Chiral effects and physics of chiral media

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In external EM field there is non-conservation of chirality - axial anomaly of QFT:

$$\partial_\mu J_5^\mu = \frac{1}{8\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{1}{2\pi^2} E \cdot B$$

This microscopic effect has macroscopic manifestation in chiral medium - chiral effects, which are transport phenomena closely tied with axial anomaly:

$$\vec{J}_5(x) = \frac{\mu}{2\pi^2}\vec{B} + \left(\frac{\mu^2+\mu_5^2}{2\pi^2} + \frac{T^2}{6}\right)\vec{\Omega}$$

$$\vec{J}(x) = \frac{\mu_5}{2\pi^2}\vec{B} + \frac{\mu\mu_5}{\pi^2}\vec{\Omega}$$

Chiral effects were firstly obtained about 80s in papers of A. Vilenkin[1]. One could consider the Dirac equation for one chirality in a dense matter and external $B \parallel x_3$:

$$\gamma \cdot (\partial + iA)\psi = 0 \quad , \quad (1 + \gamma_5)\psi = 0$$

which has LL energy spectrum $\epsilon_n^2 = p_z^2 + 2enB$. For the equilibrium current we have

$$J_z = \sum_n \int dp_y dp_z f(\epsilon_{np_z} - \mu_R)\bar{\psi}_{np_y p_z}\gamma^3\psi_{np_y p_z}$$

and it appears that the only non-zero contribution is $n = 0$ LL:

$$J_z = \frac{e^2 \mu_R B}{4\pi^2}$$

[1] see e.g. A. Vilenkin, Phys. Rev. D 22, 3080

On the other hand in the linear response theory one could use Kubo formula to evaluate the current for the same system:

$$J_i(x) = \int dx' \Pi^R_{ij} A_j(x') = \sigma_B B_i$$

$$\sigma_B = \lim_{k\to 0} \sum_{ij} \frac{i}{2k_j} \epsilon_{ijl} \Pi_{ij}|_{\omega=0} = \frac{\mu_R}{4\pi^2},$$

with the same answer as above! The answer is linear in μ_R so one could consider expand it in powers of μ_R:

$$\langle J_i(\vec{q}) J_j(-\vec{q}) \rangle = \langle J_i(\vec{q}) J_j(-\vec{q}) J_0(0) \rangle \mu_R$$

and at this point the connection to the AA is more clear. Nevertheless it is not AA in the usual sense!!

The similar question appeared in holography [2]. Here the starting point is the five dimensional Einstein-Maxwell action:

$$S = -\frac{1}{16\pi G_5}\int \left(\sqrt{-g}\,(R + 12 - F^2) - \frac{4\kappa}{3}\epsilon^{MNOPQ}A_M F_{NO} F_{PQ}\right) d^5x$$

It was shown that the theory at the boundary could be described just by the relativistic hydro:

$$\partial_\mu T^{\mu\nu} = 0\ ,\ \ T^{\mu\nu} = wu^\mu u^\nu + Pg^{\mu\nu} + \tau^{(1)\mu\nu}$$
$$\partial_\mu J^\mu = 0 \quad , \quad J^\mu = nu^\mu + \nu^{(1)\mu},$$

where $\nu^{(1)}$ and $\tau^{(1)\mu\nu}$ are corrections of the first order in spatial gradients.

[2] J. Erdmenger et al, JHEP 0901 (2009) 055

In the usual relativistic hydro the first order corrections are[3]:

$$\nu^{\mu} = -\sigma T P^{\mu\nu}\partial_{\nu}\left(\frac{\mu}{T}\right) + \sigma F^{\mu\nu} u_{\nu}$$
$$\tau^{\mu\nu} = -\eta P^{\mu\alpha}P^{\nu\beta}(\partial_{\alpha}u_{\beta} + \partial_{\beta}u_{\alpha}) - (\zeta - \frac{2}{3}\eta)P^{\mu\nu}\partial\cdot u$$

but the answer from holography was strange. It contains a previously forbidden term:

$$\nu^{\mu} = -\sigma T P^{\mu\nu}\partial_{\nu}\left(\frac{\mu}{T}\right) + \sigma F^{\mu\nu} u_{\nu} + \boldsymbol{\xi\omega^{\mu}},$$

where $\omega^{\mu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_{\nu}\partial_{\alpha}u_{\beta}$.

Is holography described by hydro?

