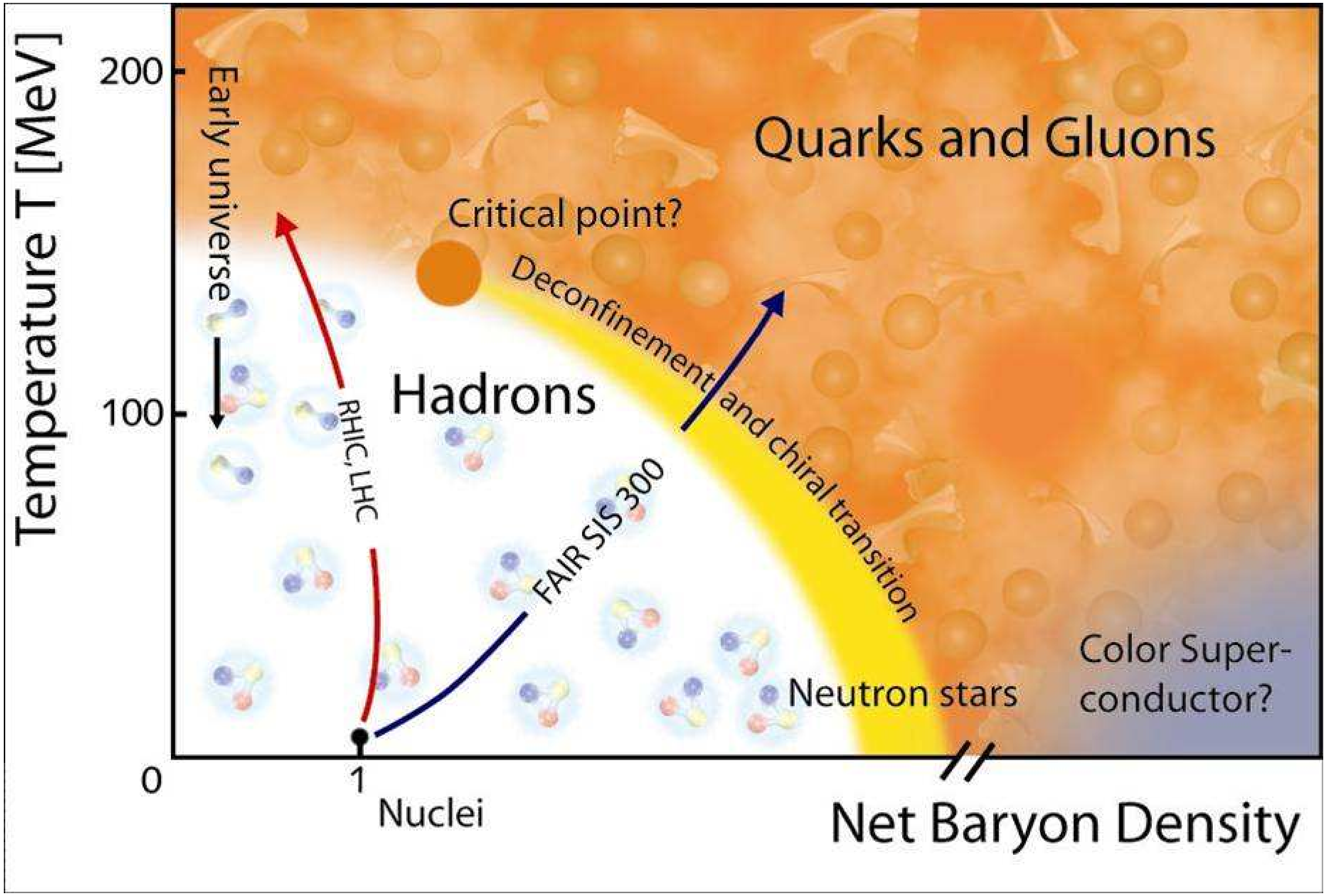
QCD at nonzero chemical potential: recent progress on the lattice

