The Weyl Consistency conditions and their consequences

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CP³ Origins
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Outline

- The local renormalisation group
- The Weyl consistency conditions and some of their consequences:
  - The perturbative “a theorem”
  - Cross-relations for ß functions
- ß functions in the Standard Model and stability of the vacuum

Scale transformations

Classically, field theories are invariant under a rescaling of all dimensionful quantities, or equivalently under a rescaling of the metric:

$$\gamma_{\mu\nu} \to \Omega\, \gamma_{\mu\nu}$$

But the scale symmetry is broken at the quantum level: the renormalised couplings depend on a scale

$$g_i(\mu) \to g_i(\Omega^{-1/2}\mu)$$

$\Longrightarrow$ scale anomaly

(≡ conformal anomaly ≡ trace anomaly ≡ Weyl anomaly)

Scale transformations

Renormalised generating functional:

$$W = \log \left[\int \mathcal{D}\Phi \, e^{iS_{\text{renormalised}} + iS_{\text{counterterms}}} \right]$$

$$S_{\text{renormalised}} = \int \mathrm{d}^4x \sqrt{-\gamma} \, \left[\mathcal{L}_{\text{free}} + g_i \mathcal{O}^i \right]$$

Transformation under a rescaling $\gamma_{\mu\nu} \to e^{2\sigma} \gamma_{\mu\nu}, \quad g(\mu) \to g(e^{-\sigma} \mu)$

$$\Delta_\sigma \equiv \sigma \left(2\gamma_{\mu\nu} \frac{\delta}{\delta \gamma_{\mu\nu}} - \beta_i \frac{\delta}{\delta g_i} \right)$$

$$\Delta_\sigma W = \sigma \left(T^\mu_\mu - \beta_i \mathcal{O}^i \right) \quad \Longrightarrow \quad \boxed{T^\mu_\mu = \beta_i \mathcal{O}^i}$$

"trace anomaly"

Conformal transformations

Conformal transformation ≡ local version of scale transformation

$$\gamma_{\mu\nu} \to \Omega(x)\gamma_{\mu\nu}$$

"Weyl transformation"

Obviously defined in curved space

But it has consequences in flat space as well!

Working in curved space means that there are new correlation functions of the form

$$\frac{\delta}{\delta\gamma_{\mu\nu}}\frac{\delta}{\delta\gamma_{\rho\tau}}W \sim \langle T^{\mu\nu}T^{\rho\tau}\rangle$$

which have to be made finite by an appropriate renormalisation

Renormalisation in curved space

In the presence of a curved background, additional counterterms are needed to make the theory finite:

$$W_{\text{curved}} = W_{\text{flat}} + \int \mathrm{d}^4x\sqrt{-\gamma}\,\left[Z_a\,E + Z_b\,R^2 + Z_c\,W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma}\right]$$

All possible 4d curvature terms: Euler density (E), Weyl tensor squared ($W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma}$)

Under a Weyl transformation $\Delta_\sigma \equiv \int \mathrm{d}^4x\,\sigma(x)\left(2\gamma_{\mu\nu}\frac{\delta}{\delta\gamma_{\mu\nu}} - \beta_i\frac{\delta}{\delta g_i}\right)$

$$\Delta_\sigma W_{\text{curved}} = \int \mathrm{d}^4x\sqrt{-\gamma}\,\sigma\left[a\,E + b\,R^2 + c\,W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma}\right]$$

Weyl anomaly

e.g. for free fields, $a = \frac{1}{90\,(8\pi)^2}\left(n_s + \frac{11}{2}n_f + 62\,n_v\right)$

D. Capper, M.J. Duff (1973), …

Space-time-dependent couplings

Under a Weyl rescaling, the coupling "constants" are not constant

$$g_i(\mu) \to g_i(\Omega(x)^{-1/2}\mu)$$

the g_i become x-dependent!

The couplings $g_i(x)$ act as auxiliary fields and are sources for the composite operators $\mathcal{O}^i$

$\Longrightarrow$ one can compute correlation functions of composite operators

$$\frac{\delta^2}{\delta g_i \delta g_j} W \sim \langle \mathcal{O}^i \mathcal{O}^j \rangle,$$

$$\langle \mathcal{O}^i \mathcal{O}^j \mathcal{O}^k \rangle, \langle \mathcal{O}^i \mathcal{O}^j \mathcal{O}^k \mathcal{O}^l \rangle$$

must also be made finite by renormalisation!

