Anomalous scaling violation in the anisotropic model of a passively advected vector field

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Description of the model: MHD, turbrlence (1)

The advection of a passive vector field, describing, for example, the evolution of the fluctuating part $\boldsymbol{\theta} \equiv \boldsymbol{\theta}(x)$ of the magnetic field in the presence of a mean component $\boldsymbol{\theta}^o$, which is supposed to be varying on a very large scale, is described by the stochastic equation

$$\partial_t \theta_i + \partial_k \left(v_k \theta_i - \mathcal{A}_0 \, v_i \theta_k \right) + \partial \mathcal{P} = \nu_0 \left(\partial_\perp^2 + f_0 \partial_\parallel^2 \right) \theta_i + f_i,$$

where $\mathcal{P}$ is the pressure term, f_i is an artificial Gaussian vector noise with zero mean and prescribed correlation function. The velocity field $\mathbf{v}(x)$ obtains preferred direction $\mathbf{n}$:

$$\mathbf{v}(t, \mathbf{x}) = \mathbf{n} \cdot v(t, \, \mathbf{x}_\perp).$$

In the real problem, the velocity field $\mathbf{v}(x)$ satisfies the NS equation, probably with additional terms that describe the feedback of the advected field $\boldsymbol{\theta}(x)$ on the velocity field. The framework of most papers is the *kinematic* problem, where the reaction of the field $\boldsymbol{\theta}(x)$ on the velocity field $\mathbf{v}(x)$ is neglected. They assume that at the initial stages $\boldsymbol{\theta}(x)$ is weak and does not affect the motions of the conducting fluid. So, we shall consider a Gaussian random velocity field with zero mean and correlation function

$$\left\langle v_i(t, \, \mathbf{x}) \, v_k(t', \, \mathbf{x}') \right\rangle = n_i n_k \cdot \delta(t - t') \int_{k>m} \frac{d\mathbf{k}}{(2\pi)^{d-1}} \, \delta(k_\parallel) \, D_0 \, e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')} \, \frac{1}{k_\perp^{d-1+\xi}}.$$

Here δ is delta-function, d is the dimensionality of $\mathbf{x}$ space, $k_\perp \equiv |\mathbf{k}_\perp|$, $1/m$ is a turbulence scale, connected to the attenuation, the exponent ξ plays the role of the RG expansion parameter and $D_0 > 0$ is an amplitude factor.

The task is to find the asymptotic behaviour of the correlation function $G_{12} = \langle F_1, \; F_2 \rangle$ in the inertial range $L \gg r \gg l$, where L is the large (external) scale, connected to the random force, and l is the internal scale, connected to the viscosity. Both F_1 and F_2 are composite operators of the type

$$F_{N, \, p, \, m} = (\theta_i \theta_i)^p \, (n_s \theta_s)^{2m}, \quad N = 2(p + m).$$

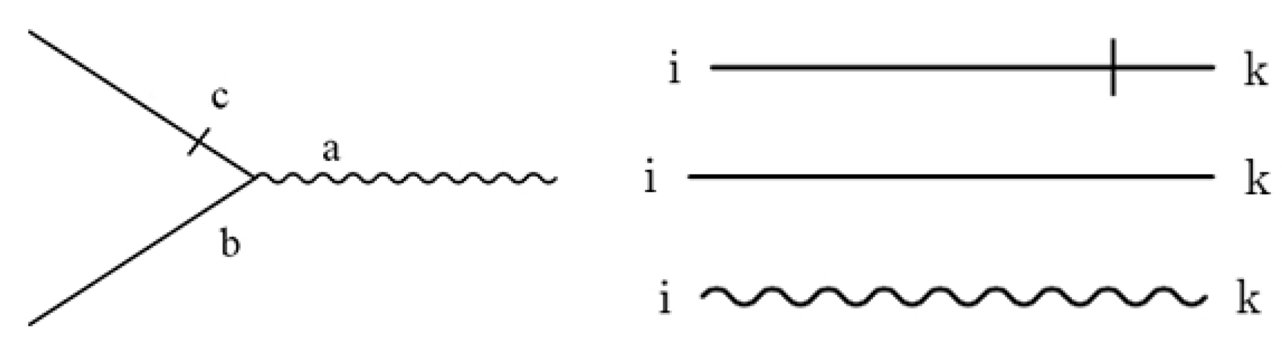
Field theoretic formulation (2)

This stochastic problem is equivalent to the field theoretic model of the set of three fields $\Phi \equiv \{\boldsymbol{\theta}, \boldsymbol{\theta}', \boldsymbol{v}\}$ with action functional

$$S(\Phi) = \frac{1}{2} \theta_i' D_\theta \theta_k' - \frac{1}{2} v_i D_v^{-1} v_k +$$

$$+ \theta_k' \left[-\partial_t \theta_k - (v_i \partial_i) \theta_k + \mathcal{A}_0 (\theta_i \partial_i) v_k + \nu_0 (\partial_\perp^2 + f_0 \partial_\parallel^2) \theta_k \right].$$

This model corresponds to a standard Feynman diagrammatic technique with the triple vertex and three bare propagators: $\langle \theta_i \theta_k' \rangle_0$ and $\langle \theta_i \theta_k \rangle_0$ and $\langle v_i v_k \rangle_0$.



