Exponentiation and Renormalization of Wilson Line Operators

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Loop functions and Wilson loops

Wilson line operators

$$P(\mathbf{r}) = \mathcal{P}\exp\left[ig\int_0^{1/T} \mathrm{d}\tau\, A_0(\tau,\mathbf{r})\right] \qquad S(\mathbf{r}) = \mathcal{P}\exp\left[ig\int_0^{1} \mathrm{d}s\, \mathbf{r}\cdot\mathbf{A}(0,s\,\mathbf{r})\right]$$

With these one can construct

Polyakov loop correlator

$$P_c(r) = \frac{1}{N_c^2}\left\langle \mathrm{Tr}\left[P(\mathbf{r})\right]\mathrm{Tr}\left[P^\dagger(0)\right]\right\rangle$$

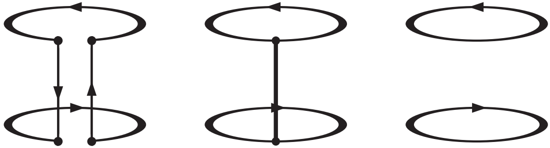
Cyclic Wilson loop

$$W_c(r) = \frac{1}{N_c}\left\langle \mathrm{Tr}\left[P(\mathbf{r})S(\mathbf{r})P^\dagger(0)S^\dagger(\mathbf{r})\right]\right\rangle$$

- static free energy of a $Q\bar{Q}$ pair given by $-T\log[P_c]$
- spectral decomposition $P_c = \frac{1}{N_c^2}\sum_n \exp[-E_n/T]$
- P_c contains both singlet and octet free energies
- W_c investigated as a (gauge invariant) function to extract singlet free energy

Divergence of the cyclic Wilson loop (in DR) [1]

- rectangular Wilson loops in the vacuum have cusp divergences
- imaginary time: $\tau = 0$ and $\tau = \beta$ are identified (periodic)
- cusps turn into intersections:



- intersection divergences are renormalized with operator mixing
- cyclic Wilson loop mixes with Polyakov loop correlator

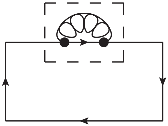
Renormalization formula

$$\begin{pmatrix} W_c^{(R)} \\ P_c^{(R)} \end{pmatrix} = \begin{pmatrix} Z & (1-Z) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} W_c \\ P_c \end{pmatrix}, \quad Z^{\mathrm{MS}} = \exp\left[-\frac{N_c \alpha_s}{\pi \varepsilon} + \dots \right]$$

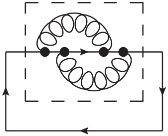
[1] [M. Berwein, N. Brambilla, J. Ghiglieri and A. Vairo, 2013]

Power Divergences [2]

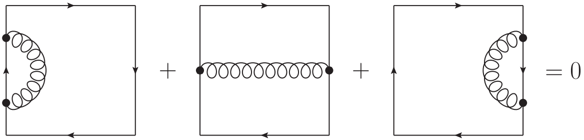
- can be neglected only in DR,
  not in lattice or other regularizations
- proportional to the length of the Wilson line
- with general UV cutoff Λ:



$$P_c^{(R)} = \exp\left[-2K\,\frac{\Lambda}{T}\right] P_c$$



- K depends on regularization scheme and colour representation
- naively expect renormalization by $\exp\left[-2K\,\frac{\Lambda}{T} - 2K\Lambda r\right]$ for W_c,
  but several divergent diagrams cancel



[2] [G.P. Korchemsky and A.V. Radyushkin, 1987]

Alternate form of $W_c - P_c$

- the intersection divergence is multiplicatively renormalizable for $W_c - P_c$
- use the identity

$$S^\dagger(\mathbf{r})T^a S(\mathbf{r}) = S_A^{ab}(\mathbf{r})T^b = T^b S_A^{\dagger\,ba}(\mathbf{r})$$

with $S_A(\mathbf{r})$ in the adjoint representation; $(T_A^c)_{ab} = -if^{abc}$

- split up a Polyakov line into components:

$$P(\mathbf{r}) = \frac{1}{N_c}\mathrm{Tr}[P(\mathbf{r})]\,I + \frac{1}{T_F}\mathrm{Tr}[P(\mathbf{r})T^a]T^a \equiv P_1(\mathbf{r})\,I + P_8^a(\mathbf{r})T^a$$

- with that one can rewrite

$$P_c(r) = \left\langle P_1(\mathbf{r})P_1^\dagger(\mathbf{0})\right\rangle\,, \quad W_c(r) - P_c(r) = \frac{T_F}{N_c}\left\langle P_8^a(\mathbf{r})S_A^{ab}(\mathbf{r})P_8^{\dagger\,b}(\mathbf{0})\right\rangle$$

- from this one can show that also the power divergences are multiplicatively renormalizable for $W_c - P_c$

