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Gluon- and Quark-Jet Average Multiplicities

OUTLINE

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absract

- I present the new results (B.Bolzoni, B.A. Kniehl and A.V.K., 2013) for gluon and quark average multiplicities, which are motivated by recent progress in timelike small-x resummation obtained in the $\overline{\mathrm{MS}}$ scheme. (C.-H.Korn, A. Vogt and K.Yeats, 2012).
  The results contain the next-to-next-to-leading-logarithmic (NNLL) resummed expressions and depend on two nonperturbative parameters with clear and simple physical interpretations.
- We did a global fit of these two quantities. Our results solved a longstandig problem of QCD: a disagreement between theoretical predictions for the ration of gluon and quark average multiplicities and the corresponding experimental data.
- We finally proposed also to use the multiplicity data as a new way

to extract the strong-coupling constant. We obtained $\alpha_s^{(5)}(M_z) = 0.1199 \pm 0.0026$ in the $\overline{\mathrm{MS}}$ scheme in an approximation equivalent to next-to-next-to-leading order (NNNLO) enhanced by the resummations of $\ln x$ terms through the NNLL level, in excellent agreement with the present world average.

Introduction

When jets are produced at colliders, they can be initiated either by a quark or a gluon. The two types of jets are expected to exhibit different properties.

The production of hadrons is a typical process where nonperturbative phenomena are involved.

However, for particular observables, this problem can be avoided. In particular, the *counting* of hadrons. In this case, one can adopt with quite high accuracy *the hypothesis of Local Parton-Hadron Duality (LPHD)*: **parton distributions are renormalized in the hadronization process without changing their shapes** (Ya.I.Azimov, Yu.L.Dokshitzer, V.A.Khoze and S.I.Troyan, 1985). Hence, if the scale Q is large enough, perturbative QCD privides predictions without an usage of phenomenological models of hadronization.

However, **the computation of average jet multiplicities indeed requires small-x resummation**, (A.H.Mueller, 1981) It was shown that the singularities for $x \sim 0$, which are encoded in large logarithms of the kind $\ln^k(1/x)$ and disappear after resummation. Usually, resummation includes the singularities from all orders according to a certain logarithmic accuracy, for which it $restores$ perturbation theory.

Small-x resummation has recently been carried out for timelike splitting fuctions in the $\overline{\mathrm{MS}}$ scheme at the next-to-leading-logarithmic (NLL) level of accuracy (A.Vogt, 2011). and at the next-to-next-to-leading-logarithmic (NNLL) level. (C.-H.Korn, A. Vogt and K.Yeats, 2012).

Thanks to these results, we are able to analytically compute the NNLL contributions to the evolutions of the average gluon and quark jet multiplicities.

1. Fragmentation functions and their evolution

The evolution of the fragmentation functions $D_a(x,\mu^2)$ for the gluon–quark-singlet system $a=g,s$. In Mellin space, is:

$$\mu^2\frac{\partial}{\partial\mu^2}\begin{pmatrix} D_s(\omega,\mu^2) \\ D_g(\omega,\mu^2)\end{pmatrix} = \begin{pmatrix} P_{qq}(\omega,a_s) & P_{gq}(\omega,a_s) \\ P_{qg}(\omega,a_s) & P_{gg}(\omega,a_s)\end{pmatrix}\begin{pmatrix} D_s(\omega,\mu^2) \\ D_g(\omega,\mu^2)\end{pmatrix}, \tag{1}$$

where $P_{ij}(\omega,a_s)$, with $i,j=g,q$, are the timelike splitting functions, $\omega=N-1$, with N being the standard Mellin moments with respect to x, and $a_s(\mu^2)=\alpha_s(\mu)/(4\pi)$ is the couplant.

The standard definition of the hadron average multiplicities in terms of the fragmentation functions is given by their integral over x, which corresponds to the first Mellin moment, with $\omega = 0$:

$$\langle n_h(Q^2) \rangle_a \equiv \left[\int_0^1 dx\, x^{\omega} D_a(x, Q^2) \right]_{\omega=0} = D_a(\omega = 0, Q^2) \quad (2)$$

The timelike splitting functions $P_{ij}(\omega, a_s)$ may be computed perturbatively in a_s,

$$P_{ij}(\omega, a_s) = \sum_{k=0}^{\infty} a_s^{k+1} P_{ij}^{(k)}(\omega). \quad (3)$$

The functions $P_{ij}^{(k)}(\omega)$ for $k = 0, 1, 2$ in the $\overline{\mathrm{MS}}$ scheme may be found through NNLO and with small-x resummation through NNLL accuracy.

2. Diagonalization

It is not in general possible to diagonalize Eq. (1) because the contributions to the timelike-splitting-function matrix do not commute at different orders.

The *usual approach* is then to write a series expansion about the leading-order (LO) solution, which can in turn be diagonalized. One thus starts by choosing a basis in which the timelike-splitting-function matrix is diagonal at LO

$$
\begin{aligned}
P(\omega, a_s) &= \begin{pmatrix} P_{++}(\omega, a_s) & P_{-+}(\omega, a_s) \\ P_{+-}(\omega, a_s) & P_{--}(\omega, a_s) \end{pmatrix} \\
&= a_s \begin{pmatrix} P^{(0)}_{++}(\omega) & 0 \\ 0 & P^{(0)}_{--}(\omega) \end{pmatrix} + a_s^2 P^{(1)}(\omega) + O(a_s^3), \qquad (4)
\end{aligned}
$$

with eigenvalues $P^{(0)}_{\pm\pm}(\omega)$.