Gert Aarts

Swansea University
Prifysgol Abertawe

QCD phase diagram



a well-known possibility

QCD phase diagram

partition function

$$Z = \int DU\, D\bar{\psi} D\psi\, e^{-S} = \int DU\, e^{-S_{\rm YM}} \det M$$

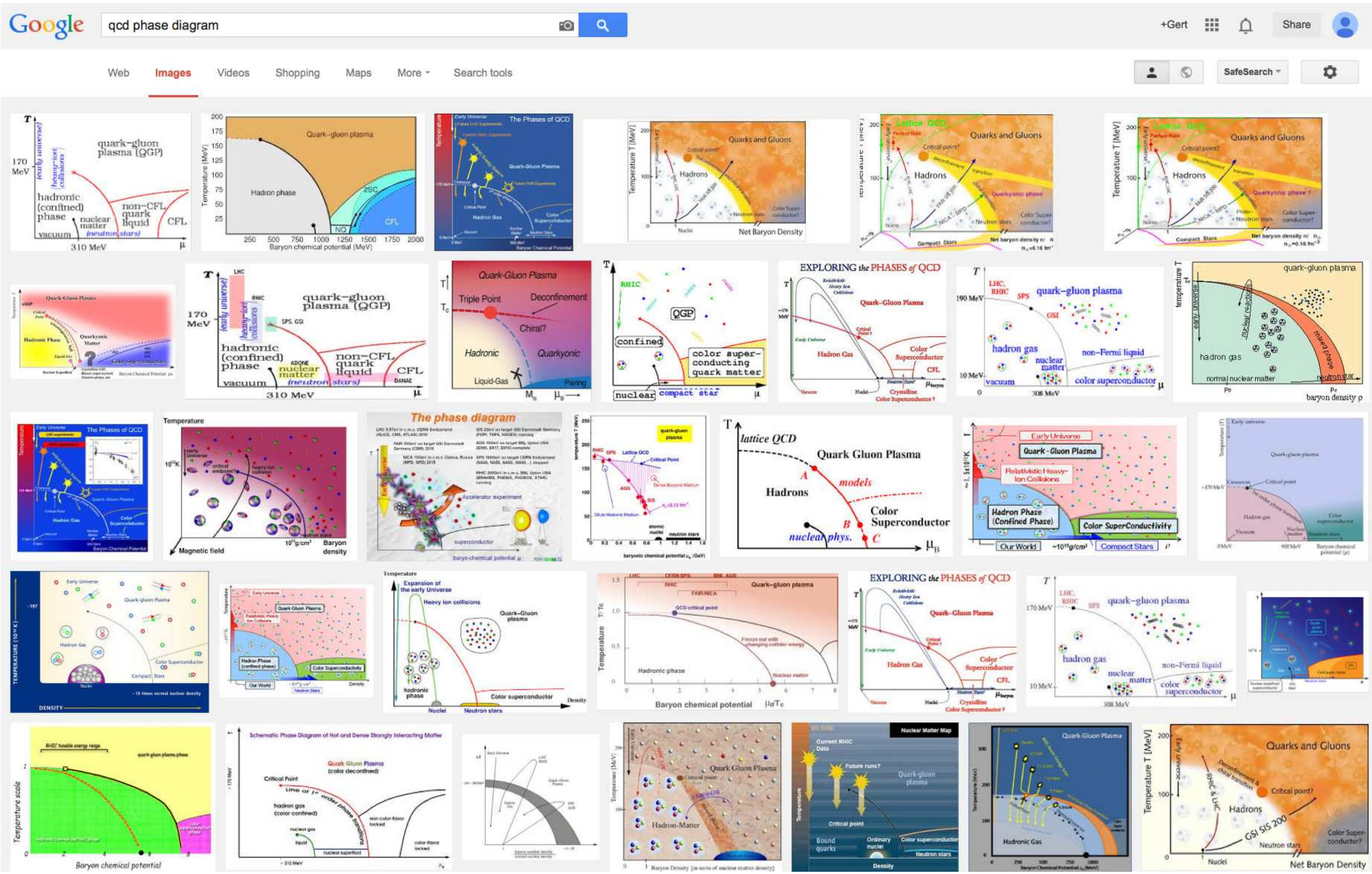
at nonzero quark chemical potential

$$[\det M(\mu)]^* = \det M(-\mu^*)$$

- fermion determinant is complex
- straightforward importance sampling not possible
- lattice QCD: sign problem

$\Rightarrow$ phase diagram has not yet been determined non-perturbatively

Many QCD phase diagrams



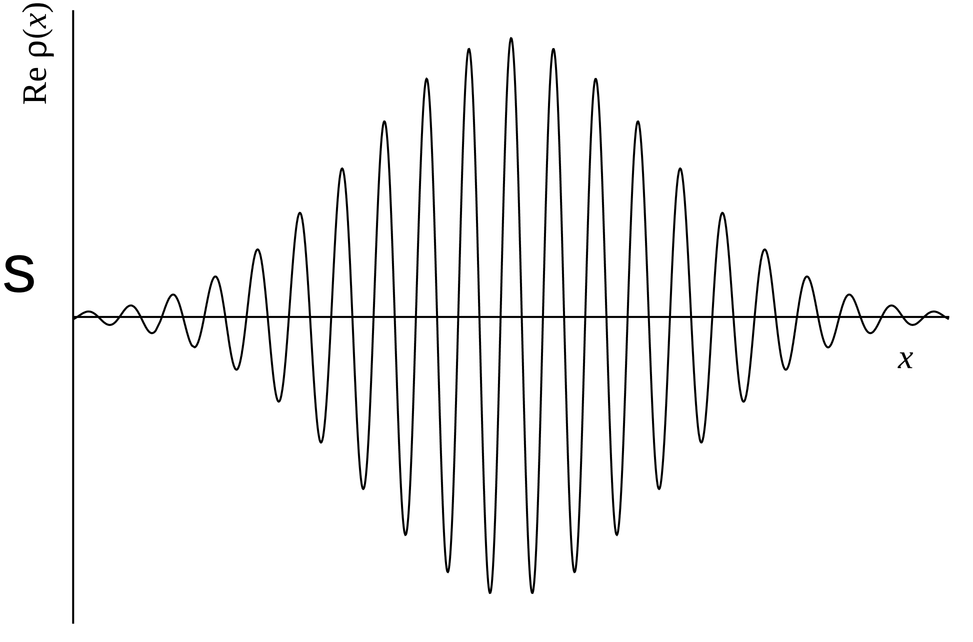
Overlap problem

- complex weight is a hard problem: cannot be ignored

$$\det M(\mu) = |\det M(\mu)| e^{i\theta}$$

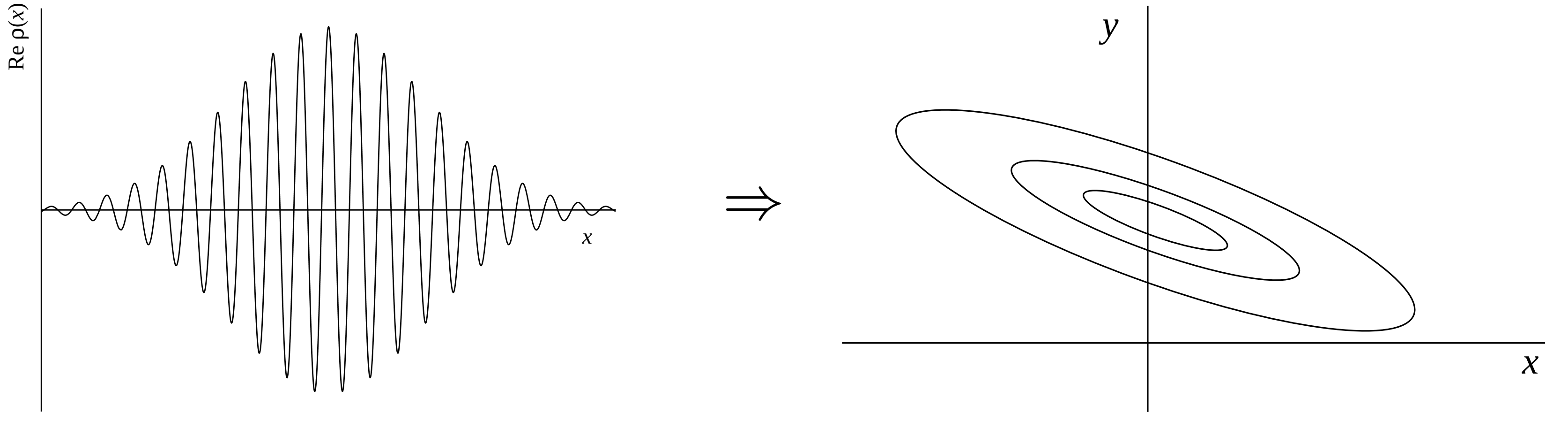
- correct physics easily destroyed
  (e.g. by phase-quenching)

dominant configurations
in the path integral?