Renormalisation with local couplings

I. Jack, H. Osborn (1987-1991)

With space-dependent couplings, even more counterterms are needed, proportional to $\partial_\mu g_i(x)$

Complete Weyl anomaly:

$$\Delta_\sigma W = \int \mathrm{d}^4x\sqrt{-\gamma}\,\left[\sigma\, a\, E + G^{\mu\nu}\left(\sigma\,\chi^{ij}\,\partial_\mu g_i\,\partial_\nu g_j + \partial_\mu\sigma\,\omega^i\,\partial_\nu g_i\right) + \ldots\right]$$

Einstein tensor $G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}\gamma^{\mu\nu}R$

$a,\ \chi^{ij},\ \omega^i,\ldots$: Functions of the couplings g_i

There are 16 diffeomorphism-invariant terms that include curvature tensors and derivatives of the couplings

We neglect here anomalous flavour currents that can lead to limit cycles

Fortin, Grinstein, Stergiou (2012)
Luty, Polchinski, Rattazzi (2012)

The Weyl consistency conditions

H. Osborn (1987-1991)

The Weyl anomaly has to be abelian:

$$\Delta_\tau \Delta_\sigma W = \Delta_\sigma \Delta_\tau W$$

Gives a number of consistency relations among the functions $a,\ \chi^{ij},\ \omega^i,\ \ldots$

One of them is particularly interesting:

$$\frac{\partial \tilde{a}}{\partial g_i} = \beta_j \left(\chi^{ij} + \frac{\partial \omega^i}{\partial g_j} - \frac{\partial \omega^j}{\partial g_i} \right) \qquad \tilde{a} = a - \omega^i \beta_i$$

Valid also in flat space and with constant couplings!

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Consequence I: the a theorem in 4d

J. Cardy (1988), Z. Komargodski, A. Schimmer (2011)

The matrix χ^{ij} can be computed in perturbation theory and happens to be positive definite at leading order, for arbitrary theories with scalar, fermions and gauge fields

$$\frac{\partial \tilde{a}}{\partial g_i} = \beta_j \left(\chi^{ij} + \frac{\partial \omega^i}{\partial g_j} - \frac{\partial \omega^j}{\partial g_i} \right) \qquad \tilde{a} = a - \omega^i \beta_i$$

$$\Longrightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}\log\mu}\tilde{a} = \beta_i \frac{\partial \tilde{a}}{\partial g_i} = \beta_i \beta_j \, \chi^{ij} \geq 0$$

The function $\tilde{a}$ is monotonic along the renormalisation group flow and coincides with a at fixed points

$\Longrightarrow$ perturbative a theorem

Consequence II: relations among β functions

The function ω^i is an exact one-form at the leading orders in perturbation theory

$$\frac{\partial \tilde{a}}{\partial g_i} \approx \chi^{ij} \beta_j$$

By positivity of χ^{ij}, this can be inverted to give

$$\beta_i \approx \chi_{ij} \frac{\partial \tilde{a}}{\partial g_j}$$

The RG flow is a gradient flow in a space with metric χ^{ij}

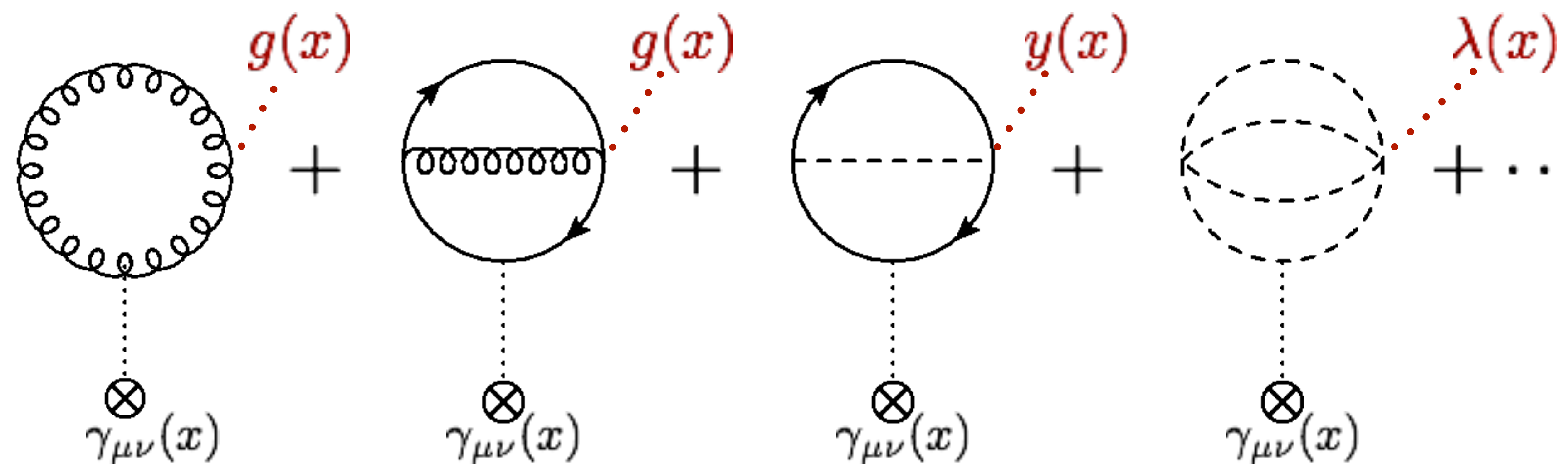
The β functions of a theory are not independent but can all be derived from a unique function, and satisfy