It is convenient to represent the change of basis for the fragmentation functions order by order for $k \geq 0$:

$$\begin{aligned} D^{+}(\omega, \mu_0^2) &= (1-\alpha_\omega) D_s(\omega, \mu_0^2) - \epsilon_\omega D_g(\omega, \mu_0^2), \\ D^{-}(\omega, \mu_0^2) &= \alpha_\omega D_s(\omega, \mu_0^2) + \epsilon_\omega D_g(\omega, \mu_0^2). \end{aligned} \tag{5}$$

This implies for the components of the timelike-splitting-function matrix that

$$\begin{aligned} P_{--}^{(k)}(\omega) &= \alpha_\omega P_{qq}^{(k)}(\omega) + \epsilon_\omega P_{qg}^{(k)}(\omega) + \beta_\omega P_{gq}^{(k)}(\omega) + (1-\alpha_\omega) P_{gg}^{(k)}(\omega), \\ P_{-+}^{(k)}(\omega) &= P_{--}^{(k)}(\omega) - \left(P_{qq}^{(k)}(\omega) + \frac{1-\alpha_\omega}{\epsilon_\omega} P_{gq}^{(k)}(\omega) \right), \\ P_{++}^{(k)}(\omega) &= P_{qq}^{(k)}(\omega) + P_{gg}^{(k)}(\omega) - P_{--}^{(k)}(\omega), \\ P_{+-}^{(k)}(\omega) &= P_{++}^{(k)}(\omega) - \left(P_{qq}^{(k)}(\omega) - \frac{\alpha_\omega}{\epsilon_\omega} P_{gq}^{(k)}(\omega) \right) \\ &= P_{gg}^{(k)}(\omega) - \left(P_{--}^{(k)}(\omega) - \frac{\alpha_\omega}{\epsilon_\omega} P_{gq}^{(k)}(\omega) \right), \end{aligned} \tag{6}$$

where the elements of the matrix for diagonalization **(LO projectors !!!)**

$$\alpha_\omega = \frac{P_{qq}^{(0)}(\omega) - P_{++}^{(0)}(\omega)}{P_{--}^{(0)}(\omega) - P_{++}^{(0)}(\omega)}, \quad \epsilon_\omega = \frac{P_{gq}^{(0)}(\omega)}{P_{--}^{(0)}(\omega) - P_{++}^{(0)}(\omega)},$$

$$\beta_\omega = \frac{P_{qg}^{(0)}(\omega)}{P_{--}^{(0)}(\omega) - P_{++}^{(0)}(\omega)}. \tag{7}$$

Our approach to solve Eq. (1) differs from *the usual one* in that we write the solution expanding about the diagonal part of the all-order timelike-splitting-function matrix in the plus-minus basis, instead of its LO contribution. For this purpose, we rewrite Eq. (4) in the following way:

$$P(\omega, a_s) = \begin{pmatrix} P_{++}(\omega, a_s) & 0 \\ 0 & P_{--}(\omega, a_s) \end{pmatrix} \\ + a_s^2 \begin{pmatrix} 0 & P_{-+}^{(1)}(\omega) \\ P_{+-}^{(1)}(\omega) & 0 \end{pmatrix} + \begin{pmatrix} 0 & O(a_s^3) \\ O(a_s^3) & 0 \end{pmatrix}. \tag{8}$$

In general, the solution to Eq. (1) in the plus-minus basis can be formally written as

$$D(\mu^2) = T_{\mu^2} \left\{ \exp \int_{\mu_0^2}^{\mu^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} P(\bar{\mu}^2) \right\} D(\mu_0^2), \tag{9}$$

where T_{μ^2} denotes the path ordering with respect to μ^2 and

$$D = \begin{pmatrix} D^+ \\ D^- \end{pmatrix}. \tag{10}$$

We make the following ansatz to expand about the diagonal part of the timelike-splitting-function matrix in the plus-minus basis: (similary to (A.Buras, 1980) in spacelike case)

$$T_{\mu^2}\left\{\exp\int_{\mu_0^2}^{\mu^2}\frac{d\bar{\mu}^2}{\bar{\mu}^2}P(\bar{\mu}^2)\right\} = Z^{-1}(\mu^2)\exp\left[\int_{\mu_0^2}^{\mu^2}\frac{d\bar{\mu}^2}{\bar{\mu}^2}P^D(\bar{\mu}^2)\right]Z(\mu_0^2), \tag{11}$$

where

$$P^D(\omega) = \begin{pmatrix} P_{++}(\omega) & 0 \\ 0 & P_{--}(\omega) \end{pmatrix} \tag{12}$$

is the diagonal part of Eq. (8) and Z is a matrix in the plus-minus basis which has a perturbative expansion of the form

$$Z(\mu^2) = 1 + a_s(\mu^2)Z^{(1)} + O(a_s^2). \tag{13}$$

In the following, we make use of the renormalization group (RG) equation for the running of $a_s(\mu^2)$,

$$\mu^2 \frac{\partial}{\partial \mu^2} a_s(\mu^2) = \beta(a_s(\mu^2)) = -\beta_0 a_s^2(\mu^2) - \beta_1 a_s^3(\mu^2) + O(a_s^4), \tag{14}$$

where

$$\begin{aligned} \beta_0 &= \frac{11}{3} C_A - \frac{4}{3} n_f T_R, \\ \beta_1 &= \frac{34}{3} C_A^2 - \frac{20}{3} C_A n_f T_R - 4 C_F n_f T_R, \end{aligned} \tag{15}$$

with $C_A = 3$, $C_F = 4/3$, and $T_R = 1/2$ being colour factors and n_f being the number of active quark flavours.

