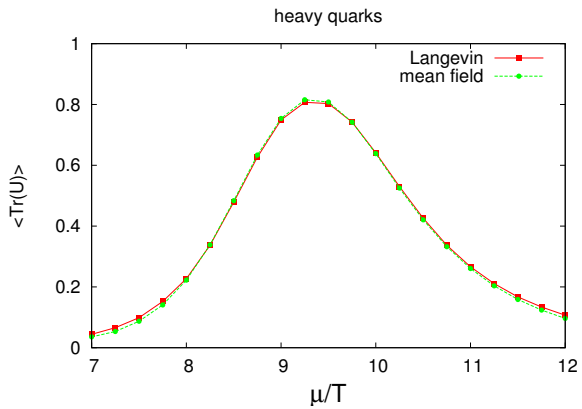


There seems to be no branch-cut crossing problem.

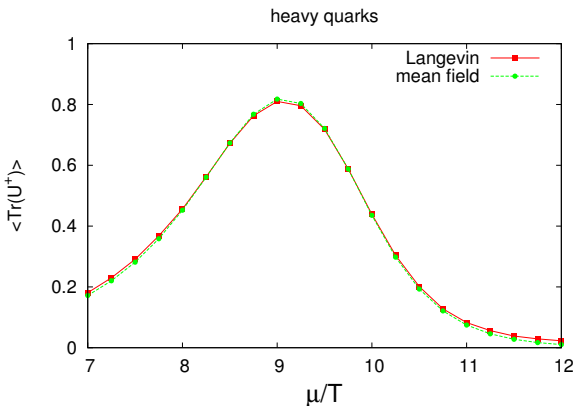
Results II - The Heavy Quark Model

$(p = 1,\ \beta = 5.6,\ h = 10^{-4},\ 16^3 \times 6)$

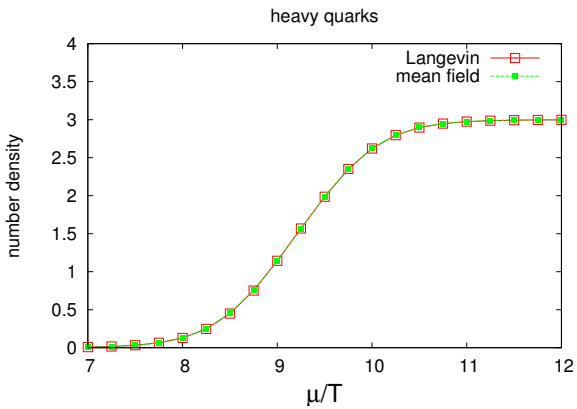
Again, near-perfect agreement:



(a) $\langle \mathrm{Tr}(U) \rangle$

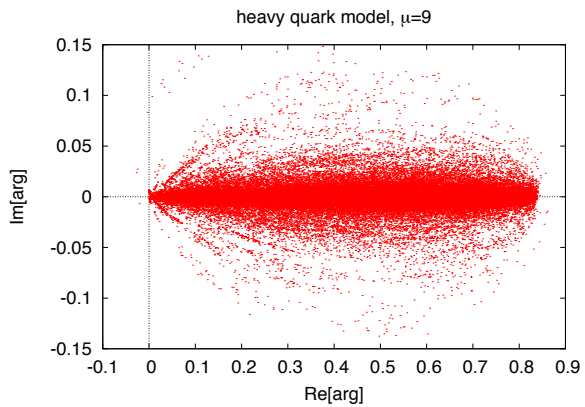


(b) $\langle \mathrm{Tr}(U^\dagger) \rangle$



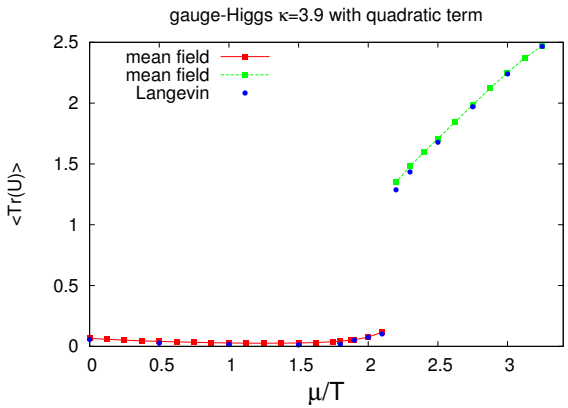
(c) density

and no branch-cut crossing problem

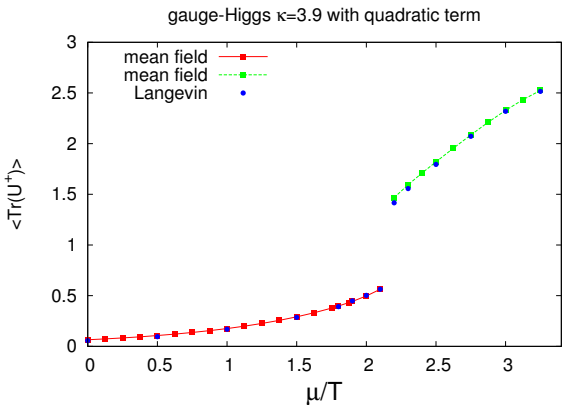


Results III - Gauge-Higgs with double winding term

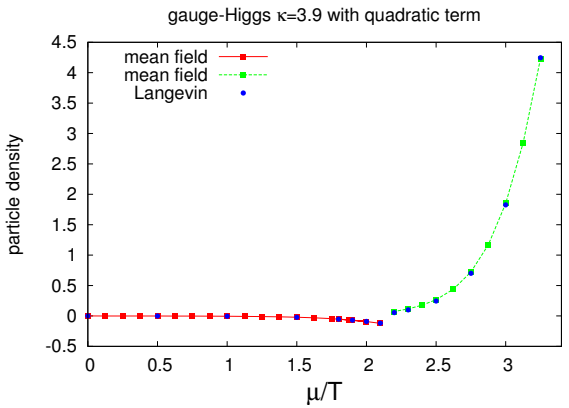
Here we see a phase transition



(a) $\langle \mathrm{Tr}(U) \rangle$

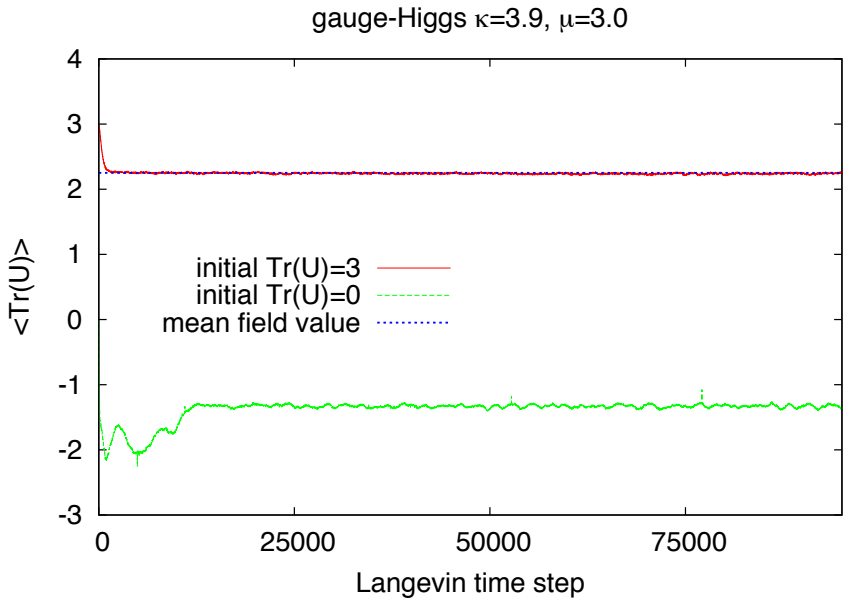


(b) $\langle \mathrm{Tr}(U^\dagger) \rangle$

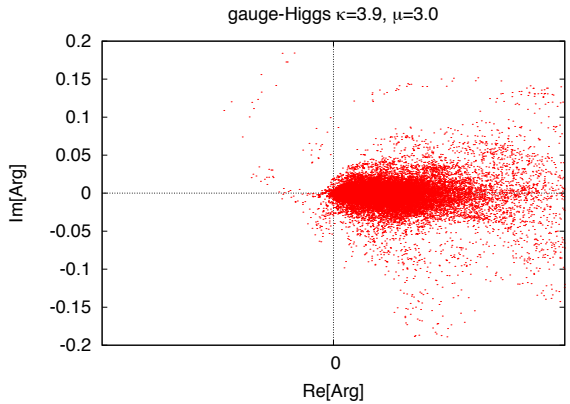


(c) density

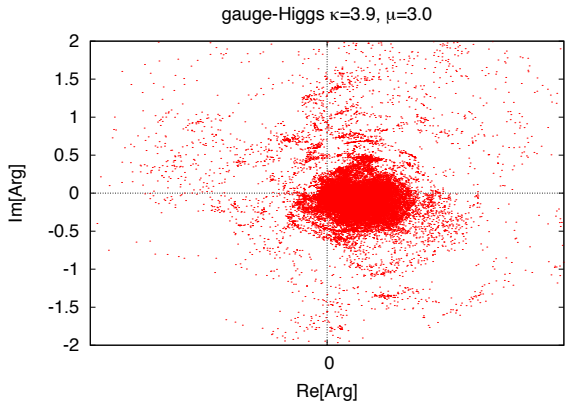