$$\frac{\partial^2 \tilde{a}}{\partial g_i \partial g_j} \approx \frac{\partial}{\partial g_i} \left(\chi^{jk} \beta_k\right) \approx \frac{\partial}{\partial g_j} \left(\chi^{ik} \beta_k\right)$$

In terms of Feynman diagrams

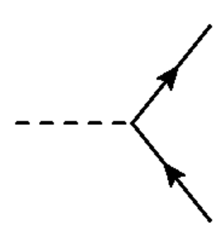
a is equal to the trace of the energy-momentum tensor on a 4-sphere:

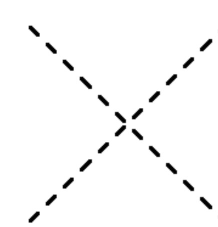
$$a = \left\langle T^{\mu}_{\mu} \right\rangle_{S^4} =$$



Partial derivatives are equivalent to removing one interaction vertex

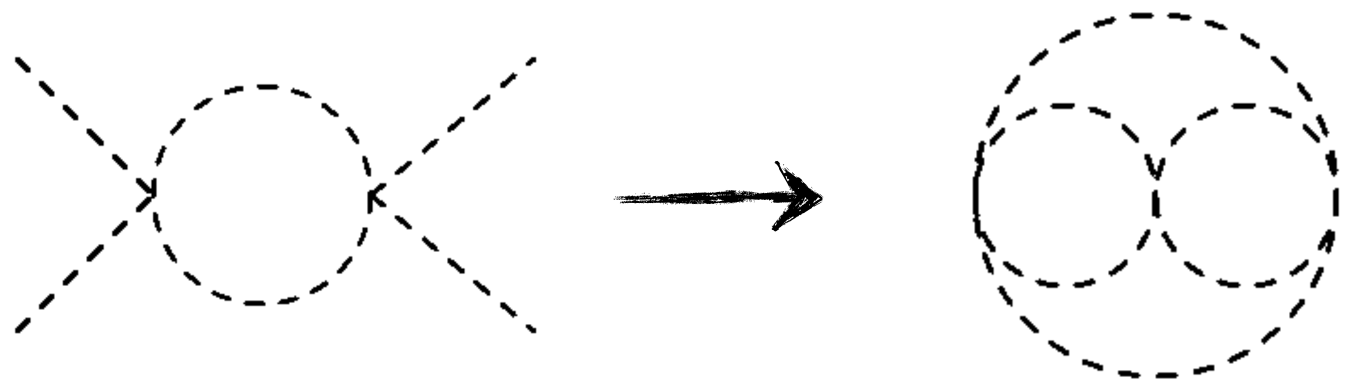
$$\beta_i \approx \chi_{ij} \frac{\partial \tilde{a}}{\partial g_j} \qquad \frac{\partial}{\partial g_i} \rightarrow \frac{\delta}{\delta g_i(x)}$$

