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Reformulations of the Yang-Mills theory toward quark confinement and mass gap

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Title: Quark confinement: dual superconductor picture based on a non-Abelian Stokes theorem and reformulations of Yang-Mills theory
(277 pages including 59 figures and 13 tables)

Contents

§ Introduction

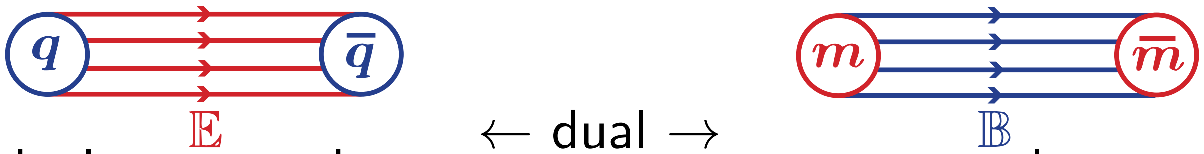
Dual superconductor picture for quark confinement

[Nambu (1974), 't Hooft (1975), Mandelstam (1976), Polyakov (1975,1977) ...]

The key ingredients for the hypothesis of dual superconductor = QCD vacuum=

* Dual Meissner effect

In the dual superconductor, chromoelectric flux is squeezed into tubes.
[$\leftarrow$ In the ordinary superconductor, magnetic flux is squeezed into tubes]



dual superconductor $\leftarrow$ dual $\rightarrow$ superconductor

* condensation of chromomagnetic monopoles

[$\leftarrow$ electric charges condense into Cooper pairs]

We must answer the following questions:

* How to introduce magnetic monopoles in the Yang-Mills theory without scalar fields?
cf. 't Hooft-Polyakov magnetic monopole
* How to define the duality in the non-Abelian gauge theory?
* How to preserve the original (non-Abelian) gauge symmetry?

§ Non-Abelian Stokes theorem (1)

We consider how the Wilson loop is related to the magnetic monopole.

First, we consider the Abelian case. The Abelian Wilson loop is cast into the surface integral by the **Stokes theorem**

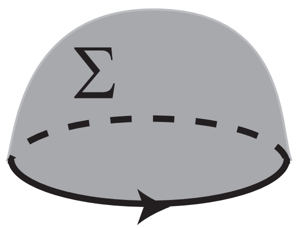
$$W_C[A] = \exp\left[ie\oint_C dx^\mu A_\mu\right] \Longrightarrow W_C[A] = \exp\left[ie\int_{\Sigma:\partial\Sigma=C} dS^{\mu\nu}(x(\sigma))F_{\mu\nu}(x(\sigma))\right].$$

Introduce the **vorticity tensor** with the support only on the surface Σ_C bounded by the loop C

$$\Theta_\Sigma^{\mu\nu}(x) := \int_{\Sigma:\partial\Sigma=C} d^2S^{\mu\nu}(x(\sigma))\delta^D(x-x(\sigma)).$$

It is rewritten into the spacetime integral:

$$W_C[A] = \exp\{ie(\Theta, F)\}, \quad (\Theta, F) := \int d^Dx\Theta_\Sigma^{\mu\nu}(x)F_{\mu\nu}(x).$$



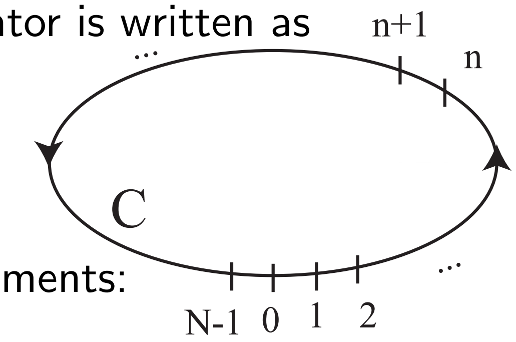
The Hodge decomposition yields the electric current j and the **magnetic current** k:

$$W_C[A] = \exp\{ie(N_\Sigma, j) + ie(\Xi_\Sigma, k)]\}, \quad N_\Sigma := \delta\Delta^{-1}\Theta_\Sigma, \quad \Xi_\Sigma := \delta\Delta^{-1*}\Theta_\Sigma.$$

Here the electric current j is non-vanishing: $j := \delta F \neq 0$, while the magnetic current k is vanishing due to the Bianchi identity and there is no magnetic contribution to the Wilson loop,

$$k := \delta^* F = {}^*dF = {}^*ddA = 0 \Longrightarrow W_C[A] = \exp\{ie(N_\Sigma, j)]\},$$

Next, we consider the non-Abelian case. The non-Abelian Wilson loop operator is written as



$$W_C[\mathscr{A}] := \mathrm{tr}_R \left\{ \mathscr{P} \exp \left[-i g_{\mathrm{YM}} \oint_C \mathscr{A} \right] \right\} / \mathrm{tr}_R(\mathbf{1}).$$

0. Define the **path ordering** $\mathscr{P}$ by dividing the path C into N infinitesimal segments:

$$W_C[\mathscr{A}] = \lim_{N\to\infty, \epsilon\to 0} \mathrm{tr}_R \left\{ \mathscr{P} \prod_{n=0}^{N-1} \exp \left[-i g_{\mathrm{YM}} \int_{x_n}^{x_{n+1}} \mathscr{A} \right] \right\} / \mathrm{tr}_R(\mathbf{1}). \tag{2}$$

The troublesome path ordering in the non-Abelian Wilson loop operator can be removed in the **Diakonov and Petrov version of the non-Abelian Stokes theorem**. [Diakonov and Petrov (1989)]

It can be obtained as the path-integral representation of the Wilson loop operator using the coherent state of the Lie group G in an unified way. [Kondo (1998), Kondo and Taira (2000), Kondo (2008)]

1. Replace the trace of the operator $\mathscr{O}$ by the integral:

$$\mathrm{tr}_{\mathrm{R}}(\mathscr{O})/\mathrm{tr}_R(\mathbf{1}) = \int d\mu(g(x_0)) \langle g(x_0), \Lambda | \mathscr{O} | g(x_0), \Lambda \rangle . \tag{3}$$

where $d\mu(g)$ is an invariant measure on G and the state is normalized $\langle g(x_n), \Lambda | g(x_n), \Lambda \rangle = 1$.

2. Insert a **complete set** of states at each partition point:

$$\mathbf{1} = \int d\mu(g(x_n)) \, |g(x_n), \Lambda\rangle \langle g(x_n), \Lambda| \quad (n = 1, \cdots, N-1), \tag{4}$$

The state $|g, \Lambda\rangle$ is constructed by operating a group element $g \in G$ to a **reference state** $|\Lambda\rangle$:

$$|g, \Lambda\rangle = g\,|\Lambda\rangle\,, \quad g \in G. \tag{5}$$

$|\Lambda\rangle$: a reference state (highest–weight state of the rep.) making a rep. of the Wilson loop we consider.
3. Take the limit $N \to \infty$ and $\epsilon \to 0$ appropriately such that $N\epsilon$ is fixed.

$$W_C[\mathscr{A}] = \lim_{N\to\infty,\epsilon\to 0} \prod_{n=0}^{N-1} \int d\mu(g(x_n)) \prod_{n=0}^{N-1} \langle g(x_{n+1}), \Lambda| \exp\left[-ig_{\mathrm{YM}} \int_{x_n}^{x_{n+1}} \mathscr{A}\right] |g(x_n), \Lambda\rangle, \tag{6}$$

For taking the limit $\epsilon \to 0$ in the final step, it is sufficient to retain the $O(\epsilon)$ terms.

