Leading chiral logarithm for the nucleon mass

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Introduction

Chiral perturbation theory (ChPT) is low-energy effective field theory.

$$\mathcal{A} = \mathcal{A}^{(0)} + \frac{q^2}{(4\pi F_\pi)^2}\mathcal{A}^{(1)} + \left(\frac{q^2}{(4\pi F_\pi)^2}\right)^2 \mathcal{A}^{(2)} + \left(\frac{q^2}{(4\pi F_\pi)^2}\right)^3 \mathcal{A}^{(3)} + ...$$

$$q = \text{momenta, masses, etc} \lesssim m_\pi$$

- Infinite effective Lagrangian with dimensional couplings (non-renormalizable theory).
- Ordering in dimension allows to systematically calculate low-energy expansion.
- With increasing chiral order the number of low-energy constants (LECs) grows.

$$\mathcal{L}^{N+\pi}_{\text{ChPT}} = \mathcal{L}^{(0)}(F_\pi, m_\pi, g_A) + \mathcal{L}^{(1)}(c_1, .., c_4) + \mathcal{L}^{(2)}(l_1, .., l_{10}, d_1, .., d_{23}) + ...$$

Logarithmical structure of chiral expansion

LLog NLLog N^2LLog

$$L = \ln\left(q^2/\mu^2\right)$$

$$\begin{aligned}
\mathcal{A} = \; & \mathcal{A}^{(0)} \\
& + \frac{q^2}{(4\pi F_\pi)^2}\left(\mathcal{A}^{(1,1)} L + \mathcal{A}^{(1,0)} \right) \\
& + \left(\frac{q^2}{(4\pi F_\pi)^2}\right)^2 \left(\mathcal{A}^{(2,2)} L^2 + \mathcal{A}^{(2,1)} L + \mathcal{A}^{(2,0)} \right) \\
& + \left(\frac{q^2}{(4\pi F_\pi)^2}\right)^3 \left(\mathcal{A}^{(3,3)} L^3 + \mathcal{A}^{(3,2)} L^2 + \mathcal{A}^{(3,1)} L + \mathcal{A}^{(3,0)} \right) \\
& + \ldots
\end{aligned}$$

$\mathcal{L}^{(0)}$ $\mathcal{L}^{(0)-(2)}$ $\mathcal{L}^{(0)-(4)}$

- Leading logarithms (LLogs) involves only the leading order Lagrangian.
- Instead of evaluation of n-loop diagrams one can apply renormalization group (RG) [Büchler,Colangelo,03]

Chiral Logs from renormalization group

The renormalization scale invariance implies

$$\mathcal{A}(\mu^2) = \exp\left(\ln\left(\frac{\mu^2}{\mu_0^2}\right)\sum_{n=0}^{\infty}\beta^{(n)}\frac{\partial}{\partial c^{(n)}}\right)\mathcal{A}(\mu_0^2)$$

The LECs obey the equation

$$\mu^2\frac{d}{d\mu^2}c^{(n)} = \beta^{(n)}\left[c^{(n-1)},\ldots,c^{(0)}\right] \qquad \leftarrow \text{infinite set of equations}$$

With help of these equations one can extract the logarithmical contributions with minimum efforts.

- For LLog contributions one needs only the one-loop β-functions for LECs of chiral order up to order of calculation.
- For NLLog contribution one needs one- and two-loop β-function
- etc.

LLogs in meson ChPT

The renormalization group technique has been successfully applied in meson sector.

$$\mathcal{L}_\pi^{(0)} = \frac{F^2}{4}\mathrm{tr}\left(\partial_\mu U \partial^\mu U^\dagger + m^2\left(U + U^\dagger\right)\right)$$

- The physical pion mass has been calculated up to six-loop order [Bijnens,et al,12]

$$m^2_{\text{phys}} = m^2\left(1 - \frac{1}{2}L + \frac{17}{8}L^2 - \frac{103}{24}L^3 + \frac{24367}{1152}L^4 - \frac{8821}{144}L^5 + \frac{1922964667}{6220800}L^6 + \cdots\right),$$

$$\text{here:} \qquad L = \frac{m^2}{(4\pi F)^2}\log\left(\frac{\mu^2}{m^2}\right).$$

- Pion decay constant, form factors and wave-lengths up to five-loop order (complete automatization limited by the computer calculation time)[Bijnens,Carlone,10]
- For the massless pions exact all-order equation ($\sim 2 - 3 \times 10^2$ loops within hour) [Kivel,et al,08]

Aim: to evaluate nucleon physical mass at LLog accuracy (up to possible highest order).