Using Eq. (14) to perform a change of integration variable in Eq. (11), we obtain

$$T_{a_s}\left\{\exp\int_{a_s(\mu_0^2)}^{a_s(\mu^2)}\frac{d\bar{a}_s}{\beta(\bar{a}_s)}P(\bar{a}_s)\right\}$$

$$= Z^{-1}(a_s(\mu^2))\exp\left[\int_{a_s(\mu_0^2)}^{a_s(\mu^2)}\frac{d\bar{a}_s}{\beta(\bar{a}_s)}P^D(\bar{a}_s)\right]Z(a_s(\mu_0^2))(16)$$

with

$$Z(\mu^2) = 1 + a_s(\mu^2)Z^{(1)} + O(a_s^2). \qquad (17)$$

After some algebra we find:

$$Z^{(1)}_{\pm\pm}(\omega) = 0, \qquad Z^{(1)}_{\pm\mp}(\omega) = \frac{P^{(1)}_{\pm\mp}(\omega)}{\beta_0 + P^{(0)}_{\pm\pm}(\omega) - P^{(0)}_{\mp\mp}(\omega)}. \quad (18)$$

Now reverting the change of basis specified in Eq. (5), we find the gluon and quark-singlet fragmentation functions to be given by

$$\begin{aligned} D_g(\omega,\mu^2) &= -\frac{\alpha_\omega}{\epsilon_\omega} D^+(\omega,\mu^2) + \left(\frac{1-\alpha_\omega}{\epsilon_\omega}\right) D^-(\omega,\mu^2), \\ D_s(\omega,\mu^2) &= D^+(\omega,\mu^2) + D^-(\omega,\mu^2). \end{aligned} \quad (19)$$

So, we can write the gluon and quark-singlet fragmentation functions in the following way:

$$D_a(\omega,\mu^2) \equiv D_a^+(\omega,\mu^2) + D_a^-(\omega,\mu^2), \qquad a=g,s, \tag{20}$$

where $D_a^+(\omega,\mu^2)$ evolves like a plus component and $D_a^-(\omega,\mu^2)$ like a minus component.

Finally, we find that

$$D_a^{\pm}(\omega,\mu^2) = \tilde{D}_a^{\pm}(\omega,\mu_0^2)\hat{T}_{\pm}(\omega,\mu^2,\mu_0^2)\, H_a^{\pm}(\omega,\mu^2), \tag{21}$$

where

$$\hat{T}_{\pm}(\omega,\mu^2,\mu_0^2) = \exp\left[\int_{a_s(\mu_0^2)}^{a_s(\mu^2)} \frac{d\bar{a}_s}{\beta(\bar{a}_s)}\, P_{\pm\pm}(\omega,\bar{a}_s)\right]. \tag{22}$$

has a RG-type exponential form and

$$\tilde{D}_g^+(\omega,\mu_0^2) = -\frac{\alpha_\omega}{\epsilon_\omega}\tilde{D}_s^+(\omega,\mu_0^2), \qquad \tilde{D}_g^-(\omega,\mu_0^2) = \frac{1-\alpha_\omega}{\epsilon_\omega}\tilde{D}_s^-(\omega,\mu_0^2),$$
$$\tilde{D}_s^+(\omega,\mu_0^2) = \tilde{D}^+(\omega,\mu_0^2), \qquad \tilde{D}_s^-(\omega,\mu_0^2) = \tilde{D}^-(\omega,\mu_0^2), \tag{23}$$

(the ratios $\tilde{D}_g^\pm(\omega,\mu_0^2)/\tilde{D}_s^\pm(\omega,\mu_0^2$ have only LO results !!!)

with $H_a^\pm(\omega,\mu^2)$ are perturbative functions given by

$$H_a^\pm(\omega,\mu^2) = 1 - a_s(\mu^2)Z^{(1)}_{\pm\mp,a}(\omega) + O(a_s^2). \tag{24}$$

and

$$Z^{(1)}_{\pm\mp,g}(\omega) = -Z^{(1)}_{\pm\mp}(\omega)\left(\frac{1-\alpha_\omega}{\alpha_\omega}\right)^{\pm 1}, \qquad Z^{(1)}_{\pm\mp,s}(\omega) = Z^{(1)}_{\pm\mp}(\omega), \tag{25}$$

where $Z^{(1)}_{\pm\mp}(\omega)$ is given by Eq. (18).

3. Resummation

Reliable computations of average jet multiplicities require resummed analytic expressions for the splitting functions because one has to evaluate the first Mellin moment (corresponding to $\omega = N - 1 = 0$), which is a divergent quantity in the fixed-order perturbative approach. As is well known, resummation overcomes this problem, as demonstrated in the pioneering works by Mueller (A.H.Mueller, 1981) and others (B.I.Ermolaev and V.S.Fadin, 1981), (Yu.L.Dokshitzer, V.S Fadin and V.A.Khoze, 1982,1983)

For future considerations, we remind the reader of an assumpion already made (S.Albino, B.Bolzoni, B.A. Kniehl and A.V.K., 2012) according to which the splitting functions $P_{--}^{(k)}(\omega)$ and $P_{+-}^{(k)}(\omega)$ are supposed to be free of singularities in the limit $\omega \to 0$. In fact, this is expected to be true to all orders. This is certainly true at the LL and NLL levels for the timelike splitting functions, as was verified in (S.Albino, B.Bolzoni, B.A. Kniehl and A.V.K., 2012). This is also true at the NNLL level, as may be explicitly checked by inserting the results of (C.-H.Korn, A. Vogt and K.Yeats, 2012).

So, the minus components are devoid of singularities as $\omega \to 0$ and thus are not resummed.