[3] Landau course vol. VI

To resolve this inconsistency one should consider an ideal hydrodynamics modified by chiral anomaly[4]:

$$\partial_\mu T^{\mu\nu} = 0 \ , \ \partial_\mu J^\mu = CE \cdot B$$

where as usually

$$T^{\mu\nu} = wu^\mu u^\nu + Pg^{\mu\nu} + \tau^{\mu\nu} \ , \ J^\mu = nu^\mu + \nu^\mu$$

and combining EOMs as $u_\nu \partial_\mu T^{\mu\nu} + \mu \partial J$ we find

$$\partial \cdot \left(su - \frac{\mu}{T}\nu \right) = -\frac{1}{T}\partial_\mu u_\nu \tau^{\mu\nu} - \nu \cdot \left(\partial \frac{\mu}{T} - \frac{E}{T} \right) - C\frac{\mu}{T} E \cdot B$$

this relation shows that $s^\alpha = su^\alpha - \frac{\mu}{T}\nu^\alpha$ is not a good choice for the entropy current

[4] D.T. Son and P. Surowka, Phys.Rev.Lett. 103 (2009) 191601

The simplest possible modification is

$$s^\alpha = su^\alpha - \frac{\mu}{T}\nu^\alpha + D\omega^\alpha + D_B B^\alpha \quad , \quad \nu^\alpha = \xi\omega^\alpha + \xi_B B^\alpha$$

It really gives possible choice (in the ideal case) for s^α and fixes by that

$$\nu^{(1)\alpha} = C\left(\mu - \frac{1}{2}\frac{\mu^2 n}{\epsilon + P}\right)B^\alpha + C\left(\mu^2 - \frac{2}{3}\frac{\mu^3 n}{\epsilon + P}\right)\omega^\alpha$$

which gives exactly **the same** vortical contribution as in holography!

Moreover generalizing to the more realistic case of vector and axial currents one finds[5]:

$$\vec{J}_5(x) = \frac{\mu}{2\pi^2}\vec{B} + \left(\frac{\mu^2 + \mu_5^2}{2\pi^2} + \frac{T^2}{6}\right)\vec{\Omega}$$

$$\vec{J}(x) = \frac{\mu_5}{2\pi^2}\vec{B} + \frac{\mu\mu_5}{\pi^2}\vec{\Omega}$$

where C is taken for the usual axial anomaly.

[5] A. Sadofyev and MI, Phys.Lett.B 697 (2011); Y. Oz and YN, JHEP 1103 (2011)

To find connection between these effects and anomalies one could consider effective field theory[6]:

$$S_{eff} = \int dx \left(i\bar{\psi}\gamma^\rho D_\rho \psi + \mu u_\mu \bar{\psi}\gamma^\mu \psi + \mu_5 u_\mu \bar{\psi}\gamma^\mu \gamma_5 \psi \right) + S_{int}.$$

After calculating of anomaly and substituting hydro averaged currents we get

$$\partial_\mu \left(n_5 u^\mu + \frac{\mu^2 + \mu_5^2}{2\pi^2}\omega^\mu + \frac{\mu}{2\pi^2} B^\mu \right) = -\frac{1}{4\pi^2}\epsilon_{\mu\nu\alpha\beta}\partial^\mu A^\nu \partial^\alpha A^\beta$$

$$\partial_\mu \left(n u^\mu + \frac{\mu\mu_5}{\pi^2}\omega^\mu + \frac{\mu_5}{2\pi^2} B^\mu \right) = 0$$

with the same coefficients as above in front of each term.

[6] A. Sadofyev et al, Phys.Rev. D83 (2011) 105025

Let's compare CME with usual electric conductivity:

$$\vec{J} = \sigma_B \vec{B} \;\;,\;\; \vec{J} = \sigma_E \vec{E}$$

and for time time reversal quantities we have

$$\vec{J}^T = -\vec{J} \;\;,\;\; \vec{E}^T = +\vec{E} \;\;,\;\; \vec{B}^T = -\vec{B}$$

it means that effectively

$$\sigma_B^T = \sigma_B \;\;,\;\; \sigma_E^T = -\sigma_E$$

while it is obvious that σ_E must be positive.

So CME is of dissipation-free nature as it is in the case of Hall current which is also time reversal $J \sim E \times B$.

Radiative corrections[7]:

$$\delta J_5 \;=\; -\frac{\alpha_{el} eB\mu}{2\pi^3}\left(\ln\frac{2\mu}{m_f} + \ln\frac{m_\gamma^2}{m_f^2} + \frac{4}{3}\right) ,$$

Back reaction of a medium results in the:

$$\mathbf{curl} B = \sigma_B B \;\;\Rightarrow\;\; (\Delta + \sigma^2)B = 0$$

where is a pole on real axe which signals the instability[8]. Moreover partial sum of diagram series results in modification of CME:

$$\vec{J}_{CME} = \lim_{k\to 0}\frac{\sigma_B k^2}{k^2 - \sigma_B^2}\vec{B} = 0$$

This result has a partner in holography with spatially modulated field-current distribution[9]

[7] E.V. Gobar et al, Phys.Rev. D88 (2013) 2, 025025

[8] VPK, AS, VIZ , arXiv:1307.0138

[9] H. Ooguri and C-S. Park, Phys.Rev.Lett. 106 (2011) 061601

Chiral effects could be also reproduced through kinetic theory[10]. To do so let's consider firstly the semiclassical limit of EOM $(\sigma \cdot p)u_p = \pm|p|u_p$ for a chiral quasiparticle:

$$\dot{x} = \mathcal{B} \times \dot{p} + \nabla_p \epsilon_p$$
$$\dot{p} = -B \times \dot{x} - \nabla_x \epsilon_p$$

where $\mathcal{B} = \nabla_p \times \mathcal{A}$ and $\mathcal{A} = -iu_p^\dagger \nabla_p u_p$ describes a phase shift in the momentum space. So for this theory one could write the effective action:

$$S = \int (p \cdot \dot{x} + A \cdot \dot{x} - \mathcal{A} \cdot \dot{p} - \epsilon_p)\, dt$$

and going to the Hamiltonian evolution in generalized phase space we get

$$\dot{\rho} + \nabla_p(\dot{p}\rho) + \nabla_x(\dot{x}\rho) = 0,$$

where $\rho = (1 + B \cdot \mathcal{B})n_p(x)$ is a modified distribution function.