Complexified field space

dominant configurations in the path integral?



conjecture: $\exists$ a real and positive distribution $P(x,y)$

$$\int dx\,\rho(x)O(x) = \int dxdy\,P(x,y)O(x+iy)$$

$\Rightarrow$ can be obtained as solution of stochastic process

complex Langevin dynamics

Parisi 83, Klauder 83

Outline

- complex Langevin dynamics
- gauge theories: from SU(N) to SL($N, \mathbb{C}$)
- recent developments:
  - heavy dense QCD
  - full QCD
  - hopping parameter expansion to all orders
  - SU(3) with a θ-term (not shown here)
- summary and outlook

with Nucu Stamatescu, Erhard Seiler, Dénes Sexty

Benjamin Jäger, Pietro Giudice, Jan Pawlowski

Lorenzo Bongiovanni, Felipe Attanasio, Frank James

reviews: 1302.3028, 1303.6425

Complex Langevin dynamics: basics

partition function $Z = \int dx\, e^{-S(x)}$ $\qquad S(x) \in \mathbb{C}$

- reach equilibrium distribution à la Brownian motion
- no importance sampling, instead stochastic process

$$\dot{x} = -\mathrm{Re}\,\partial_z S(z) + \eta \qquad \langle \eta(t)\eta(t')\rangle = 2\delta(t-t')$$
$$\dot{y} = -\mathrm{Im}\,\partial_z S(z) \qquad S(z) = S(x+iy)$$

- associated distribution $P(x,y;t)$

$$\langle O(x+iy)(t)\rangle = \int dxdy\, P(x,y;t) O(x+iy)$$

- $x(t), y(t)$ Langevin eq $\Leftrightarrow$ $P(x,y;t)$ Fokker-Planck eq

$$\dot{P}(x,y;t) = \left[\partial_x \left(\partial_x + \mathrm{Re}\,\partial_z S\right) + \partial_y \mathrm{Im}\,\partial_z S\right] P(x,y;t)$$

Complex Langevin dynamics

does it work?

- for real actions: stochastic quantisation Parisi & Wu 81
- equivalent to path integral quantisation
- for complex actions: formal proof was notably absent

recent progress:

- theoretical foundation given[1]
- practical criteria for correctness formulated[2]
- severe sign and Silver Blaze problems solved[3]
- first results for gauge theories[4] and even full QCD[5]

[1]0912.3360 [1,2]1101.3270 1306.3075 [3]0810.2089 1006.0332

[4]0807.1597 1211.3709 [5]1307.7748 1408.3770 +...

Localised distributions

crucial role played by distribution $P(x,y)$

if

- the action is holomorphic (no $\log\det$!)

and

- the distribution is localised, i.e.

$$P(x,y) = 0 \text{ for } |y| > y_{\max} \text{ [or } P(x,y) \to 0 \text{ fast enough]}$$

then

- correct result is obtained GA, Seiler & Stamatescu 0912.3360

for meromorphic drifts – with poles –, problems *may* appear

but not necessarily so

Mollgaard & Splittorff 1309.4335, Greensite 1406.4558

Gauge theories

Gauge theories

SU(N) gauge theory: complexification to SL($N, \mathbb{C}$)

- links $U \in$ SU(N): complex Langevin update

$$U(n+1) = R(n)\, U(n) \qquad\qquad R = \exp\left[i\lambda_a\left(\epsilon K_a + \sqrt{\epsilon}\eta_a\right)\right]$$

Gell-Mann matrices λ_a ($a = 1, \ldots N^2 - 1$)

- drift: $\quad K_a = -D_a(S_{\rm YM} + S_{\rm F}) \qquad S_{\rm F} = -\ln\det M$

Gauge theories

SU(N) gauge theory: complexification to SL($N,\mathbb{C}$)

- links $U \in$ SU(N): complex Langevin update

$$U(n+1) = R(n)\, U(n) \qquad\qquad R = \exp\left[i\lambda_a\left(\epsilon K_a + \sqrt{\epsilon}\eta_a\right)\right]$$

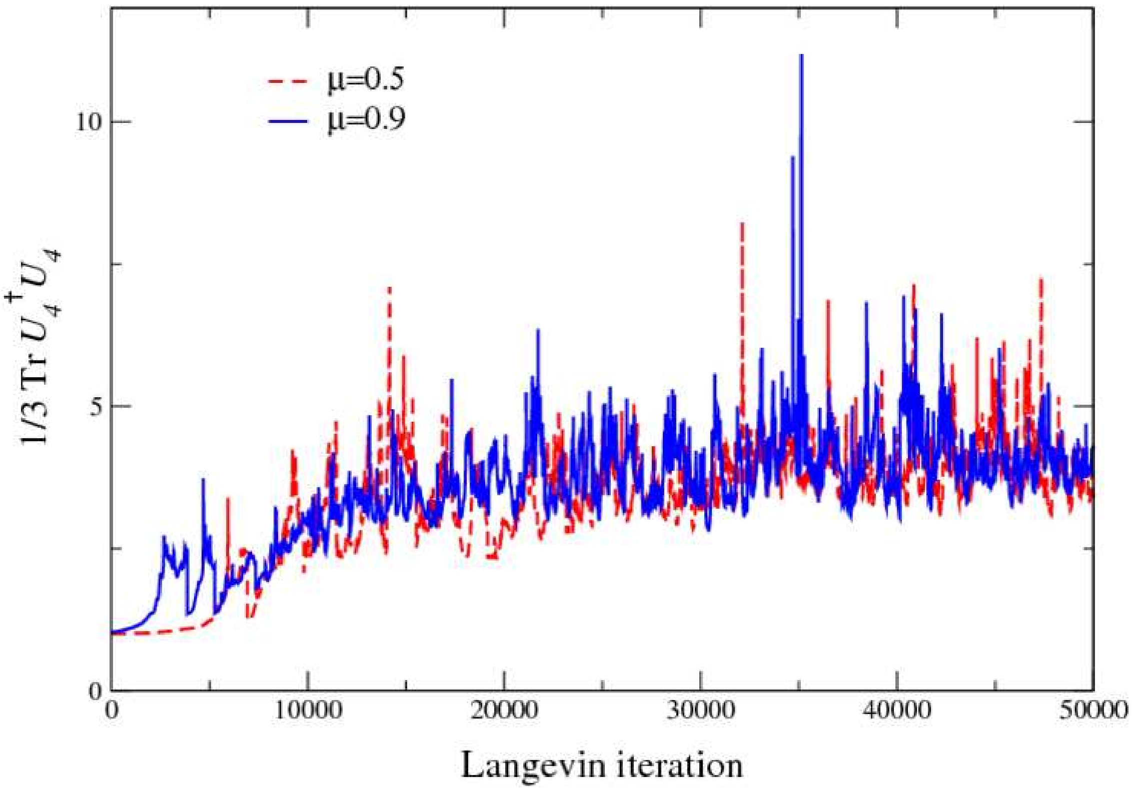
Gell-Mann matrices λ_a ($a = 1, \ldots N^2 - 1$)

- drift: $K_a = -D_a(S_{\rm YM} + S_{\rm F}) \qquad S_{\rm F} = -\ln\det M$
- complex action: $K^\dagger \neq K \Leftrightarrow U \in$ SL($N,\mathbb{C}$)
- deviation from SU(N): unitarity norms

$$\frac{1}{N}{\rm Tr}\left(UU^\dagger - \mathbb{1}\right) \geq 0 \qquad\qquad \frac{1}{N}{\rm Tr}\left(UU^\dagger - \mathbb{1}\right)^2 \geq 0$$

Gauge theories

deviation from SU(3): unitarity norm $\quad \frac{1}{3}\mathrm{Tr}\, UU^\dagger \geq 1$



heavy dense QCD, 4^4 lattice with $\beta = 5.6$, $\kappa = 0.12$, $N_f = 3$

GA & Stamatescu 0807.1597

Gauge theories

controlled evolution: stay close to SU(N) submanifold when

- small chemical potential μ
- small non-unitary initial conditions
- in presence of roundoff errors

Gauge theories

controlled evolution: stay close to SU(N) submanifold when

- small chemical potential μ
- small non-unitary initial conditions
- in presence of roundoff errors

in practice this is not the case

$\Rightarrow$ unitary submanifold is unstable!