$$\begin{aligned}
&\langle g_{n+1}, \Lambda| \exp\left[-ig_{\mathrm{YM}} \int_{x_n}^{x_{n+1}} \mathscr{A}\right] |g_n, \Lambda\rangle \\
=&\langle \Lambda| g(x_{n+1})^\dagger \exp\left[-ig_{\mathrm{YM}} \int_{x_n}^{x_{n+1}} \mathscr{A}\right] g(x_n)|\Lambda\rangle = \langle \Lambda| \exp\left[-ig_{\mathrm{YM}} \int_{x_n}^{x_{n+1}} \mathscr{A}^g\right] |\Lambda\rangle \\
&= \langle \Lambda| \left[1 - ig_{\mathrm{YM}} \int_{x_n}^{x_{n+1}} d\tau \mathscr{A}^g(\tau) + O(\epsilon^2)\right] |\Lambda\rangle \\
&= 1 - ig_{\mathrm{YM}} \int_{x_n}^{x_{n+1}} \langle \Lambda|\mathscr{A}^g|\Lambda\rangle + O(\epsilon^2) \quad (\langle \Lambda|\Lambda\rangle = 1) \\
&= \exp\left[-i\epsilon g_{\mathrm{YM}} \int_{x_n}^{x_{n+1}} \langle \Lambda|\mathscr{A}^g|\Lambda\rangle\right] + O(\epsilon^2),
\end{aligned} \tag{7}$$

Here $\mathscr{A}^g(x)$ agrees with the gauge transformation of $\mathscr{A}(x)$ by the group element g:

$$\mathscr{A}^g(x) := g(x)^\dagger \mathscr{A}(x) g(x) + i g_{\rm YM}^{-1} g(x)^\dagger dg(x). \tag{8}$$

Defining the one-form A^g from the Lie algebra valued one-form $\mathscr{A}^g$ by

$$A^g := \langle \Lambda | \mathscr{A}^g | \Lambda \rangle, \tag{9}$$

we arrive at a path-integral representation of the Wilson loop operator:

$$W_C[\mathscr{A}] = \int [d\mu(g)]_C \exp\left(-i g_{\rm YM} \oint_C A^g \right), \quad [d\mu(g)]_C := \lim_{N\to\infty, \epsilon\to 0} \prod_{n=0}^{N-1} d\mu(g(x_n)), \tag{10}$$

The path-ordering has disappeared. (pre-NAST)
Therefore, we can apply the (usual) Stokes theorem to obtain a **non-Abelian Stokes theorem** (NAST):

$$W_C[\mathscr{A}] = \int [d\mu(g)]_\Sigma \exp\left[-i g_{\rm YM} \int_{\Sigma:\partial\Sigma = C} F^g \right], \quad F^g = dA^g. \tag{11}$$

Here we replaced the integration measure on the loop C by the integration measure on the surface Σ:

$$[d\mu(g)]_\Sigma := \prod_{x\in\Sigma:\partial\Sigma=C} d\mu(g(x)), \tag{12}$$

by inserting additional integral measures, $1 = \int d\mu(g(x))$ for $x \in \Sigma - C$. [Kondo (2008)]
The explicit expression for F^g will be obtained later.

§ Field decomposition a la Cho-Duan-Ge-Faddeev-Niemi

For the **highest-weight state** $|\Lambda\rangle = (\lambda_a)$ of a representation R of a group G, we define a matrix ρ with the matrix element ρ_{ab} by

$$\rho := |\Lambda\rangle \langle\Lambda| , \quad \rho_{ab} := |\Lambda\rangle_a \langle\Lambda|_b = \lambda_a \lambda_b^*. \tag{1}$$

Since $\langle\Lambda|\Lambda\rangle = \lambda_a \lambda_a^* = 1$, the trace of ρ has a unity:

$$\mathrm{tr}(\rho) = \rho_{aa} = 1, \tag{2}$$

Moreover, the matrix element $\langle\Lambda| \mathscr{O} |\Lambda\rangle$ of an arbitrary matrix $\mathcal{O}$ is written in the trace form:

$$\langle\Lambda| \mathscr{O} |\Lambda\rangle = \mathrm{tr}(\rho\mathscr{O}), \tag{3}$$

since $\langle\Lambda| \mathscr{O} |\Lambda\rangle = \lambda_b^* \mathscr{O}_{ba} \lambda_a = \rho_{ab} \mathscr{O}_{ba} = \mathrm{tr}(\rho\mathscr{O})$.

By using the operator $\rho := |\Lambda\rangle \langle\Lambda|$, the "Abelian" field A^g is written in the trace form of a matrix:

$$\begin{aligned} A^g(x) :=& \langle\Lambda|\mathscr{A}^g(x)|\Lambda\rangle \\ =& \mathrm{tr}\{\rho\mathscr{A}^g(x)\} = \mathrm{tr}\{g(x)\rho g^\dagger(x)\mathscr{A}(x)\} + ig_{\mathrm{YM}}^{-1}\mathrm{tr}\{\rho g^\dagger(x) dg(x)\}. \end{aligned} \tag{4}$$

By introducing the traceless field $\tilde{\boldsymbol{n}}(x)$ defined by [which we call **color (direction) field**]

$$\tilde{\boldsymbol{n}}(x) := g(x)\left[\rho - \frac{\mathbf{1}}{\mathrm{tr}(\mathbf{1})}\right] g^\dagger(x) = g(x)\rho g^\dagger(x) - \frac{\mathbf{1}}{\mathrm{tr}(\mathbf{1})}, \tag{5}$$

the "Abelian" field A^g is rewritten as

$$A^g_\mu(x) = \mathrm{tr}\{\tilde{\boldsymbol{n}}(x)\mathscr{A}_\mu(x)\} + ig^{-1}_{\mathrm{YM}}\mathrm{tr}\{\rho g^\dagger(x)\partial_\mu g(x)\}, \tag{6}$$

We proceed to perform the **decomposition of the Yang-Mills field** $\mathscr{A}_\mu(x)$ into two pieces:

$$\mathscr{A}_\mu(x) = \mathscr{V}_\mu(x) + \mathscr{X}_\mu(x). \tag{7}$$

We simply require that $\mathscr{X}_\mu(x)$ satisfies the condition:[**defining equation**]

$$\text{(ii)} \quad \mathscr{X}_\mu(x)\cdot\boldsymbol{n}(x) = 2\mathrm{tr}\{\mathscr{X}_\mu(x)\boldsymbol{n}(x)\} = 0. \tag{8}$$

Then $\mathscr{X}_\mu(x)$ disappears from the Wilson loop operator, since $A^g_\mu(x)$ is written without $\mathscr{X}_\mu(x)$:

$$A^g_\mu(x) = \mathrm{tr}\{\tilde{\boldsymbol{n}}(x)\mathscr{V}_\mu(x)\} + ig^{-1}_{\mathrm{YM}}\mathrm{tr}\{\rho g^\dagger(x)\partial_\mu g(x)\}. \tag{9}$$

Consequently, the Wilson loop operator $W_C[\mathscr{A}]$ can be reproduced by the **restricted field** variable $\mathscr{V}_\mu(x)$ alone. This is the **restricted field dominance** in the Wilson loop operator. For arbitrary loop C and any representation,

$$\text{(a)} \quad W_C[\mathscr{A}] = W_C[\mathscr{V}], \tag{10}$$

This does not necessarily imply $\langle W_C[\mathscr{A}]\rangle_{\mathrm{YM}} = \langle W_C[\mathscr{V}]\rangle_{\mathrm{YM}}$, which holds only when the cross term between $\mathscr{V}$ and $\mathscr{X}$ are neglected.