Pion-nucleon ChPT

- In order to bypass the counting problem of nucleon-pion ChPT we work in heavy baryon formulation
- The lowest order Lagrangians are

$$\mathcal{L}^{(0)}_{N\pi} = \bar{N}\left(iv^{\mu}D_{\mu} + g_A S^{\mu}u_{\mu}\right)N,$$

$$\begin{aligned}\mathcal{L}^{(1)}_{\pi N} \quad &= \quad \bar{N}_v\Bigg[\frac{(v\cdot D)^2 - D\cdot D - ig_A\{S\cdot D, v\cdot u\}}{2M} + c_1\mathrm{tr}\,(\chi_+) + \left(c_2 - \frac{g_A^2}{8M}\right)(v\cdot u)^2 \\ &+ c_3 u\cdot u + \left(c_4 + \frac{1}{4M}\right)i\epsilon^{\mu\nu\rho\sigma}u_{\mu}u_{\nu}v_{\rho}S_{\sigma}\Bigg]N_v.\end{aligned}$$

- The physical mass of nucleon is known up to two-loop order [Schindler,et al,07]

Structure of nucleon mass at "LLog" accuracy [Schindler,et al,07]

$$M_{\text{phys}} = M - \boxed{4c_1 m^2} - \frac{3\pi}{2} g_A^2 \frac{m^3}{(4\pi F)^2} \boxed{+ \frac{3}{4}\left(\frac{g_A^2}{M} + c_2 + 4c_3 - 8c_1\right)\frac{m^4}{(4\pi F)^2}\ln\left(\frac{\mu^2}{m^2}\right)} + \ldots$$

$\mathcal{L}^{(1)}$

LLog contribution, $\mathcal{L}^{(0)} + \mathcal{L}^{(1)}$

Structure of nucleon mass at "LLog" accuracy [Schindler,et al,07]

$\sim \sqrt{m_q}$ term, finite part of 1-loop

elder then LLog

$$M_{\rm phys} = M - 4c_1 m^2 - \frac{3\pi}{2} g_A^2 \frac{m^3}{(4\pi F)^2} + \frac{3}{4}\left(\frac{g_A^2}{M} + c_2 + 4c_3 - 8c_1\right)\frac{m^4}{(4\pi F)^2}\ln\left(\frac{\mu^2}{m^2}\right) + \ldots$$

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$\mathcal{L}^{(1)}$

LLog contribution, $\mathcal{L}^{(0)} + \mathcal{L}^{(1)}$

$\sim \sqrt{m_q}$ term, 2-loop NLLog
elder then LLog

$$+ \frac{3\pi}{8} g_A^2 (3 - 16 g_A^2) \frac{m^5}{(4\pi F)^4} \ln\left(\frac{\mu^2}{m^2}\right) - \frac{3}{4}\left(\frac{g_A^2}{M} + c_2 + 4c_3 - 6c_1\right)\frac{m^6}{(4\pi F)^4}\ln^2\left(\frac{\mu^2}{m^2}\right) + \ldots$$

2-loop LLog contribution

- The proper "LLog" accuracy contains the true LLog contribution, as well as, LLog non-analytical in m_q contribution (which is of NLLog nature).

Course of calculation

LLog contribution

For n'th order LLog coefficient one needs:
1) Lagrangian $\mathcal{L}_{N\pi}^{(2n(+1))}(\pi^0)$ for tree diagrams
2) 1-loop β-functions for all LECs with $\chi_{\text{order}} + N_{\text{pions}} = 2n(+1)$

- Calculation of 1-loop β is performed by FORM
  (typical number of diagrams for 4-loop LLog $\sim 10^4$)
- The next order Lagrangian is generated as terms necessary for renormalization (non-minimal Lagrangian [Bijnens,Carloni,10])
  (typical number of diagrams for 4-loop LLog $\sim 5\times 10^2$)
- $$\text{LLog coef.} = \frac{1}{[n/2]!}\left(\sum_k \beta^{(k)}_{\text{1-loop}} \frac{\partial}{\partial c^{(k)}}\right)^{n/2} \text{tree}^{(n)}$$