$\beta_y =$  $+ \cdots$

$\beta_\lambda =$  $+ \cdots$

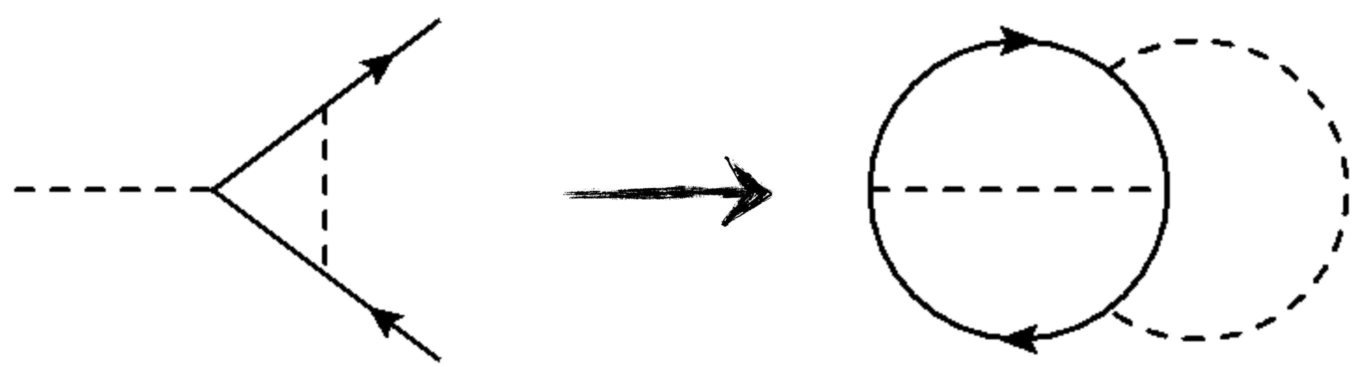
Counting loops

◇ One-loop β function of a scalar quartic interaction



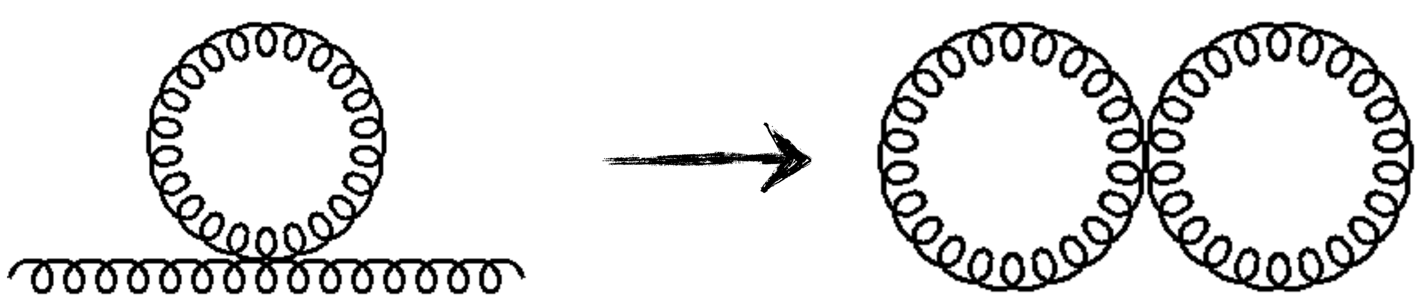
Comes from a
4-loops diagram
in the function a

◇ One-loop β function of a Yukawa interaction



3-loops diagram

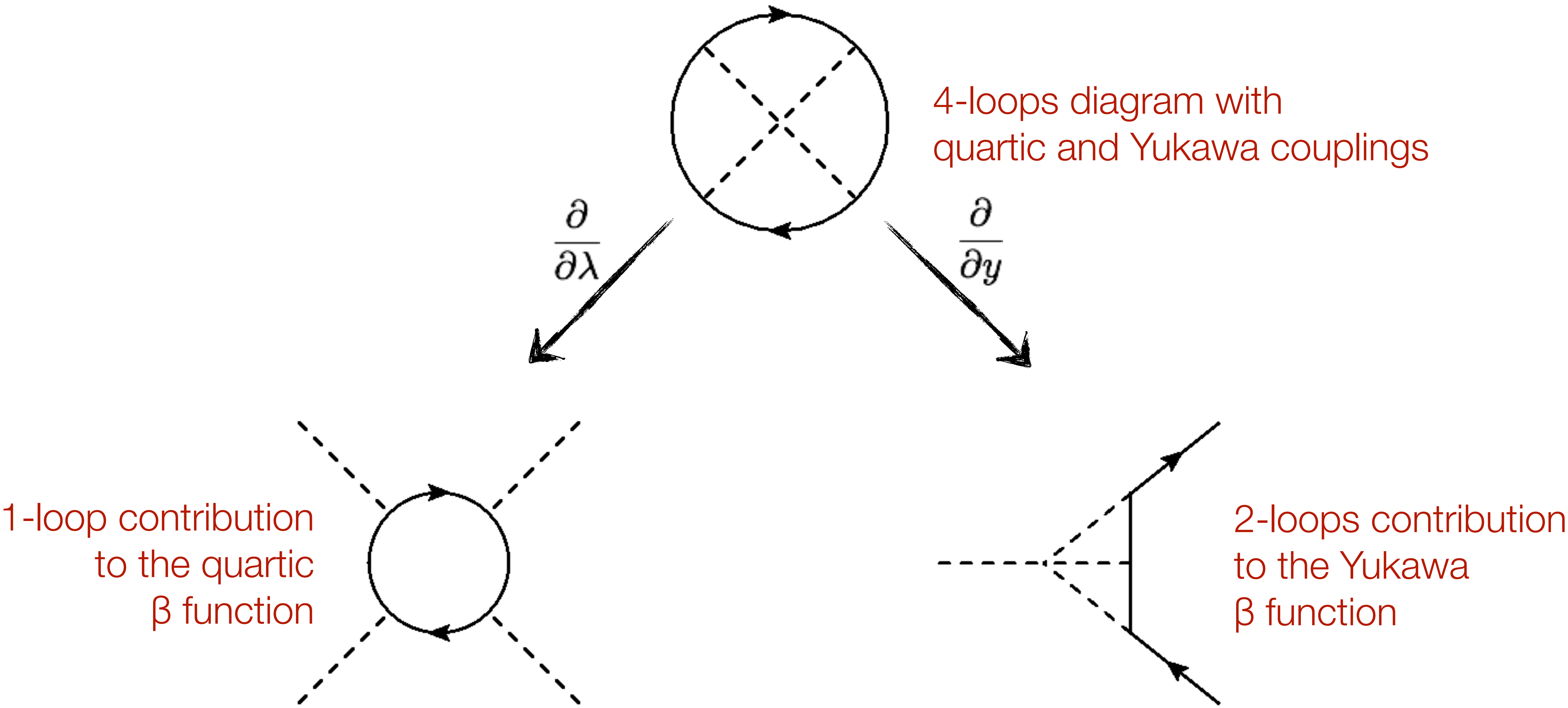
◇ One-loop β function of a gauge interaction