We look for the **gauge covariant decomposition**,

$$\mathscr{A}'_\mu(x) = \mathscr{V}'_\mu(x) + \mathscr{X}'_\mu(x). \tag{11}$$

For the condition (ii) of (8) to be gauge covariant,the transformation of the color field $\boldsymbol{n}$ given by

$$g(x) \to U(x)g(x) \Longrightarrow \boldsymbol{n}(x) \to \boldsymbol{n}'(x) = U(x)\boldsymbol{n}(x)U^\dagger(x). \tag{12}$$

requires that $\mathscr{X}_\mu(x)$ transforms like an adjoint matter field:

$$\mathscr{X}_\mu(x) \to \mathscr{X}'_\mu(x) = U(x)\mathscr{X}_\mu(x)U^\dagger(x), \tag{13}$$

This immediately means

$$\mathscr{V}_\mu(x) \to \mathscr{V}'_\mu(x) = U(x)\mathscr{V}_\mu(x)U^\dagger(x) + ig_{\mathrm{YM}}^{-1}U(x)\partial_\mu U^\dagger(x), \tag{14}$$

since $\mathscr{A}_\mu(x) \to \mathscr{A}'_\mu(x) = U(x)\mathscr{A}_\mu(x)U^\dagger(x) + ig_{\mathrm{YM}}^{-1}U(x)\partial_\mu U^\dagger(x)$.

These transformation properties impose restrictions on the requirement to be imposed on the restricted field $\mathscr{V}_\mu(x)$. Such a candidate is [**defining equation**]

$$\text{(I)} \quad \mathscr{D}_\mu[\mathscr{V}]\boldsymbol{n} = 0 \quad (\mathscr{D}_\mu[\mathscr{V}] := \partial_\mu - ig_{\mathrm{YM}}[\mathscr{V}_\mu, \cdot]), \tag{15}$$

since the covariant derivative transforms in the adjoint way: $\mathscr{D}_\mu[\mathscr{V}(x)] \to U(x)(\mathscr{D}_\mu[\mathscr{V}](x))U^\dagger(x)$.

For $G = SU(2)$, it is shown that the two conditions (8) and (15) [**defining equations** for the decomposition] are compatible and determine the decomposition uniquely.

$$\begin{aligned}\mathscr{A}_\mu(x) = \mathscr{V}_\mu(x) + \mathscr{X}_\mu(x), \\ \mathscr{V}_\mu(x) =& c_\mu(x)\boldsymbol{n}(x) + ig_{\rm YM}^{-1}[\boldsymbol{n}(x), \partial_\mu \boldsymbol{n}(x)], \quad c_\mu(x) := \mathscr{A}_\mu(x) \cdot \boldsymbol{n}(x), \\ \mathscr{X}_\mu(x) =& - ig_{\rm YM}^{-1}[\boldsymbol{n}(x), \mathscr{D}_\mu[\mathscr{A}]\boldsymbol{n}(x)]. \end{aligned} \tag{16}$$

This is the same as the **Cho–Duan-Ge–Faddeev-Niemi (CDGFN) decomposition**.
Cho(1980), Duan-Ge (1979), Faddeev-Niemi (1998)
The condition (I) means that the field strength $\mathscr{F}_{\mu\nu}^{[\mathscr{V}]}(x)$ of the field $\mathscr{V}_\mu(x)$ and $\boldsymbol{n}(x)$ commute:

$$[\mathscr{F}_{\mu\nu}^{[\mathscr{V}]}(x), \boldsymbol{n}(x)] = 0. \tag{17}$$

This follows from the identity:

$$[\mathscr{F}_{\mu\nu}^{[\mathscr{V}]}, \boldsymbol{n}] = ig_{\rm YM}^{-1}[\mathscr{D}_\mu^{[\mathscr{V}]}, \mathscr{D}_\nu^{[\mathscr{V}]}]\boldsymbol{n}, \tag{18}$$

which is derived using

$$\mathscr{F}_{\mu\nu}^{[\mathscr{V}]} = ig_{\rm YM}^{-1}[\mathscr{D}_\mu^{[\mathscr{V}]}, \mathscr{D}_\nu^{[\mathscr{V}]}], \quad \mathscr{D}_\mu^{[\mathscr{V}]} := \partial_\mu - ig_{\rm YM}[\mathscr{V}_\mu, \cdot]. \tag{19}$$

For $SU(2)$, (17) means that $\mathscr{F}_{\mu\nu}^{[\mathscr{V}]}(x)$ is proportional to $\boldsymbol{n}(x)$:

$$\mathscr{F}_{\mu\nu}^{[\mathscr{V}]}(x) = f_{\mu\nu}(x)\boldsymbol{n}(x) \Longrightarrow f_{\mu\nu}(x) = \boldsymbol{n}(x) \cdot \mathscr{F}_{\mu\nu}^{[\mathscr{V}]}(x) = 2\mathrm{tr}[\boldsymbol{n}(x)\mathscr{F}_{\mu\nu}^{[\mathscr{V}]}(x)], \tag{20}$$

§ Field decomposition: new options for $SU(N)$

For $G = SU(N)$ $(N \geq 3)$, (I) and (ii) are not sufficient to uniquely determine the decomposition. The condition (ii) [eq.(8)] must be modified:
(II) $\mathscr{X}^\mu(x)$ does not have the $\tilde{H}$-commutative part, i.e., $\mathscr{X}^\mu(x)_{\tilde{H}} = 0$:

$$\text{(II)} \quad 0 = \mathscr{X}^\mu(x)_{\tilde{H}} := \mathscr{X}^\mu(x) - \frac{2(N-1)}{N}[\boldsymbol{n}(x), [\boldsymbol{n}(x), \mathscr{X}^\mu(x)]]$$

$$\Longleftrightarrow \mathscr{X}^\mu(x) = \frac{2(N-1)}{N}[\boldsymbol{n}(x), [\boldsymbol{n}(x), \mathscr{X}^\mu(x)]]. \tag{1}$$

This condition is also gauge covariant. Note that the condition (ii)[eq.(8)] follows from (II)[eq.(1)]. For $G = SU(2)$, i.e., $N = 2$, the condition (II)[eq.(1)] reduces to (ii)[eq.(8)].

By solving (I)[eq.(15)] and (II)[eq.(1)], $\mathscr{X}_\mu(x)$ is determined as

$$\mathscr{X}_\mu(x) = -ig_{\rm YM}^{-1}\frac{2(N-1)}{N}[\boldsymbol{n}(x), \mathscr{D}_\mu[\mathscr{A}]\boldsymbol{n}(x)] \in Lie(G/\tilde{H}). \tag{2}$$

$$\begin{aligned} \mathscr{V}_\mu(x) =& \mathscr{C}_\mu(x) + \mathscr{B}_\mu(x) \in \mathscr{L}ie(G), \\ & \mathscr{C}_\mu(x) = \mathscr{A}_\mu(x) - \frac{2(N-1)}{N}[\boldsymbol{n}(x), [\boldsymbol{n}(x), \mathscr{A}_\mu(x)]] \in \mathscr{L}ie(\tilde{H}), \\ & \mathscr{B}_\mu(x) = ig_{\rm YM}^{-1}\frac{2(N-1)}{N}[\boldsymbol{n}(x), \partial_\mu \boldsymbol{n}(x)] \in \mathscr{L}ie(G/\tilde{H}). \end{aligned} \tag{3}$$

§ Non-Abelian Stokes theorem (2)