Great agreement. However, complex Langevin has at least two solutions above the transition, depending on initialization, and only one agrees with mean field.



Which to prefer? It may be determined by the branch-cut crossing problem:

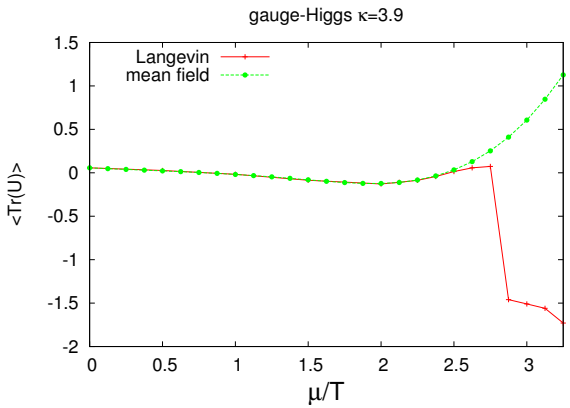


(a) initialize $\mathrm{Tr}U = 3$

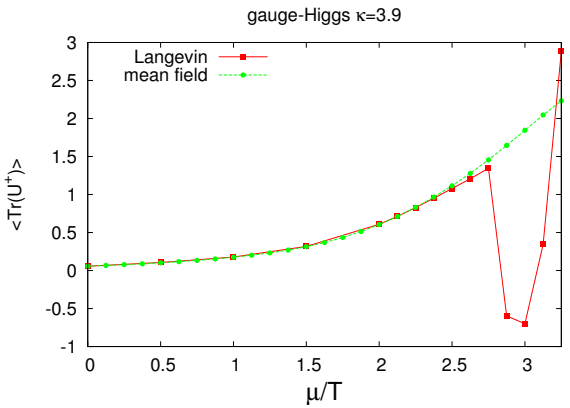


(b) initialize $\mathrm{Tr}U = 0$

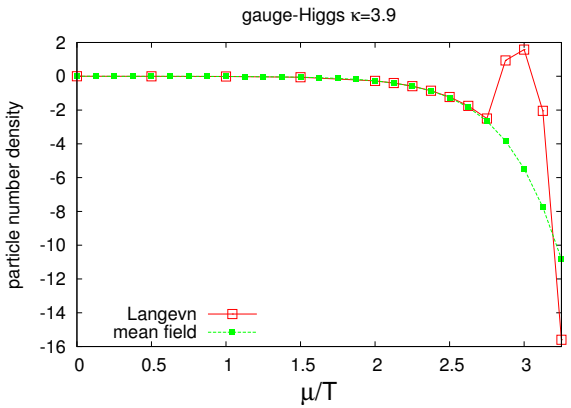
Results IV - Gauge-Higgs $\kappa = 3.9$, no double winding term



(a) $\langle \mathrm{Tr}(U) \rangle$



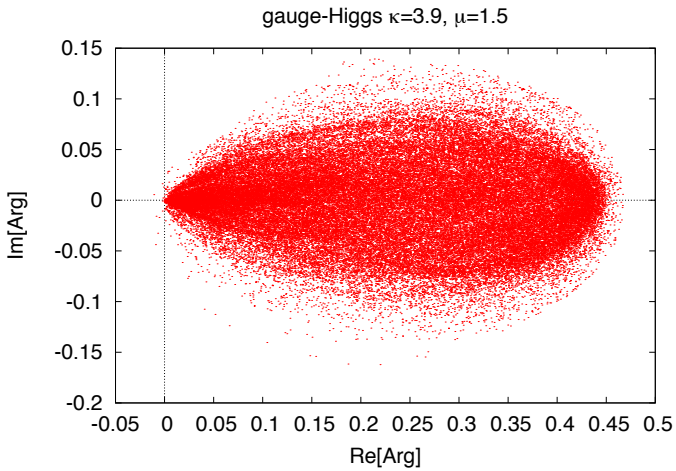
(b) $\langle \mathrm{Tr}(U^\dagger) \rangle$