Dyson equation, renormalization and fixed points (3)

From the Dyson equation it follows, that one *new* parameter u_0 is needed and

$$\nu_0 = \nu, \; \mathcal{A}_0 = \mathcal{A}, \; f_0 = f \cdot Z_f, \; u_0 = u \cdot Z_u.$$

If $\xi > 0$ and $d - 2 + \mathcal{A} > 0$ the system possesses a fixed point $g^*, \; u^*$:

$$g^* = \frac{2(d-1)}{d-2+\mathcal{A}} \cdot \xi \;\text{ and }\; u^* = \frac{(\mathcal{A}-1)^2}{d-2+\mathcal{A}}.$$

This fact implies that correlation functions of model in the IR region ($\Lambda r \gg 1$, $mr \sim 1$) exhibit a scaling behavior. The corresponding critical dimensions $\Delta[F] \equiv \Delta_F$ can be calculated as series in ξ.

Composite operators and their renormalization (4)

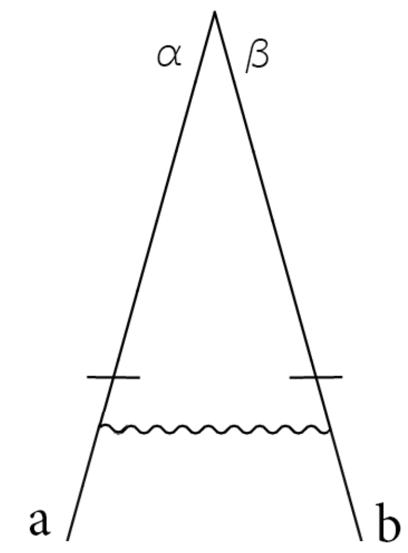
Operators $F_{N, \, p} = (\theta_i \theta_i)^p \, (n_s \theta_s)^{2m}$, where $N = 2(p + m)$, are renormalized multiplicatively:

$$F_{N,p} = Z_{N,p} \cdot F^R_{N,p}.$$

The only diagram needed to be calculated is one-loop diagram. The divergent parts of all multy-loop diagrams *are equal to zero*.

The operators are mixing in renormalization,

$$F_{N, \, p} \propto F^R_{N, \, p+1} + F^R_{N, \, p} + F^R_{N, \, p-1} + F^R_{N, \, p-2}.$$



Therefore the remormalization constant Z_F and the anomalous dimension γ_F^* are four-diagonal matrices. The critical dimension matrix, which governs the asymptotic behaviour, is

$$\Delta_N = -2(p + m) \cdot I + \gamma_N^*.$$

Nilpotency of the anomalous dimension matrix (5)

For each closed set of operators with fixed N the elements of the anomalous dimension matrix γ_F^* at critical point are

$$\gamma^*_{N, \, p+1} = \eta \cdot 2m(2m-1) \cdot \xi;$$

$$\gamma^*_{N, \, p} = \eta \cdot (2p + 8pm - 2m(2m-1)) \cdot \xi;$$

$$\gamma^*_{N, \, p-1} = \eta \cdot (4p(p-1) - 2p - 8pm) \cdot \xi;$$

$$\gamma^*_{N, \, p-2} = \eta \cdot (-4p(p-1)) \cdot \xi.$$

Therefore the critical dimension matrix Δ_N is not diagonalizable, but have a *Jordan* form!

$$\lambda_1 = \; \ldots \; = \lambda_{N/2+1} = -2(p+m) = -N,$$

$$\widetilde{\Delta}_F = \begin{pmatrix} -2(p+m) & 1 & 0 & \ldots & 0 \\ 0 & -2(p+m) & 1 & & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \ldots & & 0 & -2(p+m) \end{pmatrix}.$$

Asymptotic behaviour of the mean value of composite operators (6)

According to the solution of the RG equation, at $r \gg l$

$$\left\langle \widetilde{\mathbf{F}}^R \right\rangle \propto \nu_0^{d_F^\omega} \cdot M^{-N} \cdot (Mr)^{\widetilde{\Delta}_F} \cdot \Phi \left(\frac{f}{M^\xi} \right) \cdot \mathbf{C}_0,$$

where

$$(Mr)^{\widetilde{\Delta}_F} = \begin{pmatrix} (Mr)^\lambda & (Mr)^\lambda \cdot \ln(Mr) & \ldots & \frac{(Mr)^\lambda \cdot (\ln(Mr))^{n-1}}{(n-1)!} \\ 0 & (Mr)^\lambda & & \vdots \\ \vdots & & \ddots & (Mr)^\lambda \cdot \ln(Mr) \\ 0 & \ldots & 0 & (Mr)^\lambda \end{pmatrix}.$$

I. e. the asymptotic behaviour does not only have a power term, but is a product of a *power* and *logarithmic* dependancies!

Asymptotic behaviour of the pair correlation function (7)

According to the solution of the RG equation for pair correlation function $G_{N_1, \, N_2} = \left\langle F_{N_1, \, p_1}, \; F_{N_2, \, p_2} \right\rangle$ of composite operators and Operator Product Expansion, the desired asymptotic behaviour in the inertial range $L \gg r \gg l$ is *pure logarithmic*:

$$G = \left\langle F_{N_1,p_1} \, F_{N_2,p_2} \right\rangle \propto \nu^{d_G^\omega} \cdot [\ln \Lambda r]^{(N_1+N_2)/2} \cdot [\ln Mr]^{(N_1+N_2)/2} \cdot \widetilde{\Phi} \left(f \frac{r^\xi}{M^\xi} \right).$$

for details, see arXiv:1406.3808