Using the relationships between the components of the splitting functions in the two bases given in Eq. (6), we find that the absence of singularities for $\omega = 0$ in $P_{--}(\omega, a_s)$ and $P_{+-}(\omega, a_s)$ implies that the singular terms are related as

$$P_{gq}^{\text{sing}}(\omega, a_s) = -\frac{\epsilon_\omega}{\alpha_\omega} P_{gg}^{\text{sing}}(\omega, a_s), \tag{26}$$

$$P_{qg}^{\text{sing}}(\omega, a_s) = -\frac{\alpha_\omega}{\epsilon_\omega} P_{qq}^{\text{sing}}(\omega, a_s), \tag{27}$$

where, through the NLL level,

$$-\frac{\alpha_\omega}{\epsilon_\omega} = \frac{C_A}{C_F}\left[1 - \frac{\omega}{6}\left(1 + 2\frac{n_f T_R}{C_A} - 4\,\frac{C_F n_f T_R}{C_A^2}\right)\right] + O(\omega^2). \tag{28}$$

In fact, substituting $\omega = \omega_{\rm eff}$, where

$$\omega_{\rm eff} = 2\sqrt{2C_A a_s}, \tag{29}$$

into Eq. (28) exactly reproduces the result for the average gluon-to-quark jet multiplicity ratio $r(Q^2)$ obtained earlier (A.H. Mueller, 1984).

Here we would also like to note that, at first sight, the substitution $\omega = \omega_{\rm eff}$ should induce a Q^2 dependence in Eq. (7), which should contribute to the diagonalization matrix. This is not the case, however, because to double-logarithmic accuracy the Q^2 dependence of $a_s(Q^2)$ can be neglected, so that the factor $\alpha_\omega/\epsilon_\omega$ does not recieve any Q^2 dependence upon the substitution $\omega = \omega_{\rm eff}$. This supports the possibility to use this substitution in our analysis and gives an explanation of the good agreement with (A.H. Mueller, 1984). Nevertheless, this substitution only carries a phenomenological meaning. It should only be done in the factor $\alpha_\omega/\epsilon_\omega$, but not in the RG exponents of Eq. (22), where it would lead to a double-counting problem. In fact, the dangerous terms are already resummed in Eq. (22).

In order to be able to obtain the average jet multiplicities, we have to first evaluate the first Mellin momoments of the timelike splitting functions in the plus-minus basis. According to Eq. (6) together with the results given in (A.H. Mueller, 1981), (C.-H.Korn, A. Vogt and K.Yeats, 2012) we have

$$P_{++}^{\mathrm{NNLL}}(\omega=0)=\gamma_0(1-K_1\gamma_0+K_2\gamma_0^2), \tag{30}$$

where

$$\begin{aligned}
\gamma_0 &= P_{++}^{\mathrm{LL}}(\omega=0)=\sqrt{2C_A a_s}, &(31)\\
K_1 &= \frac{1}{12}\left[11+4\frac{n_f T_R}{C_A}\left(1-\frac{2C_F}{C_A}\right)\right], &(32)\\
K_2 &= \frac{1}{288}\left[1193-576\zeta_2-56\frac{n_f T_R}{C_A}\left(5+2\frac{C_F}{C_A}\right)\right]\\
&\quad +16\frac{n_f^2 T_R^2}{C_A^2}\left(1+4\frac{C_F}{C_A}-12\frac{C_F^2}{C_A^2}\right), &(33)
\end{aligned}$$

and

$$P_{-+}^{\mathrm{NNLL}}(\omega = 0) = -\frac{C_F}{C_A} P_{qg}^{NNLL}(\omega = 0), \tag{34}$$

where

$$P_{qg}^{\mathrm{NNLL}}(\omega = 0) = \frac{16}{3} n_f T_R a_s$$
$$-\frac{2}{3} n_f T_R \left[17 - 4 \frac{n_f T_R}{C_A} \left(1 - \frac{2C_F}{C_A} \right) \right] \left(2C_A a_s^3 \right)^{1/2}. \tag{35}$$

For the P_{+-} component, we obtain

$$P_{+-}^{\mathrm{NNLL}}(\omega=0)=O(a_s^2). \tag{36}$$

Finally, as for the P_{--} component, we note that its LO expression produces a finite, nonvanishing term for $\omega=0$ that is of the same order in a_s as the NLL-resummed results in Eq. (30), which leads us to use the following expression for the P_{--} component:

$$P_{--}^{\mathrm{NNLL}}(\omega=0)=-\frac{8n_fT_RC_F}{3C_A}a_s+O(a_s^2), \tag{37}$$

at NNLL accuracy.

We can now perform the integration in Eq. (22) through the NNLL level, which yields

$$
\begin{aligned}
\hat{T}_{\pm}^{\mathrm{NNLL}}(0, Q^2, Q_0^2) &= \frac{T_{\pm}^{\mathrm{NNLL}}(Q^2)}{T_{\pm}^{\mathrm{NNLL}}(Q_0^2)}, \\
T_{+}^{\mathrm{NNLL}}(Q^2) &= \exp\left\{\frac{4C_A}{\beta_0 \gamma^0(Q^2)} \left[1 + (b_1 - 2C_A K_2)\, a_s(Q^2)\right]\right\} \Big(a_s(Q^2)\Big)^{d_+}, \\
T_{-}^{\mathrm{NNLL}}(Q^2) &= T_{-}^{\mathrm{NLL}}(Q^2) = \Big(a_s(Q^2)\Big)^{d_-}, \qquad (38)
\end{aligned}
$$

where

$$
b_1 = \frac{\beta_1}{\beta_0}, \qquad d_- = \frac{8 n_f T_R C_F}{3 C_A \beta_0}, \qquad d_+ = \frac{2 C_A K_1}{\beta_0}. \qquad (39)
$$

4. Multiplicities

We are now ready to define the average gluon and quark jet multiplicities in our formalism, namely

$$\langle n_h(Q^2)\rangle_a \equiv D_a(0,Q^2) = D_a^+(0,Q^2) + D_a^-(0,Q^2), \qquad (40)$$

with $a = g, s$, respectively.