Finally, we can show that the field strength $F^g_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$ in NAST is cast into the form:

$$F^g_{\mu\nu}(x) = \sqrt{\frac{2(N-1)}{N}}\mathrm{tr}\{\boldsymbol{n}(x)\mathscr{F}_{\mu\nu}[\mathscr{V}](x)\} + ig_{\mathrm{YM}}^{-1}\mathrm{tr}\{\rho g^\dagger(x)[\partial_\mu, \partial_\nu]g(x)\}. \tag{1}$$

$$\begin{aligned}\mathrm{tr}\{\boldsymbol{n}(x)\mathscr{F}_{\mu\nu}[\mathscr{V}](x)\} =& \partial_\mu \mathrm{tr}\{\boldsymbol{n}(x)\mathscr{V}_\nu(x)\} - \partial_\nu \mathrm{tr}\{\boldsymbol{n}(x)\mathscr{V}_\mu(x)\} \\ &+ \frac{2(N-1)}{N} ig_{\mathrm{YM}}^{-1}\mathrm{tr}\{\boldsymbol{n}(x)[\partial_\mu \boldsymbol{n}(x), \partial_\nu \boldsymbol{n}(x)]\}.\end{aligned} \tag{2}$$

where the normalized and traceless field $\boldsymbol{n}(x)$ defined by

$$\boldsymbol{n}(x) = \sqrt{\frac{N}{2(N-1)}} g(x) \left[\rho - \frac{\mathbf{1}}{\mathrm{tr}(\mathbf{1})}\right] g^\dagger(x), \quad g(x) \in G. \tag{3}$$

Thus the Wilson loop operator can be rewritten in terms of new variables:

$$W_C[\mathscr{A}] = \int [d\mu(g)]_\Sigma \exp\left[- ig_{\mathrm{YM}}\frac{1}{2}\sqrt{\frac{2(N-1)}{N}} \int_{\Sigma:\partial\Sigma=C} 2\mathrm{tr}\{\boldsymbol{n}\mathscr{F}[\mathscr{V}]\}\right], \tag{4}$$

Incidentally, the last part $ig_{\mathrm{YM}}^{-1}\mathrm{tr}\{\rho g(x)^\dagger[\partial_\mu, \partial_\nu]g(x)\}$ in $F^g_{\mu\nu}(x)$ corresponds to the **Dirac string**.This term is not gauge invariant and does not contribute to the Wilson loop operator in the end, after the group integration $d\mu(g)$.

In this way we obtain another expression of the NAST for the Wilson loop operator in the **fundamental representation** for $SU(N)$:

$$W_C[\mathscr{A}] = \int [d\mu(g)] \exp \left\{ -ig_{\rm YM} \frac{1}{2} \sqrt{\frac{2(N-1)}{N}} [(N_{\Sigma_C}, j) + (\Xi_{\Sigma_C}, k)] \right\}, \quad (5)$$

where we have defined the $(D-3)$-form k and one-form j by

$$k := \delta^* f, \quad j := \delta f, \quad f := 2{\rm tr}\{\boldsymbol{n}\mathscr{F}[\mathscr{V}]\}, \quad (6)$$

and we have defined the $(D-3)$-form Ξ_{Σ_C} and one-form N_{Σ_C} by (Ξ_{Σ_C} is the D-dim. solid angle)

$$\Xi_{\Sigma_C} := {}^* d\Delta^{-1}\Theta_{\Sigma_C} = \delta\Delta^{-1*}\Theta_{\Sigma_C}, \quad N_{\Sigma_C} := \delta\Delta^{-1}\Theta_{\Sigma_C}, \quad (7)$$

with the inner product for two forms defined by

$$(\Xi_{\Sigma_C}, k) = \frac{1}{(D-3)!} \int d^D x k^{\mu_1 \cdots \mu_{D-3}}(x) \Xi_{\Sigma_C}^{\mu_1 \cdots \mu_{D-3}}(x), \quad (N_{\Sigma_C}, j) = \int d^D x j^\mu(x) N_{\Sigma_C}^\mu(x). \quad (8)$$

The Wilson loop operator can be expressed by the electric current j and the monopole current k. The magnetic monopole described by the current k is a topological object of **co-dimension** 3:

- $D=3$: 0-dimensional point defect $\to$ point-like magnetic monopole (cf. Wu-Yang type)
- $D=4$: 1-dimensional line defect $\to$ magnetic monopole loop (closed loop)

⊙ **SU(2) case**: $f_{\mu\nu} := 2\mathrm{tr}\{\boldsymbol{n}\mathscr{F}_{\mu\nu}[\mathscr{V}]\}$

The gauge-invariant magnetic-monopole current $(D-3)$-form k is obtained

$$k = \delta^* f,\ \ f_{\mu\nu} = \partial_\mu 2\mathrm{tr}\{\boldsymbol{n}\mathscr{A}_\nu\} - \partial_\nu 2\mathrm{tr}\{\boldsymbol{n}\mathscr{A}_\mu\} + i g_{\mathrm{YM}}^{-1} 2\mathrm{tr}\{\boldsymbol{n}[\partial_\mu \boldsymbol{n}, \partial_\nu \boldsymbol{n}]\}. \tag{9}$$

For the fundamental representation of $SU(2)$, the highest-weight state $|\Lambda\rangle$ yields

$$|\Lambda\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Longrightarrow \rho := |\Lambda\rangle\langle\Lambda| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Longrightarrow \rho - \frac{1}{2}\mathbf{1} = \frac{\sigma_3}{2}, \tag{10}$$

$$\Longrightarrow \boldsymbol{n}(x) = g(x)\frac{\sigma_3}{2}g(x)^\dagger \in SU(2)/U(1) \simeq S^2 \simeq P^1(\mathbb{C}). \tag{11}$$

Magnetic charge obeys the quantization condition a la Dirac:

$$q_m := \int d^3x k^0 = 4\pi g_{\mathrm{YM}}^{-1} \ell,\ \ \ell \in \mathbb{Z}. \tag{12}$$

This is suggested from a nontrivial Homotopy group of the map $\boldsymbol{n} : S^2 \to SU(2)/U(1)$

$$\pi_2(SU(2)/U(1)) = \pi_1(U(1)) = \mathbb{Z}. \tag{13}$$

cf. the Abelian magnetic monopole due to 't Hooft-Polyakov associated with the spontaneous breaking $G = SU(2) \to H = U(1)$:

$$\boldsymbol{n}^A \leftrightarrow \hat{\phi}^A(x)/|\hat{\phi}(x)|. \tag{14}$$

⊙ **SU(3) case**: Then the gauge–invariant magnetic–monopole current $(D-3)$-form k is given by

$$k = \delta^* f, \quad f_{\mu\nu} := \partial_\mu 2\mathrm{tr}\{\boldsymbol{n}\mathscr{A}_\nu\} - \partial_\nu 2\mathrm{tr}\{\boldsymbol{n}\mathscr{A}_\mu\} + \frac{4}{3} i g_{\mathrm{YM}}^{-1} 2\mathrm{tr}\{\boldsymbol{n}[\partial_\mu \boldsymbol{n}, \partial_\nu \boldsymbol{n}]\}. \tag{15}$$

For the fundamental rep. of $SU(3)$, the highest-weight state $|\Lambda\rangle$ yields

$$|\Lambda\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \rho := |\Lambda\rangle\langle\Lambda| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Longrightarrow \rho - \frac{1}{3}\mathbf{1} = \frac{-1}{3}\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{16}$$

$$\Longrightarrow \boldsymbol{n}(x) = g(x) \frac{-1}{2\sqrt{3}} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} g(x)^\dagger \in SU(3)/U(2) \simeq P^2(\mathbb{C}). \tag{17}$$

The matrix $\mathrm{diag.}(-2, 1, 1)$ is degenerate. Using the Weyl symmetry (discrete global or color symmetry), it is changed into λ_8.
This is a **non-Abelian magnetic monopole**, which corresponds to $SU(3) \to U(2)$.
The magnetic charge obeys the quantization condition:

$$q'_m := \int d^3x k^0 = 2\pi\sqrt{3} g_{\mathrm{YM}}^{-1} n', \; n' \in \mathbb{Z}. \tag{18}$$