Wilson line exponentiation

In components:

$$W_c(r) - P_c(r) = \frac{T^a_{ji} T^b_{lk}}{N_c T_F} \left\langle P_{ij}(\mathbf{r}) S^{ab}_A(\mathbf{r}) P^\dagger_{kl}(0) \right\rangle$$

Split Feynman diagrams into colour and kinematic part:

$$W^{ab}_{ij,kl}(D_n) = C^{ab}_{ij,kl}(D_n) K(D_n)$$

There exists an exponentiation theorem for untraced Wilson lines[3]:

$$\sum_n C^{ab}_{ij,kl}(D_n) K(D_n) = \exp\left[\sum_n \widetilde{C}^{ab}_{ij,kl}(D_n) K(D_n)\right]$$

Exponentiation is defined via the multiplication of two diagrams

$$(V \otimes W)^{ab}_{ij,kl} = V^{aa'}_{ii',kk'} W^{a'b}_{i'j,k'l}$$

[3] [E. Gardi, E. Laenen, G. Stavenga and C.D. White, 2010]

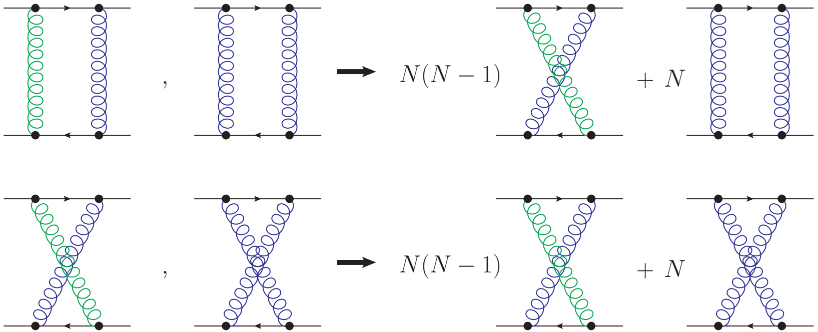
Replica trick

The exponentiated colour coefficients can be obtained through the replica trick: Take a theory with N non-interacting copies (replicas) of QCD.

$$\langle W\rangle^N = \langle W\rangle \otimes \langle W\rangle \otimes \cdots \otimes \langle W\rangle = \langle W_1 \otimes W_2 \otimes \cdots \otimes W_N\rangle$$

Then expand in N: $\quad \langle W\rangle^N = 1 + N \log\langle W\rangle + \mathcal{O}(N^2)$

So the exponentiated colour factor $\widetilde{C}$ is given by the $\mathbf{O(N)}$ **term** of the replicated colour factor C_N (Note: there is replica path ordering)



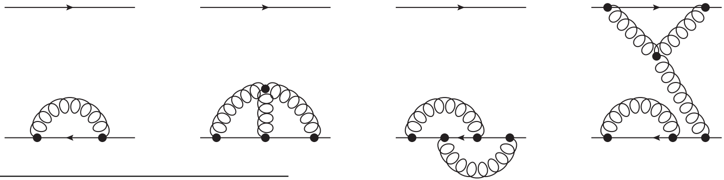
Factorization of power divergences

Generalization of the argument about power divergences to untraced Wilson lines:

- In any representation[4]:

$$f^{b_1 a_1 b_2} f^{b_2 a_2 b_3} \cdots f^{b_n a_n b_1} \left(T_R^{a_1} T_R^{a_2} \cdots T_R^{a_n}\right)_{ij} \propto \delta_{ij}$$

- all power divergent (sub)diagrams are proportional to the unit tensor
- the colour coefficients of subdiagrams factorize
- each factorized colour coefficient is at least $\mathcal{O}(N)$
- diagrams with subdiagrams have coefficients of $\mathcal{O}(N^2)$ or higher
- only power divergent diagrams without subdiagrams contribute to the exponent



[4] [V.S. Dotsenko and S.N. Vergeles, 1980]

Final renormalized expressions

Result:

The power divergences exponentiate and factorize for each Wilson line just like for closed loops.

The full expressions for the renormalized P_c and W_c then are:

$$P_c^{(R)}(r) = \exp\left[-2K_F\,\frac{\Lambda}{T}\right]\left\langle P_1(\mathbf{r})P_1^\dagger(\mathbf{0})\right\rangle$$

$$W_c^{(R)}(r) - P_c^{(R)}(r) = \exp\left[-2K_F\,\frac{\Lambda}{T} - K_A\Lambda r\right]\,Z\,\frac{T_F}{N_c}\left\langle P_8^a(\mathbf{r})S_A^{ab}(\mathbf{r})P_8^{\dagger\,b}(\mathbf{0})\right\rangle$$

Note:

- the coefficient of the power divergence depends on the representation
- in DR the ratio of $W_c - P_c$ with same T and different r is divergence-free, but not in other regularization schemes!