On the other hand, from Eqs. (21) and (23), it follows that

$$r_+(Q^2) \equiv \frac{D_g^+(0,Q^2)}{D_s^+(0,Q^2)} = -\lim_{\omega\to 0} \frac{\alpha_\omega}{\epsilon_\omega} \frac{H_g^+(\omega,Q^2)}{H_s^+(\omega,Q^2)}, \qquad (41)$$

$$r_-(Q^2) \equiv \frac{D_g^-(0,Q^2)}{D_s^-(0,Q^2)} = \lim_{\omega\to 0} \frac{1-\alpha_\omega}{\epsilon_\omega} \frac{H_g^-(\omega,Q^2)}{H_s^-(\omega,Q^2)}. \qquad (42)$$

Using these definitions and again Eq. (21), we may write general expressions for the average gluon and quark jet multiplicities:

$$
\begin{aligned}
\langle n_h(Q^2)\rangle_g &= \tilde{D}_g^{+}(0,Q_0^2)\hat{T}_{+}^{\mathrm{res}}(0,Q^2,Q_0^2)H_g^{+}(0,Q^2) \\
&\quad + \tilde{D}_s^{-}(0,Q_0^2)r_{-}(Q^2)\hat{T}_{-}^{\mathrm{res}}(0,Q^2,Q_0^2)H_s^{-}(0,Q^2), \\
\langle n_h(Q^2)\rangle_s &= \frac{\tilde{D}_g^{+}(0,Q_0^2)}{r_{+}(Q^2)}\hat{T}_{+}^{\mathrm{res}}(0,Q^2,Q_0^2)H_g^{+}(0,Q^2) \\
&\quad + \tilde{D}_s^{-}(0,Q_0^2)\hat{T}_{-}^{\mathrm{res}}(0,Q^2,Q_0^2)H_s^{-}(0,Q^2).
\end{aligned}
\tag{43}
$$

At the LO in a_s, the coefficients of the RG exponents are given by

$$
\begin{aligned}
r_{+}(Q^2) &= \frac{C_A}{C_F}, \qquad r_{-}(Q^2) = 0, \\
H_s^{\pm}(0,Q^2) &= 1, \qquad \tilde{D}_a^{\pm}(0,Q_0^2) = D_a^{\pm}(0,Q_0^2),
\end{aligned}
\tag{44}
$$

for $a = g, s$.

It would, of course, be desirable to include higher-order corrections in Eqs. (44). However, this is highly nontrivial because the general perturbative structures of the functions $H_a^{\pm}(\omega, \mu^2)$ and $Z_{\pm\mp,a}(\omega, a_s)$, which would allow us to resum those higher-order corrections, are presently unknown. **We did diagonalization before resummation!!!** Fortunatly, some approximations can be made.

On the one hand, it is well-known that the plus components by themselves represent the dominant contributions to both the average gluon and quark jet multiplicities (see, e.g., (M.Schmelling, 1995), for the gluon case and (I.M.Dremin and J.W.Gary, 2001) for the quark case).

On the other hand, Eq. (42) tells us that $D_g^-(0, Q^2)$ is suppressed with respect to $D_s^-(0, Q^2)$ because $\alpha_\omega \sim 1 + \mathcal{O}(\omega)$. These two observations suggest that keeping $r_-(Q^2) = 0$ also beyond LO should represent a good approximation. Nevertheless, we shall explain below how to obtain the first nonvanishing contribution to $r_-(Q^2)$. Furthermore, we notice that higher-order corrections to $H_a^{\pm}(0, Q^2)$ and $\tilde{D}_a^{\pm}(0, Q_0^2)$ just represent redefinitions of $D_a^{\pm}(0, Q_0^2)$ by constant factors apart from running-coupling effects. Therefore, we assume that these corrections can be neglected.

We now discuss higher-order corrections to $r_+(Q^2)$. As already mentioned above, we introduced (B.Bolzoni, B.A. Kniehl and A.V.K., 2013) an effective approach to perform the resummation of the first Mellin moment of the "plus" component of the anomalous dimension. In that approach, **resummation is performed by taking the fixed-order plus component and substituting** $\omega = \omega_{\rm eff}$, where

$$\omega_{\rm eff} = 2\sqrt{2C_A a_s} = 2\gamma_0. \tag{45}$$

We now show that this approach is exact to $O(\sqrt{a_s})$. We indeed recover Eq. (31) by substituting $\omega = \omega_{\rm eff}$ in the leading singular term of the LO splitting function $P_{++}(\omega, a_s)$,

$$P_{++}^{\rm LO}(\omega) = \frac{4C_A a_s}{\omega} + O(\omega^0). \tag{46}$$

We may then also substitute $\omega = \omega_{\mathrm{eff}}$ in the LO result for $r_+(Q^2)$ (i.e. to Eq. (41)). We thus find

$$r_+(Q^2) = \frac{C_A}{C_F}\left[1 - \frac{\sqrt{2a_s(Q^2)C_A}}{3}\left(1 + 2\frac{n_f T_R}{C_A} - 4\frac{C_F n_f T_R}{C_A^2}\right)\right] + O(a_s), \tag{47}$$

which coincides with the result obtained by Mueller (A.H.Mueller, 1984).

For this reason and because (I.M.Dremin and J.W.Gary, 1999) the average gluon and quark jet multiplicities evolve with only one RG exponent **(only with "plus" component)**, we can intepreete the result of (A.Capella, I.M.Dremin, J.W.Gary, V.A. Nechitailo and J. Tran Thanh Van, 2000) as higher-order corrections to Eq. (47).