Homotopy class of the map $\boldsymbol{n} : S^2 \to SU(3)/U(2)$

$$\pi_2(SU(3)/[SU(2)\times U(1)]) = \pi_1(SU(2)\times U(1)) = \pi_1(U(1)) = \mathbb{Z}. \tag{19}$$

For a **reference state** $|\Lambda\rangle$ of a given representation of a Lie group G, the **maximal stability subgroup** $\tilde{H}$ is a subgroup leaving $|\Lambda\rangle$ invariant (up to a phase):

$$h \in \tilde{H} \Longleftrightarrow h|\Lambda\rangle = |\Lambda\rangle e^{i\phi(h)}, \tag{20}$$

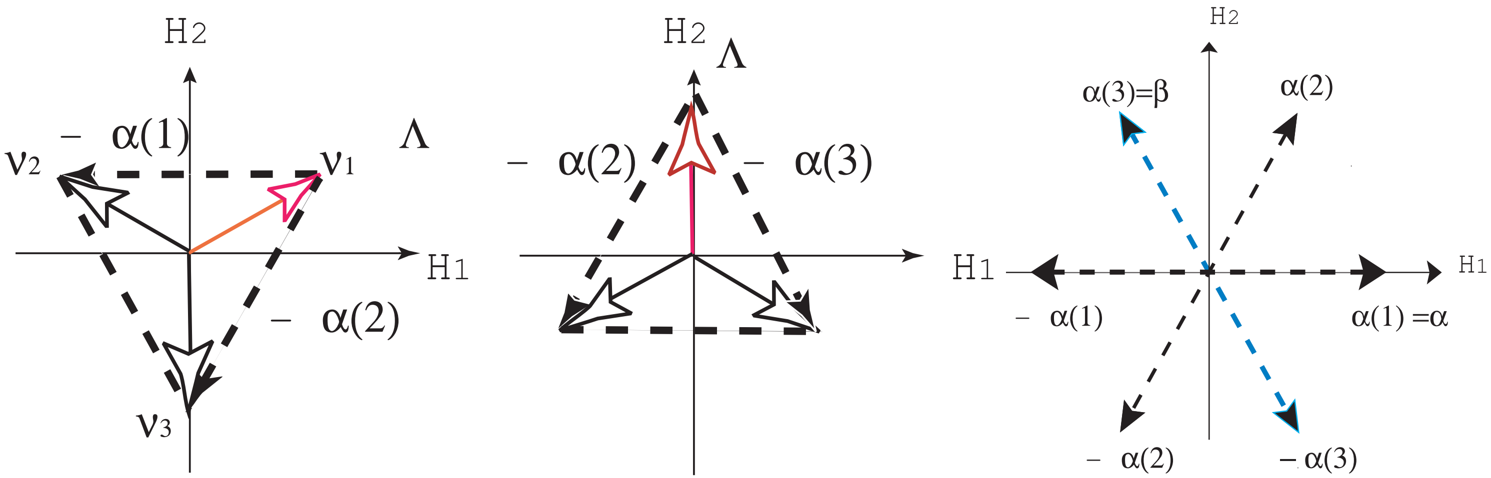
$$g = \xi h \in G, \quad \xi \in G/\tilde{H}, \quad h \in \tilde{H}. \tag{21}$$

$$|g,\Lambda\rangle := g|\Lambda\rangle = \xi h|\Lambda\rangle = \xi|\Lambda\rangle e^{i\phi(h)} = |\xi,\Lambda\rangle e^{i\phi(h)}. \tag{22}$$

Every representation R of $SU(3)$ specified by the Dynkin index [m,n] belongs to (I) or (II):

(I) [Maximal] $m \neq 0$ and $n \neq 0 \Longrightarrow \tilde{H} = H = U(1) \times U(1)$. maximal torus
e.g., adjoint rep.[1,1], $\{H_1, H_2\} \in u(1) + u(1)$,

(II) [Minimal] $m = 0$ or $n = 0 \Longrightarrow \tilde{H} = U(2)$.
when the weight vector Λ is orthogonal to some of the root vectors,
e.g., fundamental rep. [1,0], $\{H_1, H_2, E_\beta, E_{-\beta}\} \in u(2)$, where $\Lambda \perp \beta, -\beta$.



§ Reformulating Yang-Mills theory using new variables

The change of variables from $\mathscr{A}_\mu$ to new field variables $\mathscr{C}_\mu$, $\mathscr{X}_\mu$ and $\boldsymbol{n}$

$$\mathscr{A}_\mu^A \Longrightarrow (\boldsymbol{n}^\beta, \mathscr{C}_\mu^k, \mathscr{X}_\mu^b), \tag{1}$$

- $\mathscr{A}_\mu \in Lie(G) \to \#[\mathscr{A}_\mu^A] = D \cdot \mathrm{dim} G = D(N^2 - 1)$
- $\mathscr{C}_\mu \in Lie(\tilde{H}) = u(N-1) \to \#[\mathscr{C}_\mu^k] = D \cdot \mathrm{dim}\tilde{H} = D(N-1)^2$
- $\mathscr{X}_\mu \in Lie(G/\tilde{H}) \to \#[\mathscr{X}_\mu^b] = D \cdot \mathrm{dim}(G/\tilde{H}) = D(2N-2)$
- $\boldsymbol{n} \in Lie(G/\tilde{H}) \to \#[\boldsymbol{n}^\beta] = \mathrm{dim}(G/\tilde{H}) = 2(N-1)$.

The new theory written in terms of new variables $(\boldsymbol{n}^\beta, \mathscr{C}_\mu^k, \mathscr{X}_\mu^b)$ has the $2(N-1)$ extra degrees of freedom. Therefore, we must give a procedure for eliminating the $2(N-1)$ extra degrees of freedom to obtain the new theory equipollent to the original one. For this purpose, we impose $2(N-1)$ constraints $\boldsymbol{\chi} = 0$, which we call the **reduction condition**:

- $\boldsymbol{\chi} \in Lie(G/\tilde{H}) \to \#[\boldsymbol{\chi}^a] = \mathrm{dim}(G/\tilde{H}) = 2(N-1) = \#[\boldsymbol{n}^\beta]$.

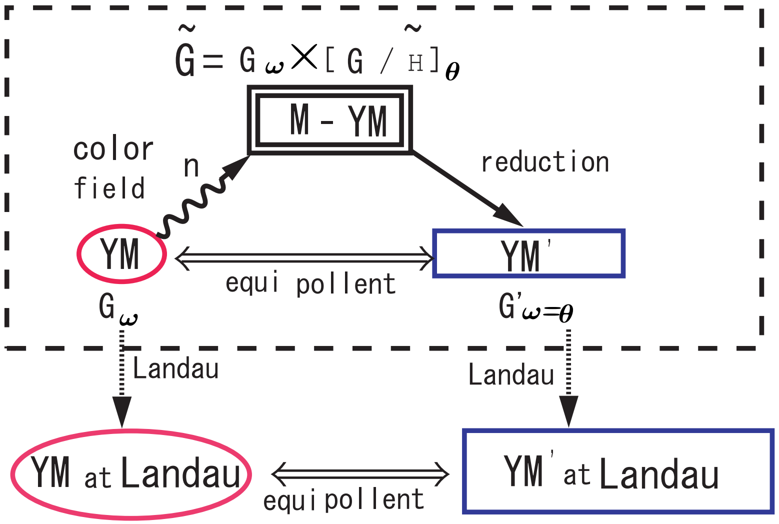


Figure 1: The relationship between the original Yang-Mills (YM) theory and the reformulated Yang-Mills (YM') theory. A single color field $\boldsymbol{n}$ is introduced to enlarge the original Yang-Mills theory with a gauge group G into the master Yang-Mills (M-YM) theory with the enlarged gauge symmetry $\tilde{G} = G \times G/\tilde{H}$. The reduction conditions are imposed to reduce the master Yang-Mills theory to the reformulated Yang-Mills theory with the equipollent gauge symmetry G'. In addition, we can impose any over-all gauge fixing condition, e.g., Landau gauge to both the original YM theory and the reformulated YM' theory.