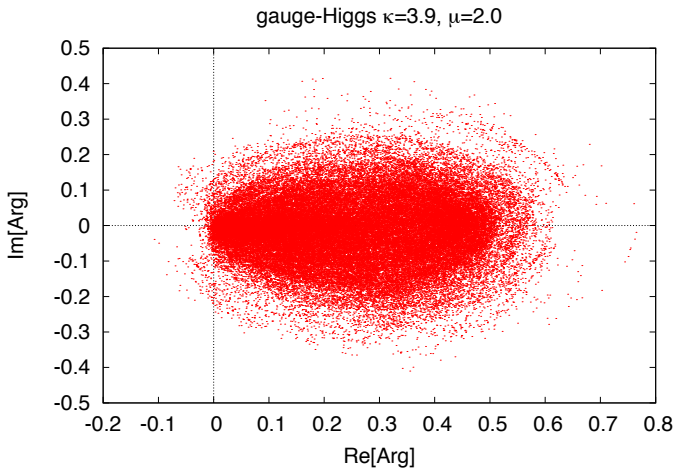
(c) density

Here there is a very strong disagreement between mean field and complex Langevin at $\mu \geq 2.75$.

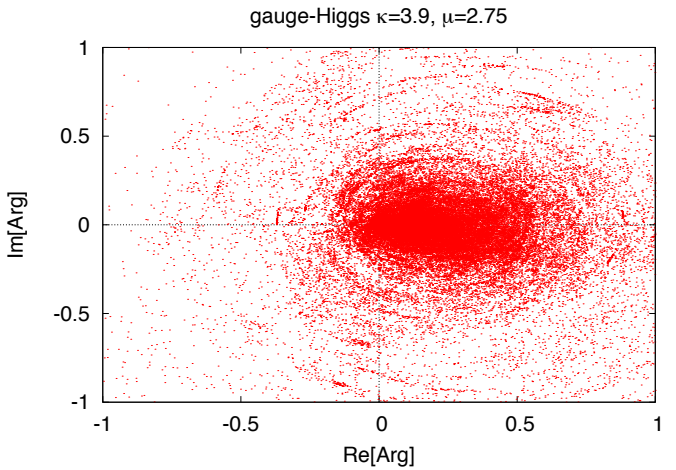
But where the results differ, complex Langevin evolution has a branch-cut problem.