- process will not stay close to SU(N)
- distributions not localised
- wrong results in practice, non-analytic around $\mu^2 \sim 0$

Unstable gauge theories

- uncontrolled dynamics in gauge directions
- unitarity norms grow exponentially
- control those with gauge cooling

Seiler, Sexty & Stamatescu 1211.3709

see also GA, Bongiovanni, Seiler, Sexty & Stamatescu 1303.6425

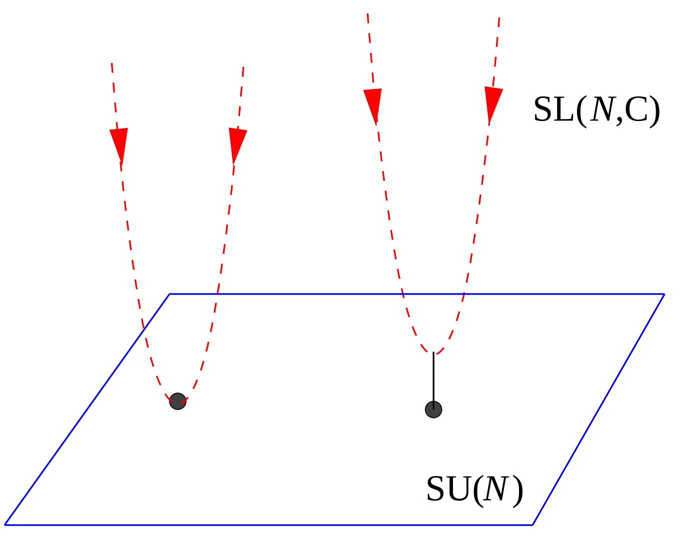
$$U_k \to \Omega_k U_k \Omega_{k+1}^{-1} \qquad \Omega_k = e^{-\alpha f_a^k \lambda_a} \qquad \alpha > 0$$

choose f_a^k as the gradient of the unitarity norm d

after one update: linearise

$$\mathrm{d}' - \mathrm{d} = -\frac{\alpha}{N}(f_a^k)^2 + \mathcal{O}(\alpha^2) \leq 0$$

reduce distance from SU(N)



Langevin with gauge cooling

in QCD:

- unitary submanifold very unstable
- gauge cooling essential
- alternate Langevin updates with cooling updates

recent and new results for

- heavy dense QCD
- full QCD
- hopping parameter expansion to all orders
- SU(3) with a θ-term (not shown here)

Heavy dense QCD

Benjamin Jäger, **Felipe Attanasio**, GA, Stamatescu, Sexty, Seiler

Heavy dense QCD

consider static quarks: fermion determinant simplifies

$$\det M = \prod_{\mathbf{x}} \det\left(1 + h e^{\mu/T} \mathcal{P}_{\mathbf{x}}\right)^2 \det\left(1 + h e^{-\mu/T} \mathcal{P}_{\mathbf{x}}^{-1}\right)^2$$

with $h = (2\kappa)^{N_\tau}$ and $\mathcal{P}^{(-1)}$ (conjugate) Polyakov loops

- full Wilson gauge action is included
- nontrivial phase diagram:
  - thermal deconfinement transition (as in pure glue)
  - μ-driven transition at $\mu_c \sim -\ln(2\kappa)$

test case for full QCD

Heavy dense QCD

consider static quarks: fermion determinant simplifies

$$\det M = \prod_{\mathbf{x}} \det\left(1 + he^{\mu/T}\mathcal{P}_{\mathbf{x}}\right)^2 \det\left(1 + he^{-\mu/T}\mathcal{P}_{\mathbf{x}}^{-1}\right)^2$$

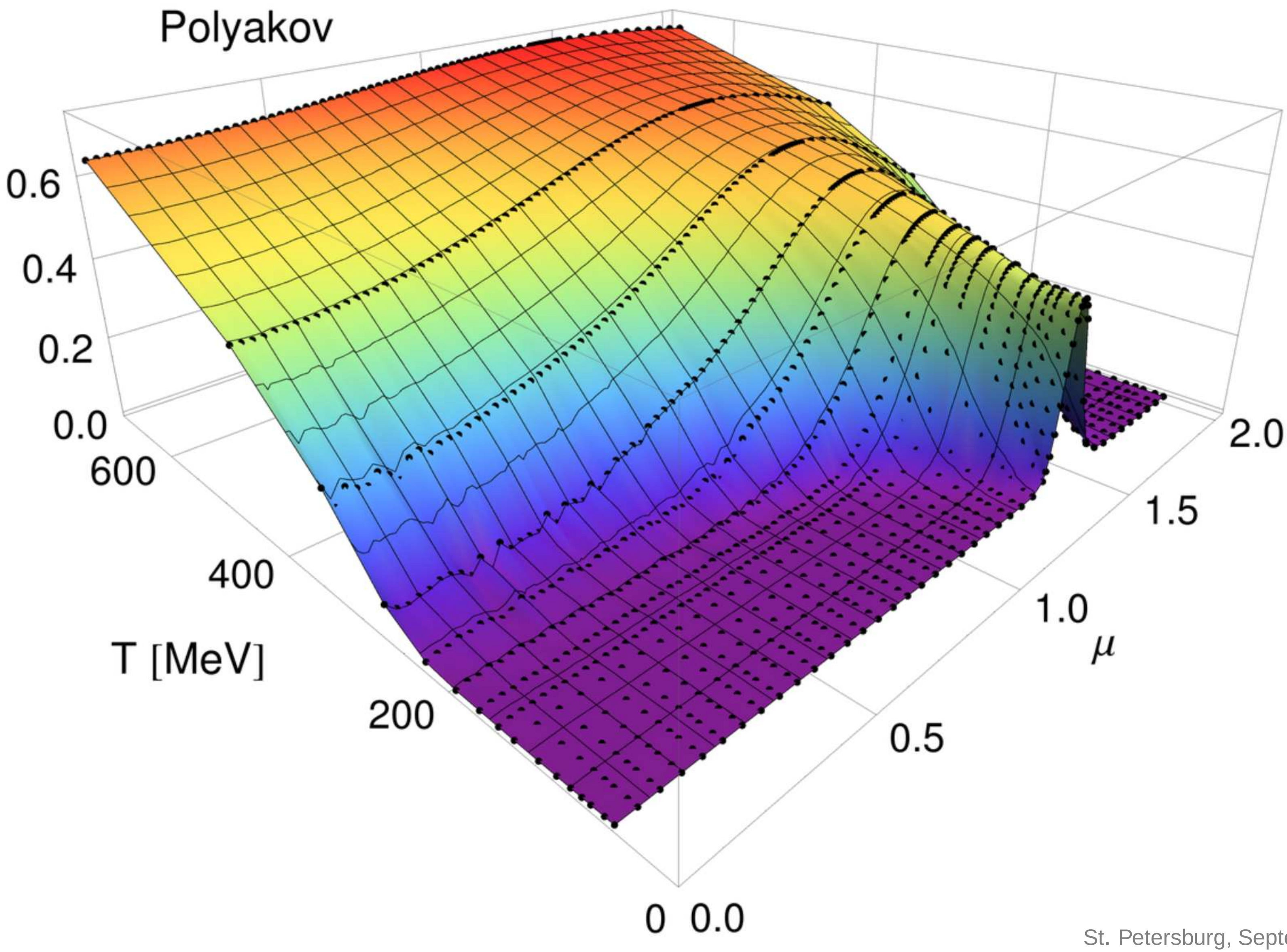
with $h = (2\kappa)^{N_\tau}$ and $\mathcal{P}^{(-1)}$ (conjugate) Polyakov loops