Exponentiation

The exponentiation of the diagrams can be put into the form of a matrix exponential

Depending on the number and representation of the Wilson lines, there is a certain number of basis tensors t_i such that $W = W_i t_i$

$$3 \otimes 3 \otimes 3 = 1 \oplus 2 \cdot 8 \oplus 10 \quad \Rightarrow \quad 1^2 + 2^2 + 1^2 = 6 \text{ basis tensors}$$

Multiplication of two bases: $t_i \otimes t_j = m_{ijk} t_k$

Multiplication of tensors:

$$(a_i t_i) \otimes (b_j t_j) = a_i b_j m_{ijk} t_k \equiv a_i M(b)_{ik} t_k \quad \text{with } M(b)_{ik} = b_j m_{ijk}$$
$$(a_i t_i) \otimes (b_j t_j) \otimes (c_k t_k) = (a_i M(b)_{il} t_l) \otimes (c_k t_k) = a_i M(b)_{il} M(c)_{ln} t_n$$

Now if $e = e_i t_i$ is the unit tensor ($e \otimes W = W \otimes e = W$), then

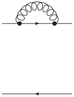
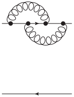
$$\exp\big[W_i t_i\big] = e_j \exp\big[M(W)\big]_{jk} t_k$$

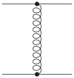
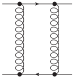
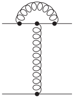
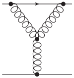
Example: Polyakov loop correlator

$$P_c(r) = \frac{\delta_{ji}\delta_{lk}}{N_c^2} \left\langle P_{ij}(\mathbf{r}) P^\dagger_{kl}(0) \right\rangle$$

- use tensors $t_1 = \delta_{ij}\delta_{kl}$ and $t_2 = \delta_{il}\delta_{kj}$
- $t_1 = e$, so $t_1 \otimes t_1 = t_1$, $t_1 \otimes t_2 = t_2 \otimes t_1 = t_2$, and $t_2 \otimes t_2 = t_1$
- then $M(W) = \begin{pmatrix} W_1 & W_2 \\ W_2 & W_1 \end{pmatrix}$, and $\frac{1}{N_c^2}(t_1)_{ii,kk} = 1$, $\frac{1}{N_c^2}(t_2)_{ii,kk} = \frac{1}{N_c}$

$$\begin{aligned} P_c(r) &= (1,0)\ \exp\begin{pmatrix} W_1 & W_2 \\ W_2 & W_1 \end{pmatrix}\begin{pmatrix} 1 \\ 1/N_c \end{pmatrix} \\ &= \exp[W_1]\left(\cosh[W_2] + \frac{1}{N_c}\sinh[W_2]\right) \end{aligned}$$

$W_1 = -\frac{1}{N_c} W_2 + \frac{2T_F}{N_c}(N_c^2 - 1)$  $+ 2T_F^2(N_c^2 - 1)$  $+ \mathcal{O}(\alpha_s^3)$

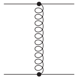
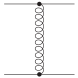
$W_2 = T_F$  $+ N_c T_F^2$  $- 2N_c T_F^2$  $+ 2iN_c T_F^2$  $+ \mathcal{O}(\alpha_s^3)$

Spectral decomposition (preliminary)

Exponentiated formula can be written in a different way:

$$P_c(r) = \frac{1}{N_c^2} e^{W_1 + N_c W_2} \left(\frac{N_c + 1}{2} e^{-(N_c - 1) W_2} - \frac{N_c - 1}{2} e^{-(N_c + 1) W_2} \right) + \frac{N_c^2 - 1}{N_c^2} e^{W_1} \left(\frac{1}{2} e^{W_2} + \frac{1}{2} e^{-W_2} \right)$$

In **Coulomb gauge** this reduces to

$$P_c(r) = \frac{1}{N_c^2} \exp \left[\frac{T_F}{N_c} (N_c^2 - 1) \right.$$  $$\left. \right] + \frac{N_c^2 - 1}{N_c^2} \exp \left[- \frac{T_F}{N_c} \right.$$  $$\left. \right] + \mathcal{O}(\alpha_s^3)$$

This corresponds nicely to the expected singlet and octet spectral decomposition:

$$P_c(r) = \frac{1}{N_c^2} \exp \left[- \frac{f_s(r)}{T} \right] + \frac{N_c^2 - 1}{N_c^2} \exp \left[- \frac{f_o(r)}{T} \right]$$

Conclusions

- the intersection divergences of the cyclic Wilson loop W_c can be removed through operator mixing with the Polyakov loop correlator P_c
- the combination $W_c - P_c$ is free of intersection divergences after multiplication with a renormalization constant Z
- $W_c - P_c$ can be expressed as the thermal average of two fundamental Polyakov lines and one adjoint string with a suitable contraction of indices
- the exponentiation theorem for Wilson lines through the replica trick was used to show that the power divergences of $W_c - P_c$ factorize
- this makes $W_c - P_c$ a multiplicatively renormalizable quantity for any kind of divergence
- the exponentiation of P_c was shown explicitly, it shows the expected spectral decomposition up to $\mathcal{O}(\alpha_s^2)$

Thank you for your attention!

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