2-loops diagram

Multiple couplings

What about diagrams involving multiple couplings?



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An example: the Standard Model

Neglecting all Yukawa coupling apart from the top one, the theory has five couplings:

$$\left\{\alpha_1, \alpha_2, \alpha_3, \alpha_t, \alpha_\lambda\right\} \equiv \left\{\frac{g_1^2}{(4\pi)^2}, \frac{g_2^2}{(4\pi)^2}, \frac{g_3^2}{(4\pi)^2}, \frac{y_t^2}{(4\pi)^2}, \frac{\lambda}{(4\pi)^2}\right\}$$

The metric is diagonal at lowest order

Jack, Osborn (1990)

$$\chi^{ij} = \mathrm{diag}\left(\frac{1}{\alpha_1^2}, \frac{3}{\alpha_2^2}, \frac{8}{\alpha_3^2}, \frac{2}{\alpha_t}, 4\right)$$

Gives a set of relations among the β functions, e.g.

$$2\frac{\partial}{\partial\alpha_t}\beta_\lambda = \frac{\partial}{\partial\alpha_\lambda}\left(\frac{\beta_t}{\alpha_t}\right) + \mathcal{O}\left(\alpha_i^2\right),$$

1-loop 2-loop

$$\frac{3}{8}\frac{\partial}{\partial\alpha_3}\left(\frac{\beta_2}{\alpha_2^2}\right) = \frac{\partial}{\partial\alpha_2}\left(\frac{\beta_3}{\alpha_3^2}\right) + \mathcal{O}\left(\alpha_i^2\right),$$

$$\ldots$$

The Standard Model β functions

$$\beta_1 = 2\alpha_1^2\left\{\frac{1}{12} + \frac{10n_G}{9} + \left(\frac{1}{4} + \frac{95n_G}{54}\right)\alpha_1 + \left(\frac{3}{4} + \frac{n_G}{2}\right)\alpha_2 + \frac{22n_G}{9}\alpha_3 + \left(\frac{163}{1152} - \frac{145n_G}{81} - \frac{5225n_G^2}{1458}\right)\alpha_1^2\right.$$

$$+ \left(\frac{87}{64} - \frac{7n_G}{72}\right)\alpha_1\alpha_2 - \frac{137n_G}{162}\alpha_1\alpha_3 + \left(\frac{3401}{384} + \frac{83n_G}{36} - \frac{11n_G^2}{18}\right)\alpha_2^2 + \left(\frac{1375n_G}{54} - \frac{242n_G^2}{81}\right)\alpha_3^2 - \frac{n_G}{6}\alpha_2\alpha_3$$

$$\left. + \alpha_t\left[-\frac{17}{12} - \frac{2827}{576}\alpha_1 - \frac{785}{64}\alpha_2 - \frac{29}{6}\alpha_3 + \left(\frac{113}{32} + \frac{101n_t}{16}\right)\alpha_t\right] + \alpha_\lambda\left(\frac{3}{4}\alpha_1 + \frac{3}{4}\alpha_2 - \frac{3}{2}\alpha_\lambda\right)\right\}$$

relations between the 2-loop gauge β functions

$$\beta_2 = 2\alpha_2^2\left\{-\frac{43}{12} + \frac{2n_G}{3} + \left(\frac{1}{4} + \frac{n_G}{6}\right)\alpha_1 + \left(-\frac{259}{12} + \frac{49n_G}{6}\right)\alpha_2 + 2n_G\alpha_3 + \left(\frac{163}{1152} - \frac{35n_G}{54} - \frac{55n_G^2}{162}\right)\alpha_1^2\right.$$

$$+ \left(\frac{187}{64} + \frac{13n_G}{24}\right)\alpha_1\alpha_2 - \frac{n_G}{18}\alpha_1\alpha_3 + \left(-\frac{667111}{3456} + \frac{3206n_G}{27} - \frac{415n_G^2}{54}\right)\alpha_2^2$$