preliminary results for Polyakov loop and density
and their susceptibilities

- $\beta = 5.8$ ($a \sim 0.15$ fm)
- $\kappa = 0.12$ ($\mu_c \sim -\ln(2\kappa) = 1.43$)
- volume $8^3 \times N_\tau$
- $N_\tau = 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16\ 18\ 20\ 24\ 28$

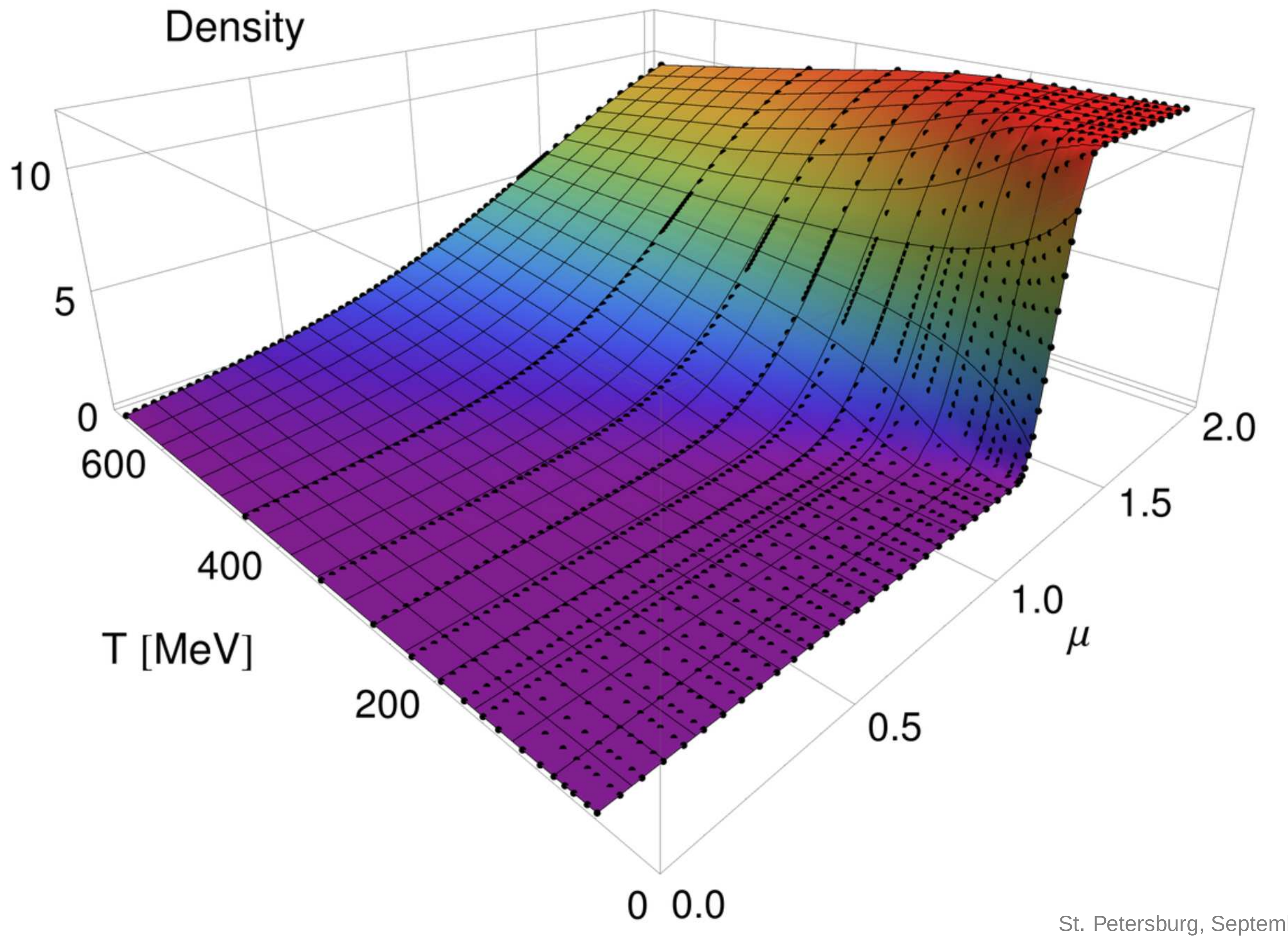
Heavy dense QCD