- Enlarged gauge symmetry by introducing n and the reduction by imposing χ

$$G \overset{\boldsymbol{n}}{\to} G \times G/\tilde{H} \overset{\chi}{\to} G. \tag{2}$$

A reduction condition in the minimal option is to minimize the functional $F_{\rm red}[\mathscr{A},\boldsymbol{n}]$

$$\int d^D x \frac{1}{2} g^2 \mathscr{X}_\mu \cdot \mathscr{X}^\mu = \frac{2(N-1)^2}{N^2} \int d^D x (\boldsymbol{n} \times D_\mu[\mathscr{A}]\boldsymbol{n})^2 = \frac{N-1}{N} \int d^D x (D_\mu[\mathscr{A}]\boldsymbol{n})^2,$$

with respect to the enlarged gauge transformation:

$$\begin{aligned} \delta \mathscr{A}_\mu &= D_\mu[\mathscr{A}]\boldsymbol{\omega} \quad (\boldsymbol{\omega} \in \mathscr{L}ie(G) \\ \delta \boldsymbol{n} &= ig[\boldsymbol{n},\boldsymbol{\theta}] = ig[\boldsymbol{n},\boldsymbol{\theta}_\perp] \quad (\boldsymbol{\theta}_\perp \in \mathscr{L}ie(G/\tilde{H})) \end{aligned} \tag{3}$$

In fact, the enlarged gauge transformation of the functional $F_{\rm red}[\mathscr{A},\boldsymbol{n}]$ is

$$\delta F_{\rm red}[\mathscr{A},\boldsymbol{n}] = \delta \int d^D x \frac{1}{2} (D_\mu[\mathscr{A}]\boldsymbol{n})^2 = g \int d^D x (\boldsymbol{\theta}_\perp - \boldsymbol{\omega}_\perp) \cdot i[\boldsymbol{n}, D^\mu[\mathscr{A}]D_\mu[\mathscr{A}]\boldsymbol{n}], \tag{4}$$

where $\boldsymbol{\omega}_\perp$ denotes the component of $\boldsymbol{\omega}$ in the direction $\mathscr{L}(G/\tilde{H})$.
For $\boldsymbol{\omega}_\perp = \boldsymbol{\theta}_\perp$ (diagonal part of $G \times G/\tilde{H}$) $\delta F_{\rm red}[\mathscr{A},\boldsymbol{n}] = 0$ imposes no condition, while for $\boldsymbol{\omega}_\perp \neq \boldsymbol{\theta}_\perp$ (off-diagonal part of $G \times G/\tilde{H}$) it implies the constraint

$$\boldsymbol{\chi}[\mathscr{A},\boldsymbol{n}] := [\boldsymbol{n}, D^\mu[\mathscr{A}]D_\mu[\mathscr{A}]\boldsymbol{n}] \equiv 0, \tag{5}$$

The number of constraint is $\#[\boldsymbol{\chi}] = \dim(G \times G/\tilde{H}) - \dim(G) = \dim(G/\tilde{H})$ as desired.
Finally, we have an equipollent Yang-Mills theory with **the residual local gauge symmetry** $G' := SU(N)^{\rm local}_{\omega'}$ with the gauge transformation parameter:

$$\boldsymbol{\omega}'(x) = (\boldsymbol{\omega}_\parallel(x), \boldsymbol{\omega}_\perp(x)) = (\boldsymbol{\omega}_\parallel(x), \boldsymbol{\theta}_\perp(x)), \quad \boldsymbol{\omega}_\perp(x) = \boldsymbol{\theta}_\perp(x), \tag{6}$$

	original YM	$\Longrightarrow$ reformulated YM
field variables	$\mathscr{A}_\mu^A \in \mathscr{L}(G)$	$\Longrightarrow \boldsymbol{n}^\beta, \mathscr{C}_\nu^k, \mathscr{X}_\nu^b$
action	$S_{\rm YM}[\mathscr{A}]$	$\Longrightarrow \tilde{S}_{\rm YM}[\boldsymbol{n}, \mathscr{C}, \mathscr{X}]$
integration measure	$\mathcal{D}\mathscr{A}_\mu^A$	$\Longrightarrow \mathcal{D}n^\beta \mathcal{D}\mathscr{C}_\nu^k \mathcal{D}\mathscr{X}_\nu^b \tilde{J}\delta(\tilde{\boldsymbol{\chi}})\Delta_{\rm FP}^{\rm red}[\boldsymbol{n}, c, \mathscr{X}]$

At the same time, the color field

$$\boldsymbol{n}(x) \in \mathscr{L}ie(G/\tilde{H})$$

must be obtained by solving the reduction condition $\boldsymbol{\chi} = 0$ for a given $\mathscr{A}$, e.g.,

$$\boldsymbol{\chi}[\mathscr{A}, \boldsymbol{n}] := [\boldsymbol{n}, D^\mu[\mathscr{A}]D_\mu[\mathscr{A}]\boldsymbol{n}] \in \mathscr{L}ie(G/\tilde{H}). \tag{7}$$

Here $\tilde{\boldsymbol{\chi}} = 0$ is the reduction condition written in terms of the new variables:

$$\tilde{\boldsymbol{\chi}} := \tilde{\boldsymbol{\chi}}[\boldsymbol{n}, \mathscr{C}, \mathscr{X}] := D^\mu[\mathscr{V}]\mathscr{X}_\mu, \tag{8}$$

and $\Delta_{\rm FP}^{\rm red}$ is the Faddeev-Popov determinant associated with the reduction condition:

$$\Delta_{\rm FP}^{\rm red} := \det\left(\frac{\delta\boldsymbol{\chi}}{\delta\boldsymbol{\theta}}\right)_{\boldsymbol{\chi}=0} = \det\left(\frac{\delta\boldsymbol{\chi}}{\delta\boldsymbol{n}^\theta}\right)_{\boldsymbol{\chi}=0}. \tag{9}$$

which is obtained by the BRST method as $\Delta_{\rm FP}^{\rm red}[\boldsymbol{n}, c, \mathscr{X}] = \det\{-D_\mu[\mathscr{V}+\mathscr{X}]D_\mu[\mathscr{V}-\mathscr{X}]\}$.
The Jacobian $\tilde{J}$ is very simple, irrespective of the choice of the reduction condition:

$$\tilde{J} = 1. \tag{10}$$

[Kondo, Shinohara & Murakami, Prog.Theor.Phys. **120**, 1–50 (2008). arXiv:0803.0176]