So, we use the results of (A.Capella, I.M.Dremin, J.W.Gary, V.A. Nechitailo and J. Tran Thanh Van, 2000) for the ratio gluon and quark average multiplicities as the estimation for $r_+(Q^2)$.

Since we showed that our approach reproduces exact analytic results at $O(\sqrt{a_s})$, we may safely apply it to predict the first non-vanishing correction to $r_-(Q^2)$ defined in Eq. (42), which yields

$$r_-(Q^2) = -\frac{4n_f T_R}{3}\sqrt{\frac{2a_s(Q^2)}{C_A}} + O(a_s). \tag{48}$$

For the reader's convenience, we list here expressions with numerical coefficients for $r_+(Q^2)$ through $O(a_s^{3/2})$ and for $r_-(Q^2)$ through $O(\sqrt{a_s})$ in QCD with $n_f = 5$:

$$r_+(Q^2) = 2.25 - 2.18249\sqrt{a_s(Q^2)} - 27.54\,a_s(Q^2) + 10.8462\,a_s^{3/2}(Q^2) + O(a$$

$$r_-(Q^2) = -2.72166\sqrt{a_s(Q^2)} + O(a_s). \quad (4$$

In all the approximations considered here, we may summarize our main theoretical results for the avarage gluon and quark jet multiplicities in the following way:

$$\langle n_h(Q^2)\rangle_g = n_1(Q_0^2)\hat{T}_+^{\rm res}(0,Q^2,Q_0^2) + n_2(Q_0^2)\, r_-(Q^2)\hat{T}_-^{\rm res}(0,Q^2,Q_0^2),$$
$$\langle n_h(Q^2)\rangle_s = n_1(Q_0^2)\frac{\hat{T}_+^{\rm res}(0,Q^2,Q_0^2)}{r_+(Q^2)} + n_2(Q_0^2)\, \hat{T}_-^{\rm res}(0,Q^2,Q_0^2), \qquad (50)$$

where

$$n_1(Q_0^2) = r_+(Q_0^2)\frac{D_g(0,Q_0^2) - r_-(Q_0^2)D_s(0,Q_0^2)}{r_+(Q_0^2) - r_-(Q_0^2)},$$
$$n_2(Q_0^2) = \frac{r_+(Q_0^2)D_s(0,Q_0^2) - D_g(0,Q_0^2)}{r_+(Q_0^2) - r_-(Q_0^2)}. \qquad (51)$$

The average gluon-to-quark jet multiplicity ratio may thus be written as

$$r(Q^2) \equiv \frac{\langle n_h(Q^2)\rangle_g}{\langle n_h(Q^2)\rangle_s} = r_+(Q^2)\left[\frac{1+r_-(Q^2)R(Q_0^2)\frac{\hat{T}_-^{\rm res}(0,Q^2,Q_0^2)}{\hat{T}_+^{\rm res}(0,Q^2,Q_0^2)}}{1+r_+(Q^2)R(Q_0^2)\frac{\hat{T}_-^{\rm res}(0,Q^2,Q_0^2)}{\hat{T}_+^{\rm res}(0,Q^2,Q_0^2)}}\right], \tag{52}$$

where

$$R(Q_0^2) = \frac{n_2(Q_0^2)}{n_1(Q_0^2)}. \tag{53}$$

It follows from the definition of $\hat{T}_\pm^{\rm res}(0, Q^2, Q_0^2$ in Eq. (**??**) and from Eq. (51) that, for $Q^2 = Q_0^2$, Eqs. (50) and (52) become

$$\begin{aligned} &\langle n_h(Q_0^2)\rangle_g = D_g(0, Q_0^2), \quad \langle n_h(Q_0^2)\rangle_q = D_s(0, Q_0^2), \\ &r(Q_0^2) = \frac{D_g(0, Q_0^2)}{D_s(0, Q_0^2)}. \end{aligned} \tag{54}$$

The NNLL-resummed expressions for the average gluon and quark jet multiplicites given by Eq. (50) only depend on two nonperturbative constants, namely $D_g(0, Q_0^2)$ and $D_s(0, Q_0^2)$. These allow for a simple physical interpretation. In fact, according to Eq. (54), they are the average gluon and quark jet multiplicities at the arbitrary scale Q_0.

4. Analysis

We are now in a position to perform a global fit to the available experimental data of our formulas in Eq. (50) in the $\mathrm{LO}+\mathrm{NNLL}$ ($r_+ = C_A/C_F = 2.25$, $r_- = 0$), $\mathrm{N^3LO_{approx+NNLL}}$ ($r_+ = r_{\mathrm{Capella}}$, $r_- = 0$), , and $\mathrm{N^3LO_{approx+NLO+NNLL}}$ ($r_+ = r_{\mathrm{Capella}}$, $r_- = -2.72166\sqrt{a_s(Q^2)}$) approximations, so as to extract the nonperturbative constants $D_g(0, Q_0^2)$ and $D_s(0, Q_0^2)$.