Polyakov loop



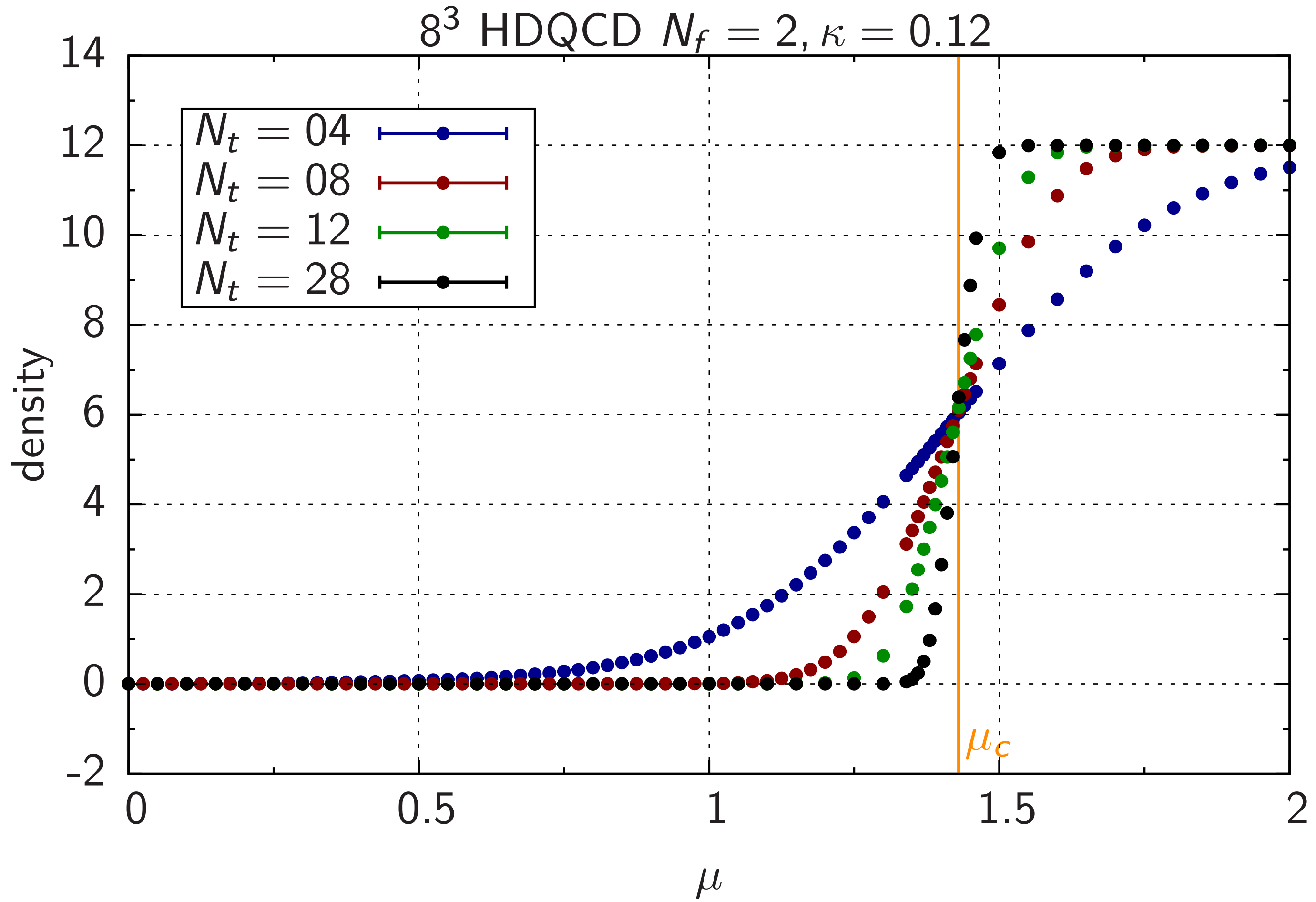
Heavy dense QCD

density



Heavy dense QCD

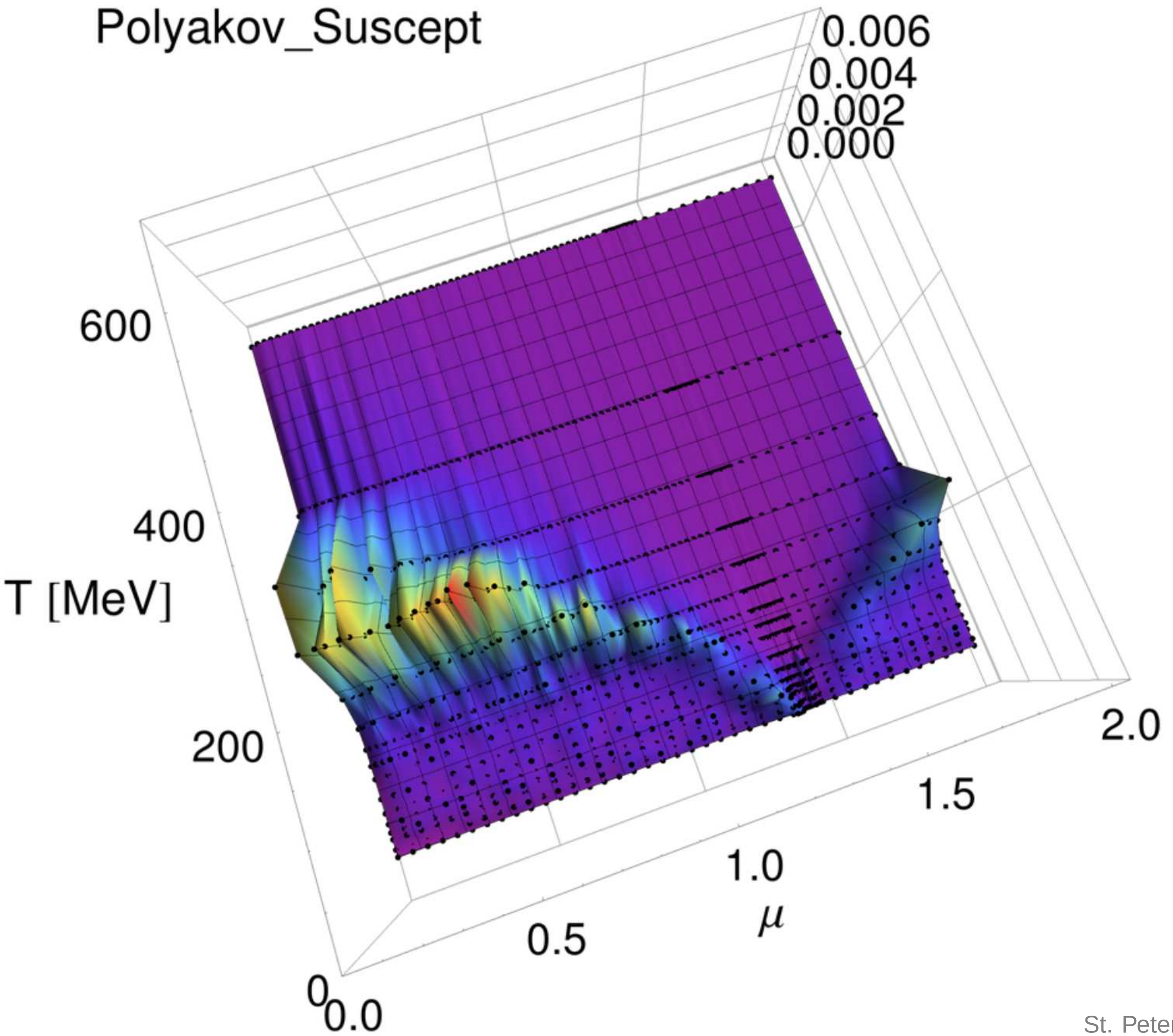
density $\mu_c = -\ln(2\kappa) = 1.43$ $n_{\text{sat}} = 12$



first order transition at $T = 0$ (expected) Silver Blaze

Heavy dense QCD

Polyakov loop susceptibility (and phase diagram)