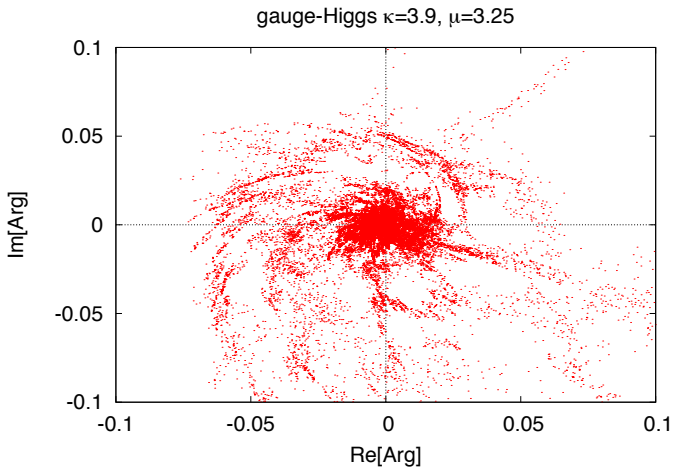
(a) $\mu/T = 1.5$



(b) $\mu/T = 2.0$



(c) $\mu/T = 2.75$

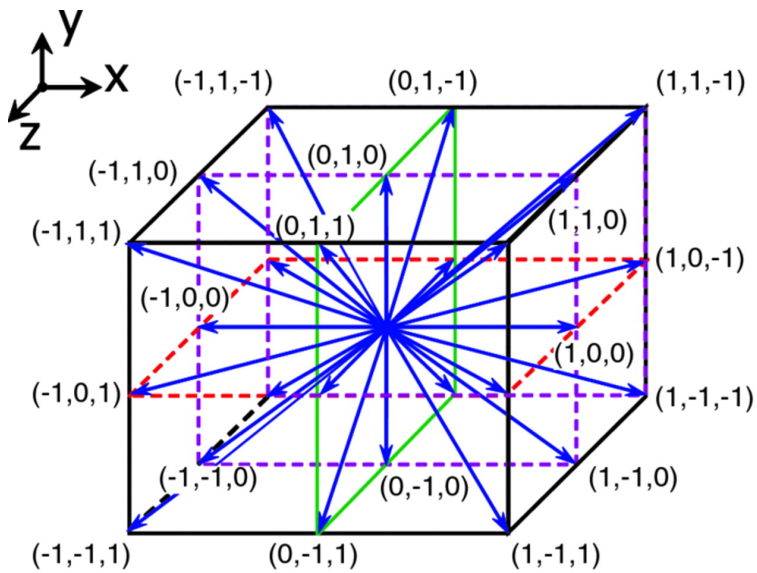


(d) $\mu/T = 3.25$

Why is mean field so good?

Perhaps because many spins – not just nearest neighbors – are coupled to a given spin, through the non-local kernel $K(\mathbf{x} - \mathbf{y})$.

The basic idea behind mean field theory, i.e. that each spin is effectively coupled to the average spin on the lattice, may be a very good approximation to the true situation



Improve complex Langevin for the effective theories?

One possibility is to complexify $U_{\mathbf{x}}$, as in the the Langevin approach to lattice gauge theory, rather than the angles $\theta_{1,2}(\mathbf{x})$.

Then there is no logarithm of the measure in the action, and no Møllgård-Splittorff problem.

It is still necessary to monitor $U_{\mathbf{x}} U_{\mathbf{x}}^{\dagger}$, to check that it doesn't wander too far from the unit matrix.

CONCLUSIONS

- We have developed a method for determining the effective Polyakov line action.
- At $\mu = 0$ there is excellent agreement for the Polyakov line correlators computed in the effective theory and underlying lattice gauge theory.
- At $\mu > 0$ we can solve the effective theory by either mean field or complex Langevin methods.
- Where the two methods agree, they agree almost perfectly. Where they disagree, complex Langevin has a Møllgård-Splittorff branch cut crossing problem.

Next Steps

Go on to dynamical fermions. First heavy, then light.

Prescription:

Find the effective action via relative weights, solve by mean field.

Given S_P, there may be no need to resort to any further numerical simulation at finite μ.

EXTRA SLIDES

Mean Field Theory for the effective actions

We follow the approach of *Splittorff and JG (2012)*.
The idea is to localize the part of the action S_P^0 containing products of terms at different sites:

$$\begin{aligned} S_P^0 &= \frac{1}{9}\sum_{\mathbf{xy}} \mathrm{Tr}[U_{\mathbf{x}}]\mathrm{Tr}[U_{\mathbf{y}}^\dagger] K(\mathbf{x}-\mathbf{y}) \\ &= \frac{1}{9}\sum_{(\mathbf{xy})} \mathrm{Tr}[U_{\mathbf{x}}]\mathrm{Tr}[U_{\mathbf{y}}^\dagger] K(\mathbf{x}-\mathbf{y}) + a_0 \sum_{\mathbf{x}} \mathrm{Tr}[U_{\mathbf{x}}]\mathrm{Tr}[U_{\mathbf{x}}^\dagger] \end{aligned}$$

where we have introduced the notation for the double sum, excluding $\mathbf{x} = \mathbf{y}$,

$$\sum_{(\mathbf{xy})} \equiv \sum_{\mathbf{x}} \sum_{\mathbf{y}\neq\mathbf{x}} \quad \text{and} \quad a_0 \equiv \frac{1}{9}K(0)$$

Next, introduce parameters u, v

$$\mathrm{Tr}U_{\mathbf{x}} = (\mathrm{Tr}U_{\mathbf{x}} - u) + u \quad , \quad \mathrm{Tr}U_{\mathbf{x}}^\dagger = (\mathrm{Tr}U_{\mathbf{x}}^\dagger - v) + v$$