$$+ \frac{13n_G}{2}\alpha_2\alpha_3 + \left(\frac{125n_G}{6} - \frac{22n_G^2}{9}\right)\alpha_3^2$$

$$\left. + \alpha_t\left[-\frac{3}{4} - \frac{593}{192}\alpha_1 - \frac{729}{64}\alpha_2 - \frac{7}{2}\alpha_3 + \left(\frac{57}{32} + \frac{45n_t}{16}\right)\alpha_t\right] + \alpha_\lambda\left(\frac{1}{4}\alpha_1 + \frac{3}{4}\alpha_2 - \frac{3}{2}\alpha_\lambda\right)\right\}$$

$\vdots$

relations between the 3-loop gauge and 1-loop Higgs quartic β functions

$$\beta_\lambda = \frac{9}{16}\alpha_2^2 - \frac{9}{2}\alpha_\lambda\alpha_2 + \frac{3}{16}\alpha_1^2 - \frac{3}{2}\alpha_\lambda\alpha_1 + \frac{3}{8}\alpha_1\alpha_2 + 12\alpha_\lambda^2 + 6\alpha_\lambda\alpha_t - 3\alpha_t^2 + \ldots$$

Precision running in the Standard Model

Knowing the value of the Standard Model couplings at an arbitrary energy scale is crucial: vacuum stability, grand unification, cosmology...

The state-of-the-art computations make use of the gauge, top Yukawa and Higgs quartic β functions at 3-loops order

Degrassi et al. (2012), Buttazzo et al. (2013)

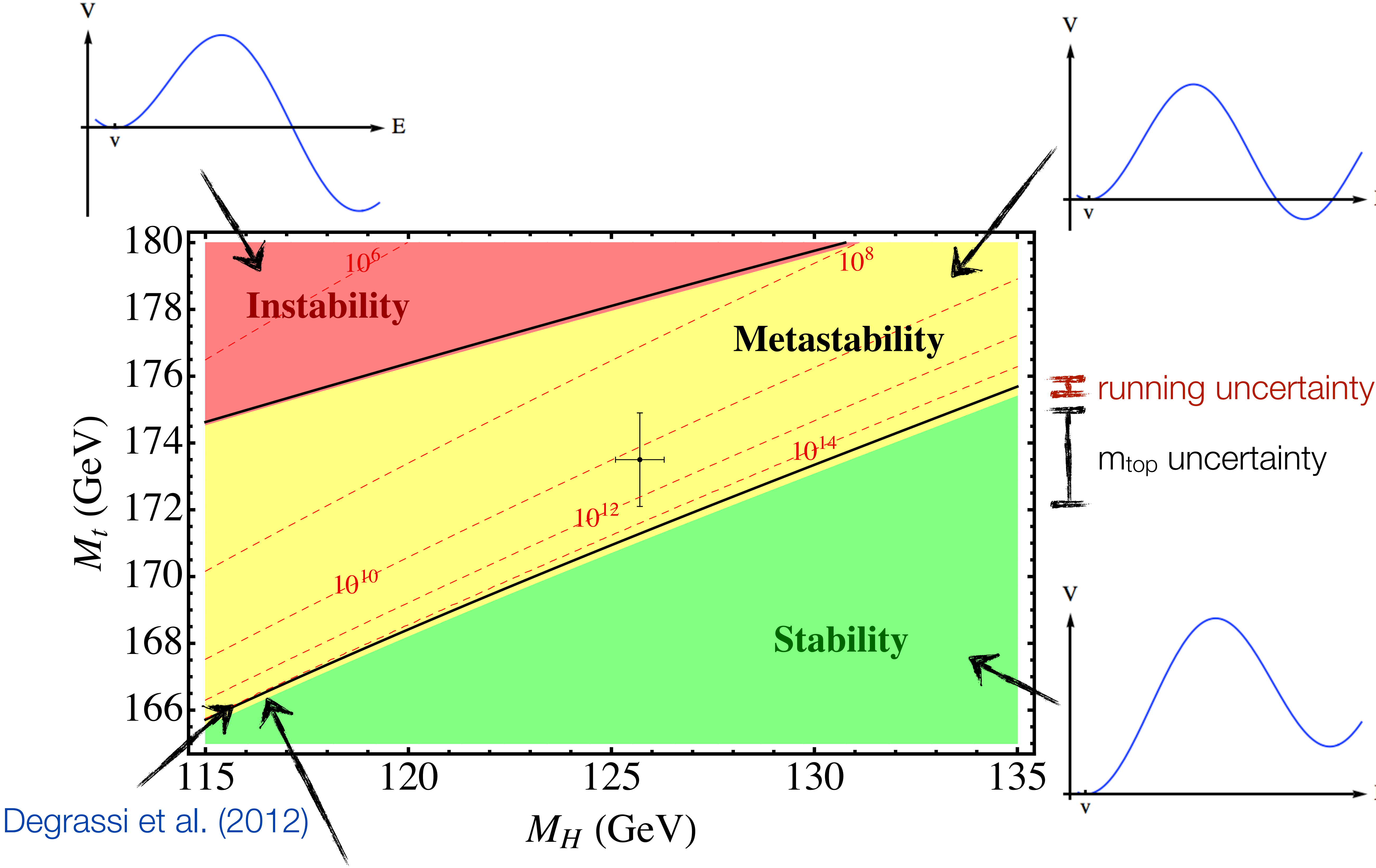
Inconsistent with the Weyl symmetry!