Non-analytical in m_q terms

For n'th order LLog non-analytical in m_q terms one additionally needs:
3) Finite part of one-loop diagrams without external pions

$$\sim \sqrt{m_q}\,\text{LLog coef.} = \frac{1}{[(n-1)/2]!}\left(\sum_k \beta^{(k)}_{\text{1-loop}} \frac{\partial}{\partial c^{(k)}}\right)^{(n-1)/2} (\text{finite part 1-loop})^{(n-1)}$$

$$M_{\text{phys}} = M + k_2 \frac{m^2}{M} + k_3 \frac{\pi m^3}{(4\pi F)^2} + k_4 \frac{m^4}{(4\pi F)^2 M} \ln\left(\frac{\mu^2}{m^2}\right) + k_5 \frac{\pi m^5}{(4\pi F)^4} \ln\left(\frac{\mu^2}{m^2}\right) + \cdots$$

LLog coefficients

k_2 $-4c_1 M$

k_4 $\frac{3}{4}\left(g_A^2 + (c_2 + 4c_3 - 8c_1)M\right)$ [Bernard,et al,92]

k_6 $-\frac{3}{4}\left(g_A^2 + (c_2 + 4c_3 - 6c_1)M\right)$ [Schindler,et al,07]

k_8 $\frac{27}{8}\left(g_A^2 + (c_2 + 4c_3 - \frac{16}{3}c_1)M\right)$

k_{10} $-\frac{257}{32}\left(g_A^2 + (c_2 + 4c_3 - 5c_1)M\right)$ (4-loop)

Non-analytical in m_q LLog coefficients

k_3 $-\frac{3}{2}g_A^2$ [Bernard,et al,92]

k_5 $\frac{3g_A^2}{8}\left(3 - 16g_A^2\right)$ [McGovern,Birse,99]

k_7 $g_A^2\left(-18g_A^4 + \frac{35g_A^2}{4} - \frac{443}{64}\right)$

k_9 $\frac{g_A^2}{3}\left(-116g_A^6 + \frac{2537g_A^4}{20} - \frac{3569g_A^2}{24} + \frac{55609}{1280}\right)$

k_{11} $\frac{g_A^2}{2}\left(-95g_A^8 + \frac{5187407g_A^6}{20160} - \frac{449039g_A^4}{945} + \frac{16733923g_A^2}{60480} - \frac{298785521}{1935360}\right)$ (5-loop)

k_2	$-4c_1 M$
k_4	$\frac{3}{4}\left(g_A^2 + (c_2 + 4c_3 - 8c_1)M\right)$
k_6	$-\frac{3}{4}\left(g_A^2 + (c_2 + 4c_3 - 6c_1)M\right)$
k_8	$\frac{27}{8}\left(g_A^2 + (c_2 + 4c_3 - \frac{16}{3}c_1)M\right)$
k_{10}	$-\frac{257}{32}\left(g_A^2 + (c_2 + 4c_3 - 5c_1)M\right)$

Peculiarity 1

No higher powers of g_A, only g_A^2.

- Consequence of Lorentz invariance (?)
- It implies that: *diagrams with odd-number-of-pion vertices do not contribute to the LLog coefficient of nucleon mass.*
  Note: these diagrams are non-zero, but cancel with each other within the solution of pole-equation.
- Supposing that this behavior holds at higher orders we can suppress higher power during calculation (significant speed improvement!)

k_2	$-4c_1 M$
k_4	$\frac{3}{4}\left(g_A^2 + (c_2 + 4c_3 - 8c_1)M\right)$
k_6	$-\frac{3}{4}\left(g_A^2 + (c_2 + 4c_3 - 6c_1)M\right)$
k_8	$\frac{27}{8}\left(g_A^2 + (c_2 + 4c_3 - \frac{16}{3}c_1)M\right)$
k_{10}	$-\frac{257}{32}\left(g_A^2 + (c_2 + 4c_3 - 5c_1)M\right)$
k_{12}	$\frac{115}{3}\left(g_A^2 + (c_2 + 4c_3 - \frac{24}{5}c_1)M\right)$

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  Note: these diagrams are non-zero, but cancel with each other within the solution of pole-equation.
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k_2	$-4c_1 M$
k_4	$\frac{3}{4}\left(g_A^2 + (c_2 + 4c_3 - 4c_1)M\right) - 3c_1 M$
k_6	$-\frac{3}{4}\left(g_A^2 + (c_2 + 4c_3 - 4c_1)M\right) + \frac{3}{2}c_1 M$
k_8	$\frac{27}{8}\left(g_A^2 + (c_2 + 4c_3 - 4c_1)M\right) - \frac{9}{2}c_1 M$
k_{10}	$-\frac{257}{32}\left(g_A^2 + (c_2 + 4c_3 - 4c_1)M\right) + \frac{257}{32}c_1 M$
k_{12}	$\frac{115}{3}\left(g_A^2 + (c_2 + 4c_3 - 4c_1)M\right) - \frac{92}{3}c_1 M$