Then

$$S_P^0 \quad = \quad J_0 \sum_{\mathbf{x}} (v \mathrm{Tr} U_{\mathbf{x}} + u \mathrm{Tr} U_{\mathbf{x}}^\dagger) - uv J_0 V + a_0 \sum_{\mathbf{x}} \mathrm{Tr}[U_{\mathbf{x}}] \mathrm{Tr}[U_{\mathbf{x}}^\dagger] + E^0$$

where $V = L^3$ is the lattice volume, and we have defined

$$\begin{aligned} E^0 &= \sum_{(\mathbf{xy})} (\mathrm{Tr} U_x - u)(\mathrm{Tr} U_{\mathbf{y}}^\dagger - v) \frac{1}{9} K(\mathbf{x} - \mathbf{y}) \, , \\ J_0 &= \frac{1}{9} \sum_{\mathbf{x} \neq 0} K(\mathbf{x}) \end{aligned}$$

If we drop E_0, the action is local and and the group integrations can be carried out analytically.

The trick is to choose u and v such that E_0 can be treated as a perturbation, to be ignored as a first approximation. In particular, $\langle E_0 \rangle = 0$ when

$$u = \langle \mathrm{Tr} U_x \rangle \quad , \quad v = \langle \mathrm{Tr} U_x^\dagger \rangle$$

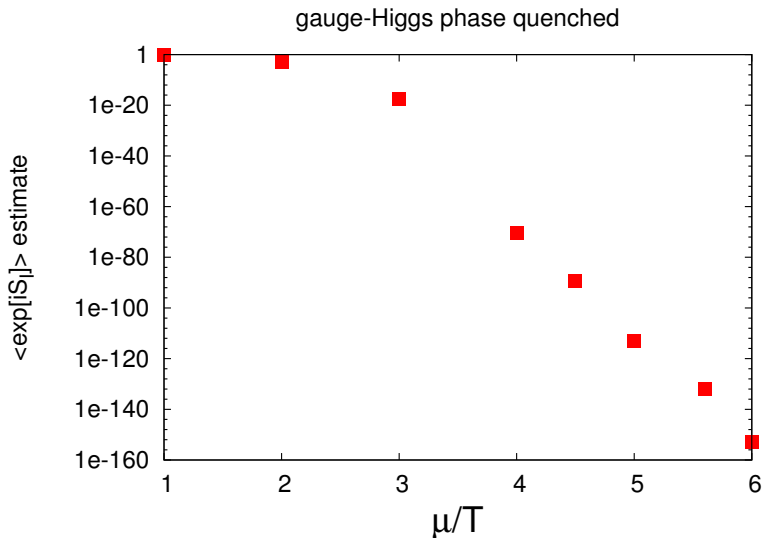
This is *equivalent to stationarity* of the mean field free energy, with respect to variations in u, v, and is solved numerically.

How bad is the sign problem here?

We estimate $\langle e^{iS_I} \rangle_{pq}$ in the "phase quenched" measure e^{-S_R}, using the crude approximation

$$\langle e^{iS_i} \rangle_{pq} \sim \exp\left[-\frac{1}{2} \langle S_i^2 \rangle_{pq} \right]$$

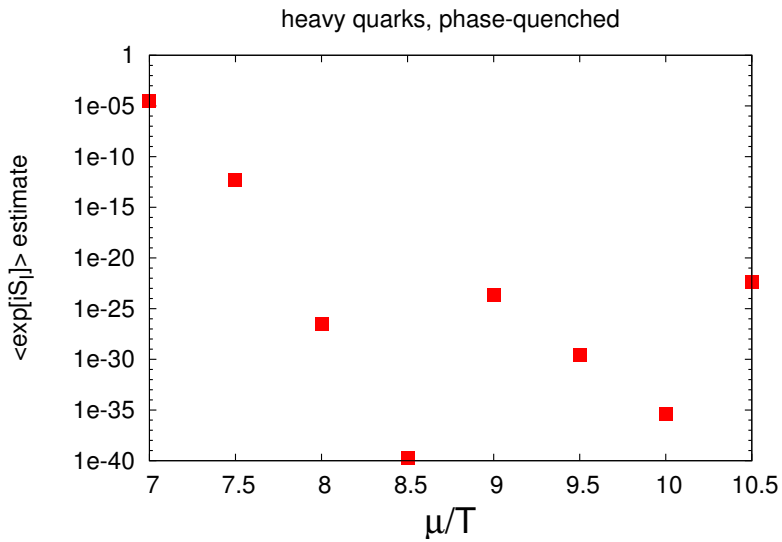
The rhs is the lowest-order term in the cumulant expansion.