We have to make a choice for the scale Q_0, which, in principle, is arbitrary. The perturbative series appears to be more rapidly converging at relatively large values of Q_0. Therefore, **we adopt $Q_0 = 50$ GeV in the following.**

	$\mathrm{LO} + \mathrm{NNLL}$	$\mathrm{N}^3\mathrm{LO}_{\mathrm{approx+NNLL}}$	$\mathrm{N}^3\mathrm{LO}_{\mathrm{approx+NLO+NNLL}}$
$\langle n_h(Q_0^2)\rangle_g$	24.31 ± 0.85	24.02 ± 0.36	24.17 ± 0.36
$\langle n_h(Q_0^2)\rangle_q$	15.49 ± 0.90	15.83 ± 0.37	15.89 ± 0.33
χ^2_{dof}	18.09	3.71	2.92

Table 1: Fit results for $\langle n_h(Q_0^2)\rangle_g$ and $\langle n_h(Q_0^2)\rangle_q$ at $Q_0 = 50$ GeV with 90% CL errors and minimum values of χ^2_{dof} achieved in the $\mathrm{LO} + \mathrm{NNLL}$, $\mathrm{N}^3\mathrm{LO}_{\mathrm{approx+NNLL}}$, and $\mathrm{N}^3\mathrm{LO}_{\mathrm{approx+NLO+NNLL}}$ approximations.

We included the measurements of average gluon jet multiplicities and those of average quark jet multiplicities, which include 27 and 51 experimental data points, respectively. The results for $\langle n_h(Q_0^2)\rangle_g$ and $\langle n_h(Q_0^2)\rangle_q$ at $Q_0 = 50$ GeV together with the χ^2_{dof} values obtained in our $\mathrm{LO} + \mathrm{NNLL}$, $\mathrm{N}^3\mathrm{LO}_{\mathrm{approx+NNLL}}$, and $\mathrm{N}^3\mathrm{LO}_{\mathrm{approx+NLO+NNLL}}$ fits are listed in Table 1. The errors correspond to 90% CL as explained above. All these fit results are in agreement with the experimental data.

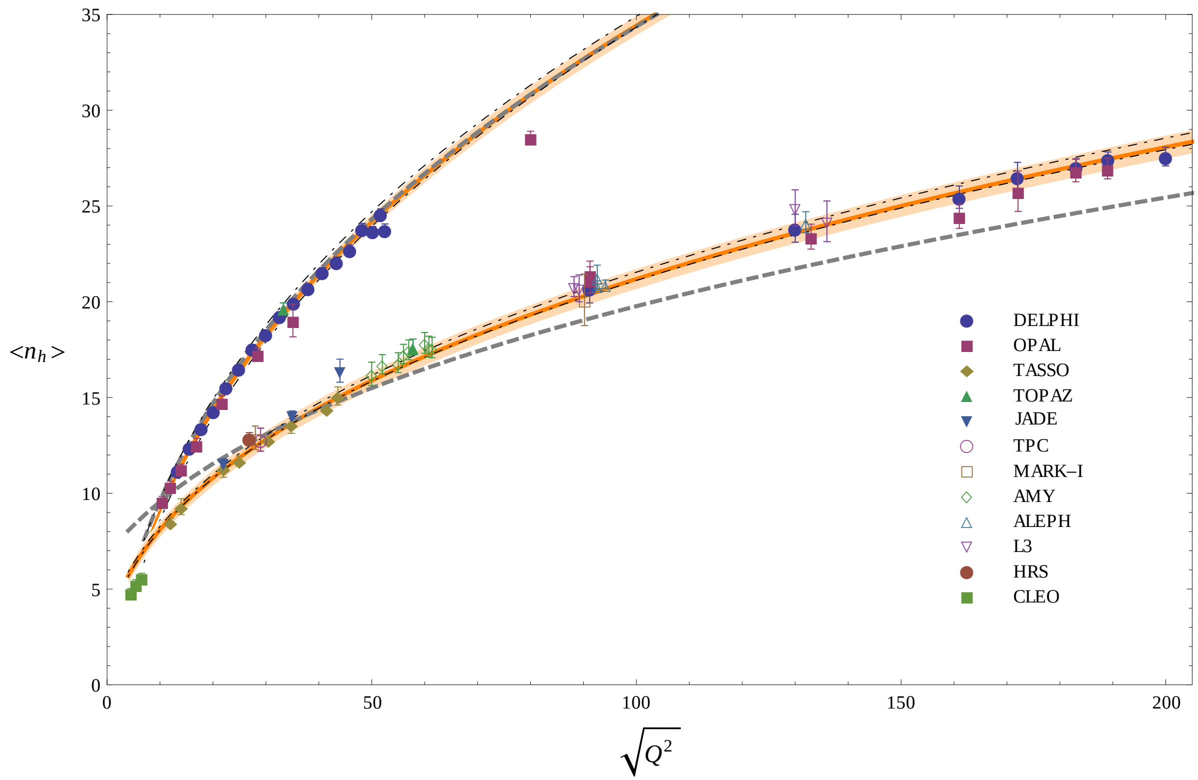

Figure 1: The average gluon (upper curves) and quark (lower curves) jet multiplicities evaluated from Eq. (50), respectively, in the LO + NNLL (dashed/gray lines) and $\mathrm{N^3LO_{approx+NLO+NNLL}}$ (solid/orange lines) approximations using the corresponding fit results for $\langle n_h(Q_0^2)\rangle_g$ and $\langle n_h(Q_0^2)\rangle_q$ from Table 1 are compared with the experimental data included in the fits. The experimental and theoretical uncertainties in the $\mathrm{N^3LO_{approx+NLO+NNLL}}$ results are indicated by the shaded/orange bands and the bands enclosed between the dot-dashed curves, respectively.

In Fig. 1, we show as functions of Q the average gluon and quark jet multiplicities evaluated from Eq. (50) at $\mathrm{LO + NNLL}$ and $\mathrm{N^3LO_{approx+NLO+NNLL}}$ using the corresponding fit results for $\langle n_h(Q_0^2)\rangle_g$ and $\langle n_h(Q_0^2)\rangle_q$ at $Q_0 = 50$ GeV from Table 1.