[10] see e.g. D.T. Son and N. Yamamoto, Phys.Rev. D87 (2013) 8, 085016

After some algebra the kinetic eq. is modified to be

$$\partial_t n = \nabla \cdot j = -\int \frac{d^3p}{(2\pi)^3} \left(\mathcal{B} \cdot \frac{\partial n_p}{\partial p} \right) E \cdot B$$

where the current is

$$j = -\int \frac{d^3p}{(2\pi)^3} \left(\epsilon \frac{\partial n_p}{\partial p} + \left(\mathcal{B} \cdot \frac{\partial n_p}{\partial p} \right) \epsilon_p B + \epsilon_p \mathcal{B} \times \frac{\partial n_p}{\partial x} \right)$$

and it reduces in the equilibrium (homogeneous n_p) to

$$j = -\int \frac{d^3p}{(2\pi)^3} \left(\mathcal{B} \cdot \frac{\partial n_p}{\partial p} \right) \mu B = \pm \frac{\mu}{4\pi^2} B$$

...again the same contribution to the current...

Now one could consider modified Maxwell equations by extracting non-equilibrium behaviour from the result above:

$$\left(k^2 g^{\mu\nu} - k^\mu k^\nu + \Pi^{\mu\nu}\right) A_\nu = 0$$

where $\Pi = \Pi_+ + \Pi_-$

$$\Pi_+^{\mu\nu} = -m_D^2 \left(\delta^{\mu 0}\delta^{\mu 0} - \omega \int \frac{d^3 v}{4\pi} \frac{v^\mu v^\nu}{v \cdot k + i0}\right)$$

$$\Pi_-^{ij} = \frac{e^2 \mu}{4\pi^2} i\epsilon^{ijk} k^k \left(1 - \frac{\omega^2}{\vec{k}^2}\right) [1 - \frac{\omega}{2|\vec{k}|} \ln \frac{\omega + |\vec{k}|}{\omega - |\vec{k}|}]$$

and after solving of the dispersion relation in the limit $|\omega| \ll |\vec{k}|$ one finds

$$\omega = \pm \frac{4 i \alpha \mu}{\pi^2 m_D^2} \vec{k}^2 \left(1 - \frac{\pi |\vec{k}|}{\alpha} \mu\right)$$

thus there is **a growing EM mode** in the system!![11]

[11] N. Yamamoto and YA, Phys.Rev.Lett. 111 (2013) 052002

Taking the anomaly into account the axial charge is modified to

$$\partial_t \left(Q_5 + \frac{1}{4\pi^2} \int \vec{A} \cdot \vec{B} d^3 x \right) = 0$$

In the medium there is no corrections to the anomaly. But on the other hand it was shown that there is a formal modification in hydro of the form

$$A_\nu \to A_\nu + \mu u_\nu$$

and then the generalized helical charge takes the form:

$$\partial_t \left(Q_5^{(0)} + Q_m + Q_{fm} + Q_f \right) = 0$$

where

$$Q_5^{(0)} = \int_x n_5 u_0 \quad , \quad Q_{fm} = \frac{\mu}{2\pi^2} \int_x v \cdot B \quad , \quad Q_f = \frac{\mu^2}{4\pi^2} \int_x v \cdot \Omega$$

and we could see a chirality transfer to macroscopic helicities of fluid flow and EM field.

Then the generalized helical charge takes the form:

$$\partial_t \left(Q_5 + Q_m + Q_{fm} + Q_f \right) = 0$$

But only the second term is actually proportional to e^2 and has anomalous nature. Thus one of possible regimes is anomaly screening corresponding to the non-dissipative case

$$\eta \to 0 \quad , \quad \sigma_E \to \infty \tag{1}$$

where helicities are classically conserved and anomaly is screened by the medium

$$E \cdot B \to 0 \quad , \quad \frac{\partial}{\partial t} Q_{fm} = \frac{\partial}{\partial t} Q_f = 0 \tag{2}$$

In this limit non-dissipative properties appeared from a classical consideration.

- We have seen that conservation of the axial charge in classically chiral media might be related to the perfect liquid limit, with no dissipation.
- It is interesting to consider physics of CE in quantum states (e.g. some kinds of superfluids and superconductors).
- Holographic fluids are good place to check these constraints and instabilities since of nearly ideal behaviour.
- There is a close connection between CME and α-dynamo so it is interesting to check consequences for the turbulence phase of chiral liquids.

Thanks!