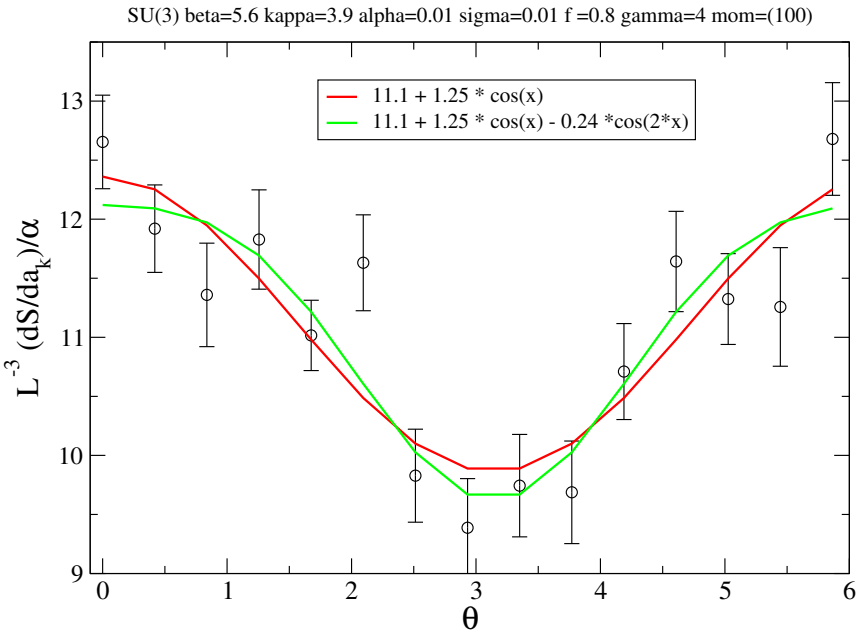
How bad is the sign problem in this model?

Once again, estimate $\langle e^{iS_I} \rangle_{pq}$ in the "phase quenched" measure e^{-S_R},

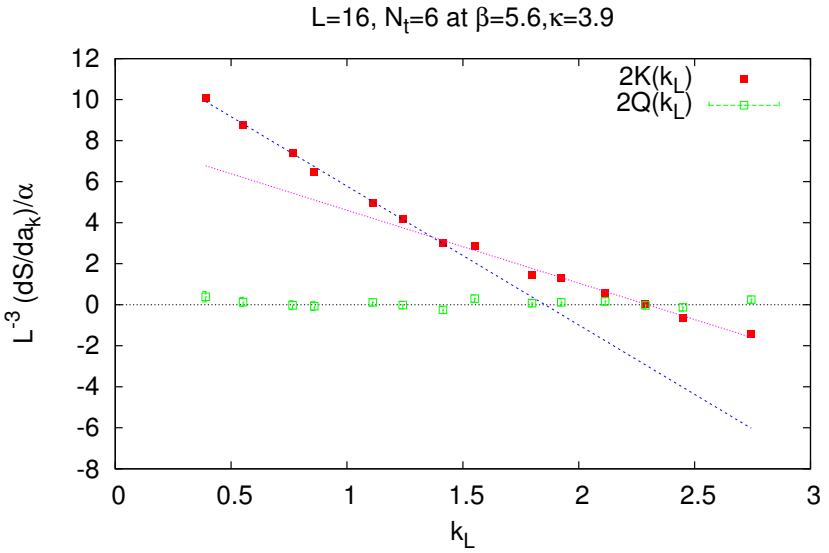
$$\langle e^{iS_i} \rangle_{pq} \sim \exp \left[-\frac{1}{2} \langle S_i^2 \rangle_{pq} \right]$$



$\widetilde{Q}(k_L, \mu)$ seems calculable, but the magnitude is small and the errorbars are large:



(a) $a_{\mathbf{k}}$ derivative at smallest $k_L \neq 0$, vs. θ



(b) $\widetilde{K}(k_L)$ and (estimate of) $\widetilde{Q}_1(k_L)$ vs. k_L

For now we will ignore the $Q(\mathbf{x} - \mathbf{y}; \mu)$ term in the action.