Full QCD at nonzero density

Dénes Sexty 1307.7748

Full QCD

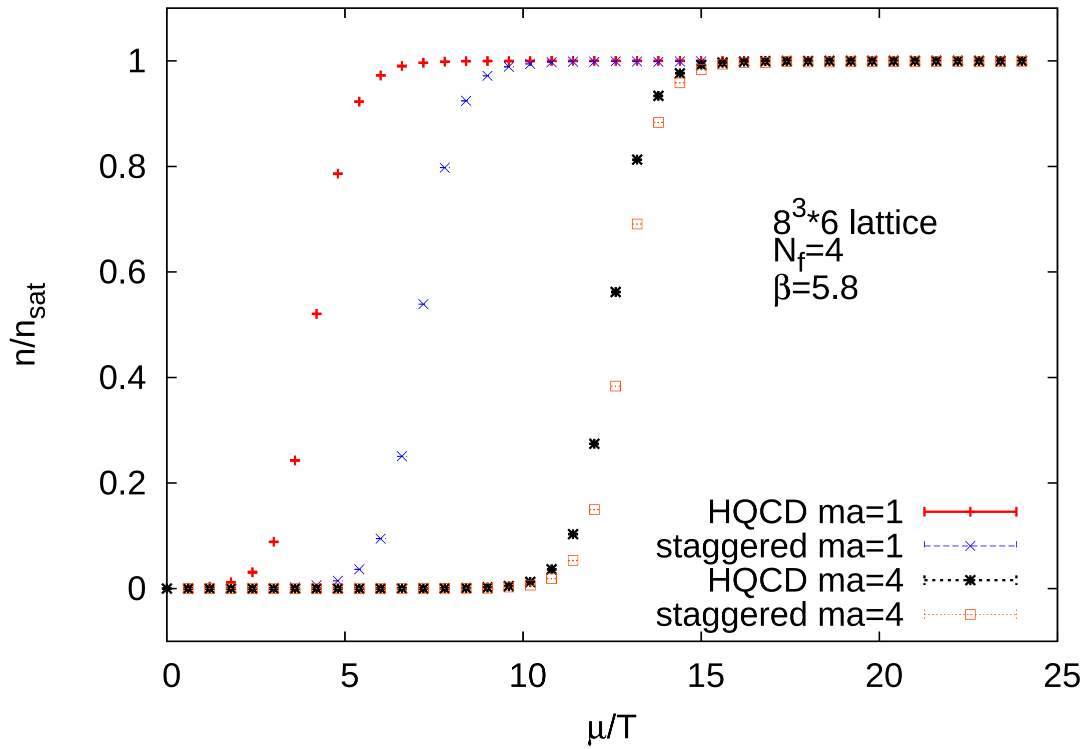
first application to full QCD

- fermion determinant: additional drift term in CLE
- requires inversion of fermion matrix
- stochastic inversion using conjugate gradient

staggered fermions with 4 flavours
(Wilson fermions as well)

- monitor unitarity norm, log det, distributions, . . .
- compare with HDQCD for heavy quarks and reweighting for light quarks