Peculiarity 2

Universal structure of the expression

$$k_{2n} = b_n \left(\frac{-3c_1 M}{n-1} + \frac{3}{4}\left(g_A^2 + (c_2 + 4c_3 - 4c_1)M\right) \right)$$

- **Scientific guess:** Coefficients b_n are related to the pure pion physics.
- **Indeed:** They coincides with the LLog expansion of m^4_{phys} (accidentally?)

k_2	$-4c_1 M$
k_4	$\frac{3}{4}\left(g_A^2 + (c_2 + 4c_3 - 4c_1)M\right) - 3c_1 M$
k_6	$-\frac{3}{4}\left(g_A^2 + (c_2 + 4c_3 - 4c_1)M\right) + \frac{3}{2}c_1 M$
k_8	$\frac{27}{8}\left(g_A^2 + (c_2 + 4c_3 - 4c_1)M\right) - \frac{9}{2}c_1 M$
k_{10}	$-\frac{257}{32}\left(g_A^2 + (c_2 + 4c_3 - 4c_1)M\right) + \frac{257}{32}c_1 M$
k_{12}	$\frac{115}{3}\left(g_A^2 + (c_2 + 4c_3 - 4c_1)M\right) - \frac{92}{3}c_1 M$

LLog expression for all orders

We have all-order conjecture for LLog expression for nucleon mass

$$M = M_{\text{phys}} + \frac{3}{4} m_{\text{phys}}^4 \frac{\log\left(\frac{\mu^2}{m_{\text{phys}}^2}\right)}{(4\pi F)^2} \left(\frac{g_A^2}{M_{\text{phys}}} - 4c_1 + c_2 + 4c_3\right) - \frac{3c_1}{(4\pi F)^2} \int_{m_{\text{phys}}^2}^{\mu^2} m_{\text{phys}}^4(\mu') \, \frac{d\mu'^2}{\mu'^2}.$$

- Knowledge of pion mass LLog expansion up to six-order, allows us to guess two more coefficients.

k_2	$-4c_1 M$
k_4	$\frac{3}{4}\left(g_A^2 + (c_2 + 4c_3 - 4c_1)M\right) - 3c_1 M$
k_6	$-\frac{3}{4}\left(g_A^2 + (c_2 + 4c_3 - 4c_1)M\right) + \frac{3}{2}c_1 M$
k_8	$\frac{27}{8}\left(g_A^2 + (c_2 + 4c_3 - 4c_1)M\right) - \frac{9}{2}c_1 M$
k_{10}	$-\frac{257}{32}\left(g_A^2 + (c_2 + 4c_3 - 4c_1)M\right) + \frac{257}{32}c_1 M$
k_{12}	$\frac{115}{3}\left(g_A^2 + (c_2 + 4c_3 - 4c_1)M\right) - \frac{92}{3}c_1 M$
k_{14}	$-\frac{186515}{1536}\left(g_A^2 + (c_2 + 4c_3 - 4c_1)M\right) + \frac{186515}{2304}c_1 M$
k_{16}	$\frac{153149887}{259200}\left(g_A^2 + (c_2 + 4c_3 - 4c_1)M\right) - \frac{153149887}{453600}c_1 M$ (7-loop)

LLog expression for all orders

We have all-order conjecture for LLog expression for nucleon mass

$$M = M_{\text{phys}} + \frac{3}{4} m_{\text{phys}}^4 \frac{\log\left(\frac{\mu^2}{m_{\text{phys}}^2}\right)}{(4\pi F)^2} \left(\frac{g_A^2}{M_{\text{phys}}} - 4c_1 + c_2 + 4c_3\right) - \frac{3c_1}{(4\pi F)^2} \int_{m_{\text{phys}}^2}^{\mu^2} m_{\text{phys}}^4(\mu') \, \frac{d\mu'^2}{\mu'^2}.$$

Conclusion

- The RG technique is elaborated for the heavy-baryon nucleon-pion ChPT.
- LLog and (N)LLog non-analytical in m_q coefficients for the nucleon mass up to 4- and 5-loop order.
- Using conjectures which follows from the form of LLog coefficients we have obtained coefficient up to 7-loop order.
- We suggests an all order LLog expression for the nucleon mass

Nearest future result

- Relativistic calculation (infrared-renormalization scheme)
- LLog coefficients for axial coupling, form factors and nucleon-pion wave-lengths.