Already going to 2 loops in the Higgs quartic β functions means including diagrams that contributes to the 4-loop gauge β functions

The best Weyl-consistent running based on the existing computations:

◇ 3 loops in the gauge β functions
◇ 2 loops in the top Yukawa β function
◇ 1 loop in the Higgs quartic β function

Standard Model vacuum stability



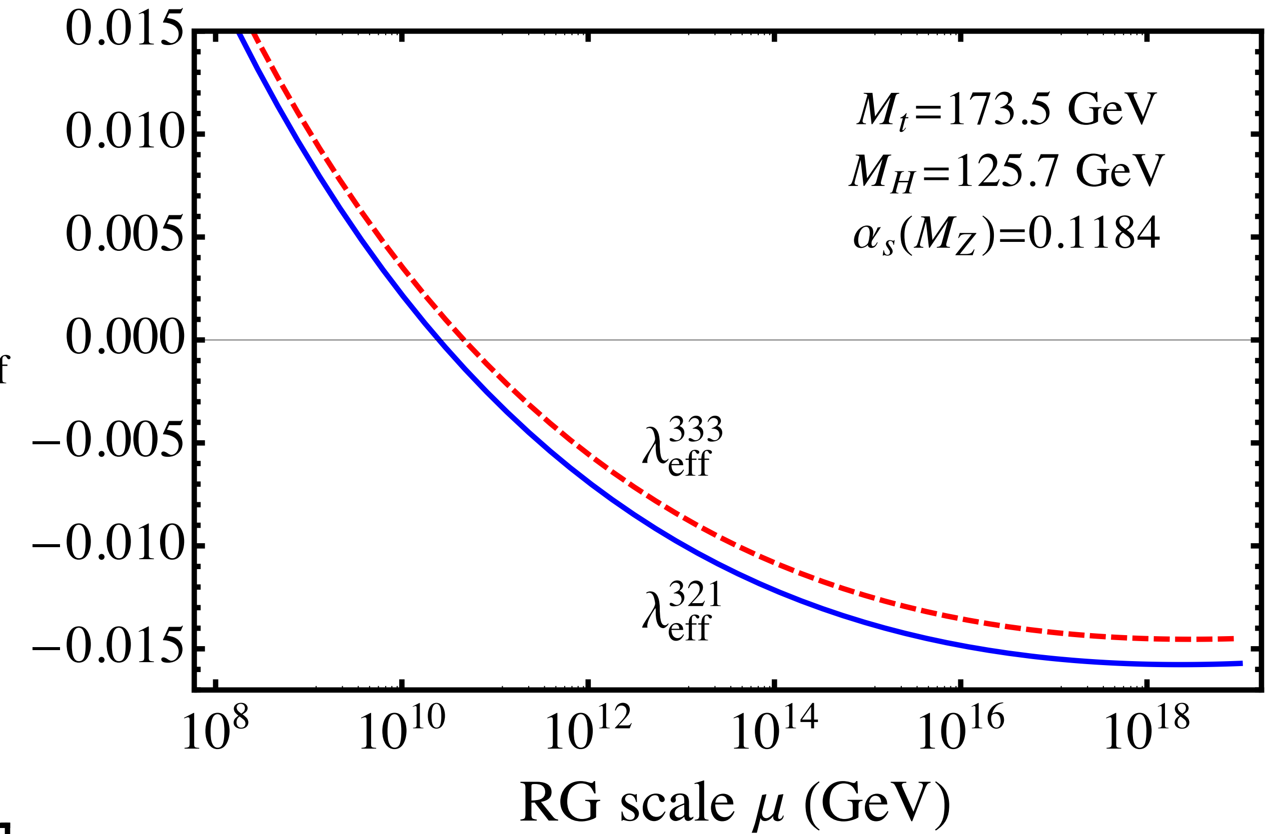
3-2-1 counting scheme

Importance of precision running

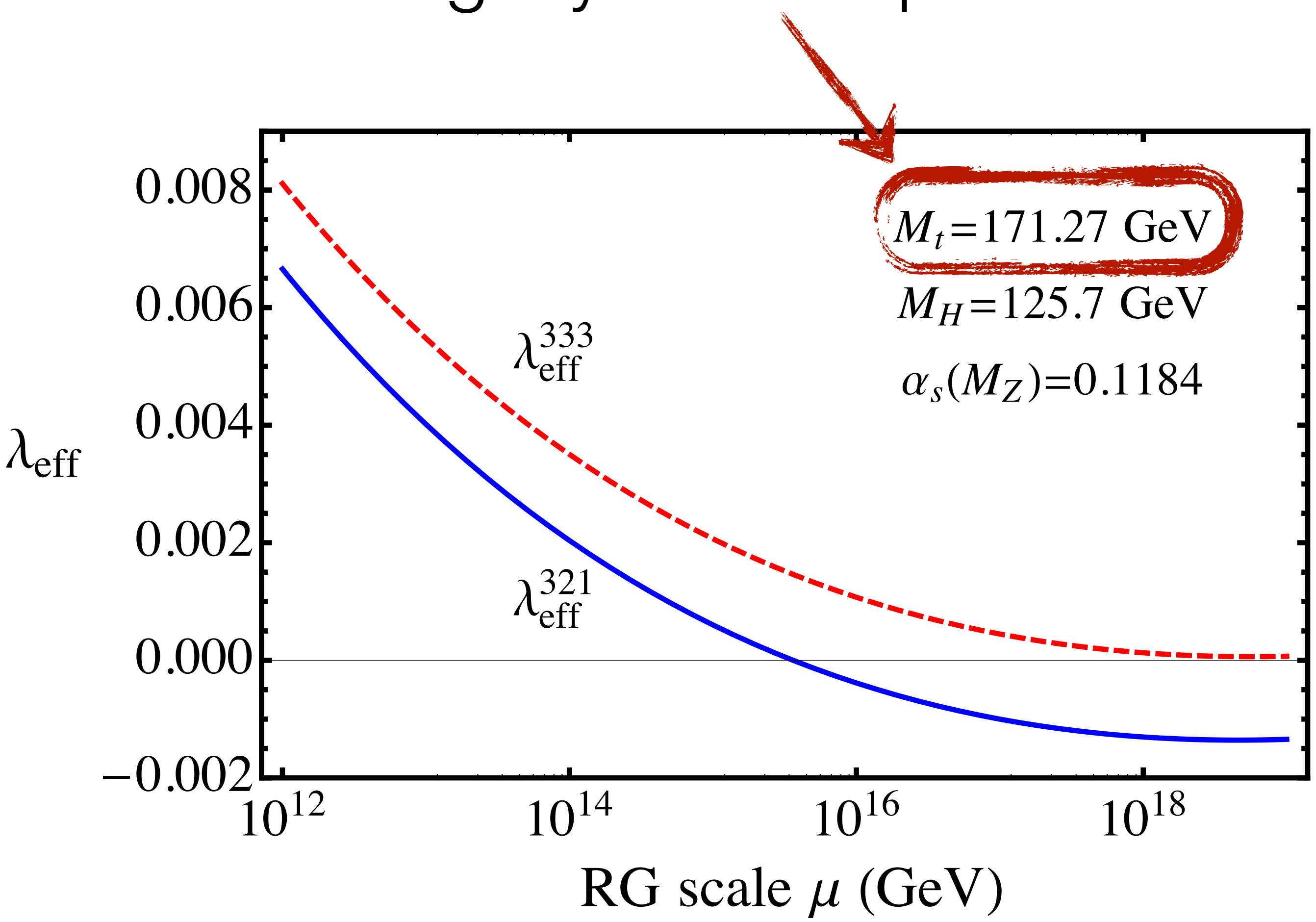
"Coincidence" in the SM:

$$\lambda(\mu) = 0 \quad \text{and} \quad \frac{\mathrm{d}\lambda}{\mathrm{d}\mu}(\mu) = 0$$

happen around the same scale
→ Higgs inflation?



With a slightly lower top mass...



Important uncertainties also in:

◇ matching of $\overline{\text{MS}}$ parameters at the electroweak scale

◇ tunneling probability

Summary & Outlook

- The Weyl symmetry constrains the RG flow of any theory
- For theories with multiple couplings, it provides relations among the β functions at different loop order
- Precision computations should make use of a loop counting scheme consistent with the Weyl symmetry

More consequences of the local RGE:

see talks by O. Antipin and E. Mølgaard

- Important for the search of perturbative fixed points in gauge-Yukawa theories
- Work in progress: Weyl consistency conditions for dim.-6 operators in the Standard Model
- Work in progress: local RGE and semiclassical solutions
- …