The Wilson loop average in the original theory:

$$W(C) := \langle W_C[\mathscr{A}] \rangle_{\rm YM} = Z_{\rm YM}^{-1} \int \mathcal{D}\mathscr{A} e^{-S_{\rm YM}[\mathscr{A}]} W_C[\mathscr{A}]. \tag{11}$$

is defined in the reformulated Yang-Mills theory:

$$\begin{aligned}\langle W_C[\mathscr{A}] \rangle_{\rm YM'} =& Z_{\rm YM'}^{-1} \int [d\mu(g)] \int \mathcal{D}n^\beta \mathcal{D}\mathscr{C}_\nu^k \mathcal{D}\mathscr{X}_\nu^b \tilde{J} \delta(\tilde{\boldsymbol{\chi}}) \Delta_{\rm FP}^{\rm red} e^{-\tilde{S}_{\rm YM}[\boldsymbol{n},\mathscr{C},\mathscr{X}]} \\ & \times \exp \left\{ i g_{\rm YM} \sqrt{\frac{2(N-1)}{N}} [(j, N_\Sigma) + (k, \Xi_\Sigma)] \right\}, \\ Z_{\rm YM'} =& \int \mathcal{D}n^\beta \mathcal{D}\mathscr{C}_\nu^k \mathcal{D}\mathscr{X}_\nu^b \tilde{J} \delta(\tilde{\boldsymbol{\chi}}) \Delta_{\rm FP}^{\rm red} e^{-S_{\rm YM'}[\boldsymbol{n},\mathscr{C},\mathscr{X}]}.\end{aligned} \tag{12}$$

Remark: For $SU(2)$, when we fix the color field $\mathbf{n}(x) = (0,0,1)$ or $\boldsymbol{n}(x) = \sigma_3/2$, the reduction condition $D^\mu[\mathscr{V}]\mathscr{X}_\mu = 0$ reduces to the conventional Maximally Abelian gauge.
For $SU(3)$, this is not the case: This reduction does not reduce to the conventional Maximally Abelian gauge for $SU(3)$, even if the color field is fixed to be uniform. Therefore, the results to be obtained are nontrivial.

§ Conclusion and discussion

We have combined a non-Abelian Stokes theorem and reformulations of the Yang-Mills theory.
1) In $SU(2)$ Yang-Mills theory without adjoint scalar fields, we do not need to use the Abelian projection ['t Hooft,1981] to define magnetic monopoles as gauge-fixing defects. In fact, a gauge-invariant magnetic monopole can be defined as proposed by [Cho, 1980] and [Duan & Ge, 1979] independently. For any representation of $G = SU(2)$, Abelian magnetic monopole described by $\boldsymbol{n} \in SU(2)/U(1) = P^1(\mathbb{C})$

2) In $SU(N)$ Yang-Mills theory without adjoint scalar fields, we have defined a gauge-invariant magnetic monopole k inherent in the Wilson loop operator by using a non-Abelian Stokes theorem for the Wilson loop operator.
The type of magnetic monopole depends on the representation of quarks defining the Wilson loop, related to the target space of the color field $\boldsymbol{n}(x)$:

- For quarks in the fundamental representation, $\tilde{H} = U(2)$

  $G = SU(3)$ a non-Abelian magnetic monopole $\boldsymbol{n}(x) \in SU(3)/U(2) = P^2(\mathbb{C})$

- For quarks in the adjoint representation, $\tilde{H} = H = U(1) \times U(1)$

  $G = SU(3)$ two Abelian magnetic monopoles $\boldsymbol{n}(x) \in SU(3)/[U(1) \times U(1)] = F_2$

3) We have constructed a new reformulation of Yang-Mills theory using new field variables, which gives an optimal description of the magnetic monopole defined through the Wilson loop operator. The idea of using new variables is originally due to Cho, and Faddeev & Niemi, where $N-1$ color fields $\boldsymbol{n}_{(j)}$ $(j=1,...,N-1)$ are introduced. However, our reformulation in the minimal option is new for $SU(N), N \geq 3$: we introduce only a single color field $\boldsymbol{n}$ for any N, which is enough for reformulating the quantum Yang-Mills theory to describe confinement of the fundamental quark. The reformulation allows options discriminated by the maximal stability group $\tilde{H}$.

For $G = SU(3)$, two options are possible:

• The maximal option with $\tilde{H} = H = U(1) \times U(1)$, the new theory reduces to a manifestly gauge-independent reformulation of the conventional Abelian projection in the maximal Abelian gauge. [Cho,1980] and [Faddeev & Niemi,1999]
• The minimal option with $\tilde{H} = U(2)$ gives an optimized description of quark confinement through the Wilson loop in the fundamental representation. [Kondo, Shinohara and Murakami, 2008]

4) By constructing a lattice version of the reformulation of the $SU(N)$ Yang-Mills theory and performing numerical simulations on a lattice, [Talk by Akihiro Shibata]

4a) For $SU(3)$, we have confirmed the infrared dominance of the restricted variables $\mathscr{V}$ and the non-Abelian magnetic monopole dominance for quark confinement (in the string tension),
cf. [infrared Abelian dominance and Abelian magnetic monopole dominance in MA gauge]

4b) We have shown the numreical evidences of the dual Meissner effect caused by non-Abelian magnetic monopoles in $SU(3)$ Yang-Mills theory: simultaneous formation of the tube-shaped flux of the chromo-electric field originating from the restricted field (including the non-Abelian magnetic monopoles) and the associated magnetic current induced around the flux tube.

To confirm the non-Abelian dual superconductivity picture in $SU(3)$ Yang-Mills theory, we plan to do further checks, e.g., determination of the type of dual superconductor, measurement of the penetrating depth, induced magnetic current around color flux due to magnetic monopole condensations, and so on.

Applications of the reformulation to other topics:

• relationship between magnetic monopoles and instantons or merons. SU(2) case
Kondo, Fukui, Shibata & Shinohara, Phys.Rev.D78, 065033 (2008). arXiv:0806.3913[hep-th], dimeron
Fukui, Kondo, Shibata & Shinohara, Phys.Rev.D82, 045015 (2010). arXiv:1005.3157[hep-th], 2-instanton

• Extension to finite temperature case: SU(2) case
K.-I. Kondo, Phys.Rev.D82, 065024 (2010). arXiv:1005.0314 [hep-th],

• Green functions for quark and gluon confinement: SU(2) case
K.-I. Kondo, Phys.Rev.D84, 061702 (2011). arXiv:1103.3829 [hep-th],

• relationship between magnetic monopoles and vortex.
We can define a gauge-invariant vortex which ends on the non-Abelian magnetic monopole.
K.-I. Kondo, J. Phys. G: Nucl. Part. Phys. **35**, 085001 (2008). arXiv:0802.3829 [hep-th],

• Skyrme-Fadeev-Niemi model as an low-energy effective theory,
L.A. Ferreira, P. Klimas & W.J. Zakrzewski, arXiv:1111.2338 [hep-th] , JHEP 1112, 098 (2011).
K.-I. Kondo, A. Ono, A. Shibata, T. Shinohara and T. Murakami, J. Phys. A: Math. Gen. **39**, 13767–13782 (2006). [hep-th/0604006],

Questions:

• dual gauge symmetry, spontaneous symmetry breaking, dual Meissner effect,

• Large N analysis

• Casimir scaling

Thank you very much for your attention!