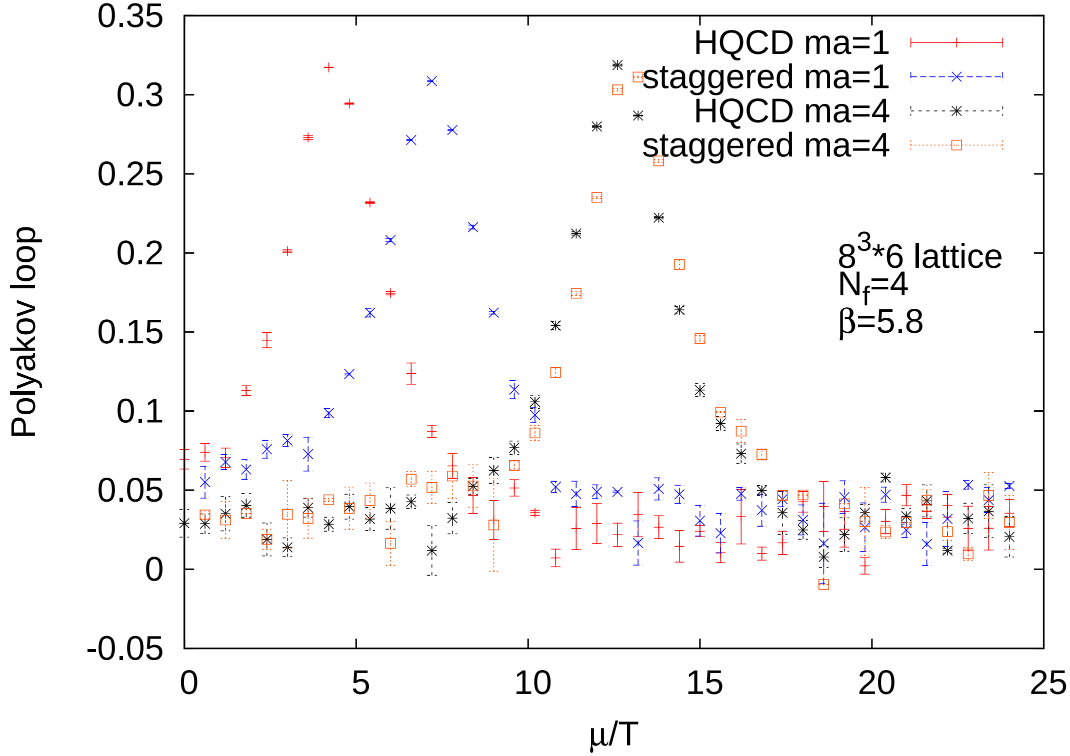
Full QCD



density / density$_{\mathrm{sat}}$ vs μ/T

- for heavier quarks and lighter quarks
- comparison with HDQCD

Full QCD

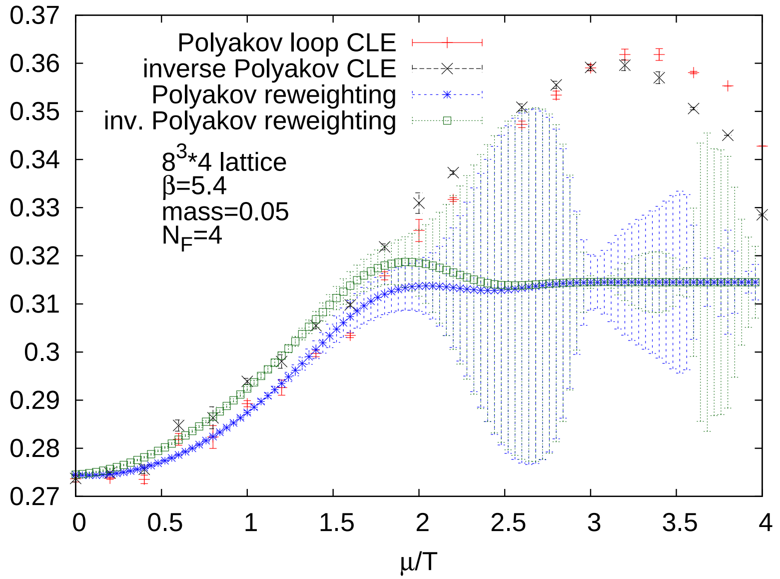


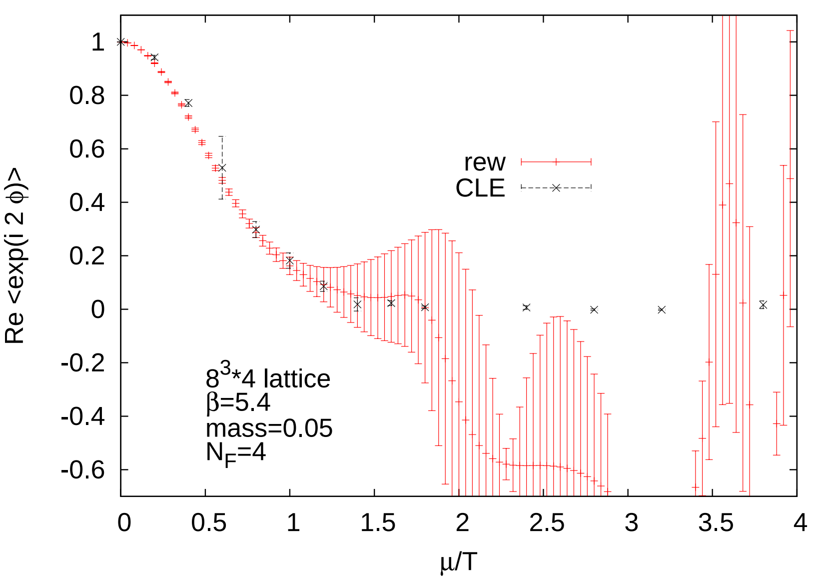
Polyakov loop vs μ/T

- for heavier quarks and lighter quarks
- comparison with HDQCD

Full QCD

comparison with reweighting





Polyakov loop vs μ/T — average sign

- for light quarks
- agreement until reweighting breaks down

Fodor, Katz & Sexty in prep

From HDQCD to full QCD:

hopping parameter expansion to all orders

Dénes Sexty, GA, Seiler, Stamatescu 1408.3770

From HDQCD to full QCD

heavy dense QCD

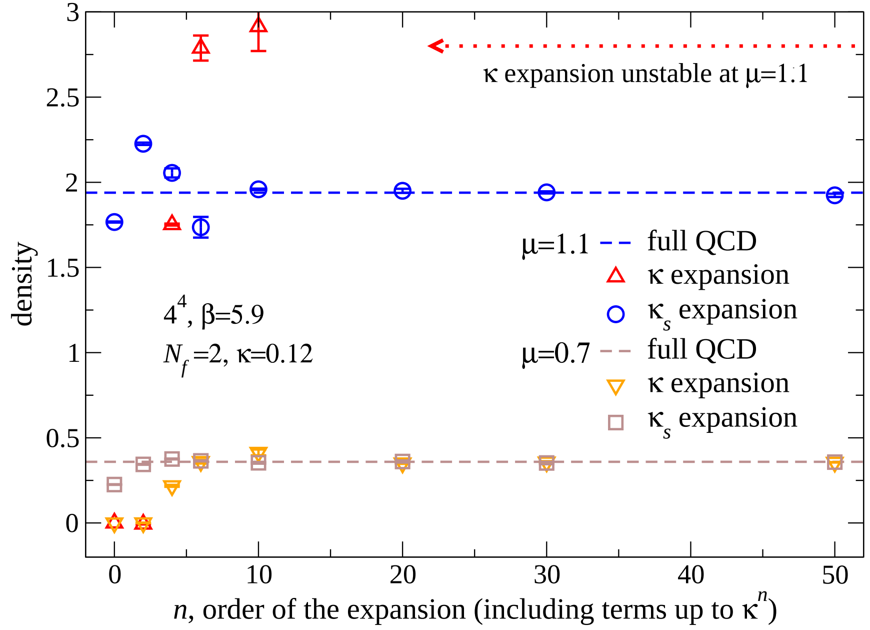
- leading order term in expansion in inverse quark kinetic mass: static limit
- several shortcomings: e.g. $m_B/3 = m_\pi/2 = m_q$, immediate saturation after onset

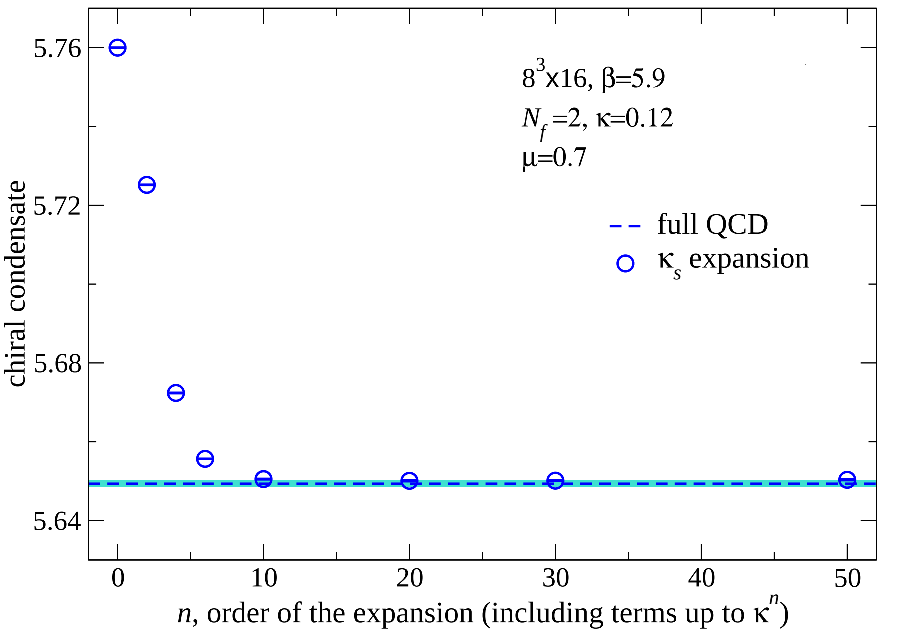
improve and make connection with full QCD

- systematic expansion of quark determinant to all orders in spatial hopping parameter κ_s
- truncate at high order, up to $\mathcal{O}(\kappa_s^{50})$
- determinant still complex: simulate with Langevin
- compare with full QCD results for Wilson fermions

From HDQCD to full QCD

convergence of hopping parameter expansion





density (4^4) chiral condensate ($8^3 \times 16$)

- agreement between expansion and full result
- important cross check
- implies $\log\det$ not a problem in this case

Summary and outlook

complex Langevin: recent progress for gauge theories

- better mathematical *and* practical understanding
- gauge cooling for SU(N) gauge theories
- work in progress for
  - heavy dense QCD
  - full QCD at nonzero density
  - hopping parameter expansion to all orders
  - SU(3) $+\ \theta$-term
  - role of determinant

towards the phase diagram of QCD

many things to do!