For a given $\boldsymbol{n} \in Lie(G/\tilde{H})$, any $su(N)$ Lie algebra valued function $\mathscr{F}(x)$ is decomposed into the $\tilde{H}$-commutative (or parallel) part $\mathscr{F}_{\tilde{H}}$ and the remaining (or orthogonal) part $\mathscr{F}_{G/\tilde{H}}$ as

$$\mathscr{F} = \mathscr{F}_{\tilde{H}} + \mathscr{F}_{G/\tilde{H}} \in su(N), \quad [\boldsymbol{n}, \mathscr{F}_{\tilde{H}}] = 0, \quad \boldsymbol{n} \cdot \mathscr{F}_{G/\tilde{H}} := 2\mathrm{tr}(\boldsymbol{n}\mathscr{F}_{G/\tilde{H}}) = 0 \tag{1}$$

where

$$\mathscr{F}_{G/\tilde{H}} = \frac{2(N-1)}{N}[\boldsymbol{n}, [\boldsymbol{n}, \mathscr{F}]] \in su(N) - u(N-1), \tag{2}$$

while

$$\mathscr{F}_{\tilde{H}} = \mathscr{F} - \mathscr{F}_{G/\tilde{H}} = \mathscr{F} - \frac{2(N-1)}{N}[\boldsymbol{n}, [\boldsymbol{n}, \mathscr{F}]] \in u(N-1). \tag{3}$$

$\mathscr{F}_{\tilde{H}}$ is written as

$$\mathscr{F}_{\tilde{H}} = \boldsymbol{n}(\boldsymbol{n} \cdot \mathscr{F}) + \bar{\mathscr{F}} \in u(N-1), \tag{4}$$

Here $\bar{\mathscr{F}}$ is defined by a subset of generators $\boldsymbol{u}^k \in su(N-1)$ such that $[\boldsymbol{n}, \boldsymbol{u}^k] = 0$ $(k = 1, \dots, (N-1)^2 - 1)$ and $[\boldsymbol{n}, \bar{\mathscr{F}}] = 0$. In particular, we can choose

$$\boldsymbol{n}(x) := g(x) H_r g^\dagger(x) \in G/\tilde{H} = SU(3)/U(2), \; H_r = \frac{1}{\sqrt{2N(N-1)}}\mathrm{diag}(1, \cdots, 1, -N+1),$$

For $SU(3)$, $u^1, u^2, u^3 \to \lambda_1, \lambda_2, \lambda_3$, and $H_r \to \lambda_8$. (5)

The stability group $U(2) = SU(2)_{1,2,3} \times U(1)_8$

magnetic charge $\pi_1(U(2)) = \pi_1(SU(2)_{1,2,3} \times U(1)_8) = \pi_1(U(1)_8) = \mathbb{Z}$

$$G = SU(2), \tilde{H} = U(1) = H,$$

$$G = SU(3), \tilde{H} = U(1) \times U(1), \quad U(2),$$

$$G = SU(4), \tilde{H} = U(1) \times U(1) \times U(1), \quad U(1) \times U(2), \quad SU(2) \times U(2), \quad U(3),$$

$$G = SU(N), \tilde{H} \subset H = U(1)^{N-1}, ..., U(N-1), \tag{6}$$

The target space of the color field is specified by the maximal stability group $\tilde{H}$:

$$\boldsymbol{n}(x) = g(x)\mathrm{diag.}(\lambda_1, \lambda_2, \lambda_3)g(x)^\dagger \in G/\tilde{H}, \tag{7}$$

The gauge-invariant magnetic monopoles inherent in the SU(3) Wilson loop operator for the fundamental rep. are non-Abelian $U(2)$ magnetic monopole in the sense of Goddard–Nuyts–Olive–Weinberg.

c.f. Abelian projection method=the partial gauge fixing from an original gauge group G to the maximal torus subgroup H:

$$G = SU(3) \to H = U(1) \times U(1) \tag{8}$$

$$\pi_2(SU(3)/[U(1) \times U(1)]) = \pi_1(U(1) \times U(1)) = \mathbb{Z}^2. \tag{9}$$

$\Longrightarrow$ two kinds of Abelian U(1) magnetic monopoles for any rep.
No such difference for SU(2). For any rep. of $SU(2)$, magnetic monopole is U(1), since $\tilde{H} = H = U(1)$.

For $SU(2)$, if we choose a special gauge, unitary-like gauge, in which the color field is uniform:

$$\boldsymbol{n}(x) = (n_1(x), n_2(x), n_3(x)) = \boldsymbol{n}_0, \quad \boldsymbol{n}_0 := (0,0,1), \tag{10}$$

then the Wilson loop operator reduces to the "Abelian-projected" form:

$$W_C[\mathscr{A}] = \exp\left[ig_{\rm YM}\int_{\Sigma:\partial\Sigma=C} F\right] = \exp\left[ig_{\rm YM}\int_{\Sigma:\partial\Sigma=C}\frac{1}{2}f\right], f_{\mu\nu} = \partial_\mu \mathscr{A}_\nu^3 - \partial_\nu \mathscr{A}_\mu^3 \tag{11}$$

For the gauge group $SU(2)$, in particular, arbitrary representation is characterized by a single index $J = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \cdots$. The $SU(2)$ Wilson loop operator in the representation J obeys

$$\begin{aligned} W_C[\mathscr{A}] =& \int [d\mu(g)]_\Sigma \exp\left\{-ig_{\rm YM}J\int_{\Sigma:\partial\Sigma=C} dS^{\mu\nu}f_{\mu\nu}^g\right\}, \\ f_{\mu\nu}^g(x) =& \partial_\mu[n^A(x)\mathscr{A}_\nu^A(x)] - \partial_\nu[n^A(x)\mathscr{A}_\mu^A(x)] \\ & - g_{\rm YM}^{-1}\epsilon^{ABC}n^A(x)\partial_\mu n^B(x)\partial_\nu n^C(x), \\ n^A(x)\sigma^A =& g(x)\sigma^3 g^\dagger(x),\ g(x) \in SU(2)\ (A,B,C \in \{1,2,3\}), \end{aligned} \tag{12}$$

and $[d\mu(g)]_\Sigma$ is the product measure of an invariant measure on $SU(2)/U(1)$ over Σ:

$$[d\mu(g)]_\Sigma := \prod_{x\in\Sigma} d\mu(\boldsymbol{n}(x)),\ d\mu(\boldsymbol{n}(x)) = \frac{2J+1}{4\pi}\delta(\boldsymbol{n}^A(x)\boldsymbol{n}^A(x)-1)d^3\boldsymbol{n}(x). \tag{13}$$

This is the Diakonov-Petrov version of the $SU(2)$ non-Abelian Stokes theorem.

We consider quark confinement based on the Wilson criterion.

Wilson loop operator $W_C[\mathscr{A}]$ for the non-Abelian Yang-Mills field $\mathscr{A}_\mu(x)$

$$W_C[\mathscr{A}] := \mathrm{tr}\left\{ \mathscr{P} \exp\left[ig \oint_C dx^\mu \mathscr{A}_\mu(x) \right] \right\} / \mathrm{tr}(\mathbf{1}), \quad \mathscr{A}_\mu(x) = \mathscr{A}_\mu^A(x) T_A. \tag{14}$$

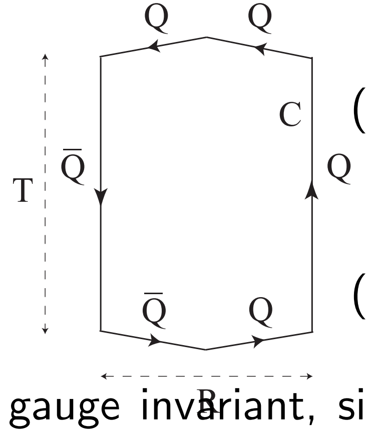
Wilson loop average $W(C)$ in the Yang-Mills theory

$$W(C) = \langle W_C[\mathscr{A}] \rangle_{\mathrm{YM}} = Z_{\mathrm{YM}}^{-1} \int [d\mathscr{A}] e^{-S_{\mathrm{YM}}[\mathscr{A}]} W_C[\mathscr{A}]. \tag{15}$$

static quark-antiquark potential

$$V_{Q\bar{Q}}(R) = \lim_{T \to \infty} \frac{-1}{T} \ln W(C). \tag{16}$$

$$W(C) \simeq \exp[-T V_{Q\bar{Q}}(R)], \quad (T \gg R \gg 1). \tag{17}$$



The static quark-antiquark potential $V_{Q\bar{Q}}(R)$ obtained in this way is obviously gauge invariant, since the Wilson loop operator is gauge invariant by construction.