- Gluon average multiplicity is fitted well like in the previous analysis (A.Capella, I.M.Dremin, J.W.Gary, V.A. Nechitailo and J. Tran Thanh Van, 2000). In a sence, the result (based on the plus components in our approach) should be close to ones obtained in the framework of the famous modified leading-logarithmic approximation (MLLA).

- The fit of quark average multiplicity is good because minus component: there is the additional contribution with the additional free parameter $D_s(0, Q^2)$.

  The quark-singlet minus component comes with an arbitrary normalization and has a slow Q^2 dependence. Consequently, its numerical contribution may be approximately mimicked by a constant introduced to the average quark jet multiplicity as in (P.Abreu et al. [DELPHI Collab.], 1998)

- We can compare our results with the data for the ratio of the gluon and quark average multiplicities. It is not a fit because all our parameters have been already fixed.

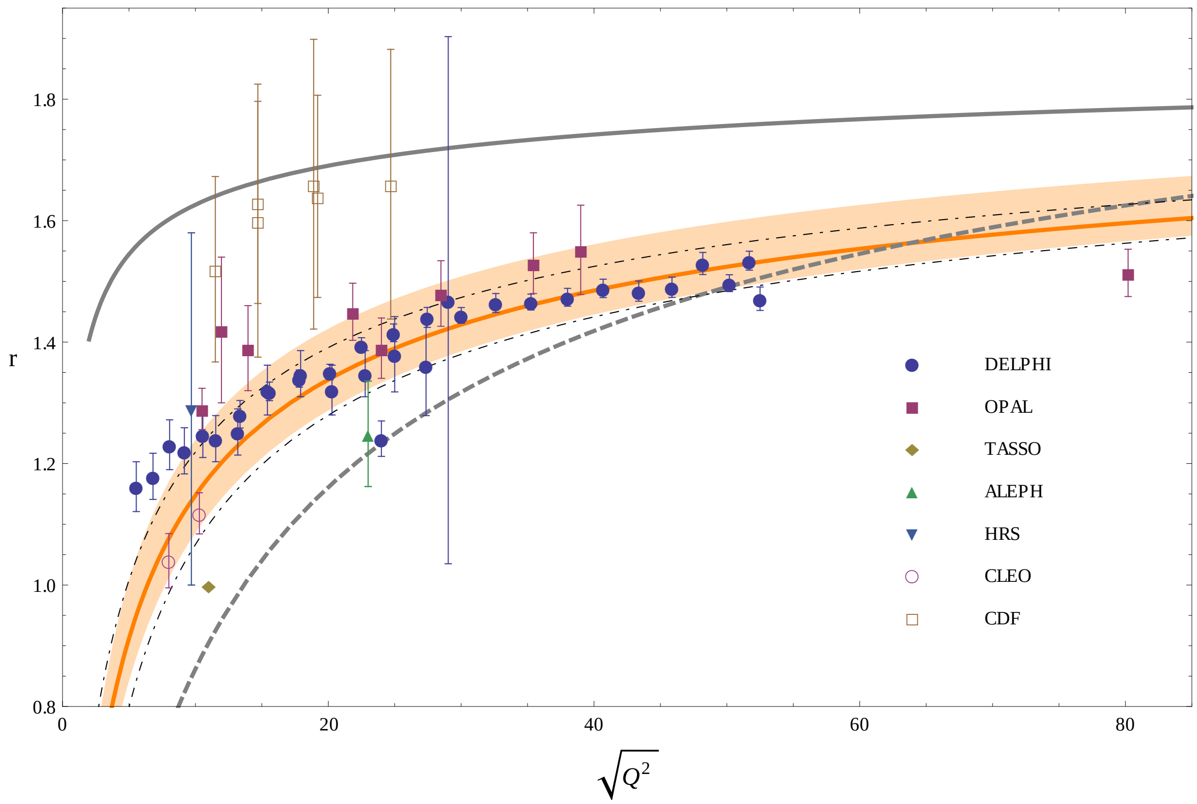

Figure 2: The average gluon-to-quark jet multiplicity ratio evaluated from Eq. (52) in the LO + NNLL (dashed/g $N^3LO_{\mathrm{approx+NLO+NNLL}}$ (solid/orange lines) approximations using the corresponding fit results for $\langle n_h(Q_0^2)\rangle_g$ and $\langle n_h(Q_0^2)\rangle_q$ from Table with experimental data. The experimental and theoretical uncertainties in the $N^3LO_{\mathrm{approx+NLO+NNLL}}$ result are indicated by the bands and the bands enclosed between the dot-dashed curves, respectively. The prediction given by analysis in (A.Capella, I.M.Dre V.A. Nechitailo and J. Tran Thanh Van, 2000) is indicated by the continuous/gray line.

4. Determination of strong-coupling constant

Before we took $\alpha_s^{(5)}(m_Z^2)$ to be a fixed input parameter for our fits. Motivated by the excellent goodness of our $\mathrm{N^3LO_{approx+NNLL}}$ and $\mathrm{N^3LO_{approx+NLO+NNLL}}$ fits, **we now include it among the fit parameters**, the more so as the fits should be sufficiently sensitive to it in view of the wide Q^2 range populated by the experimental data fitted to.

	$\mathrm{N}^3\mathrm{LO}_{\mathrm{approx+NNLL}}$	$\mathrm{N}^3\mathrm{LO}_{\mathrm{approx+NLO+NNLL}}$
$\langle n_h(Q_0^2)\rangle_g$	24.18 ± 0.32	24.22 ± 0.33
$\langle n_h(Q_0^2)\rangle_q$	15.86 ± 0.37	15.88 ± 0.35
$\alpha_s^{(5)}(m_Z^2)$	0.1242 ± 0.0046	0.1199 ± 0.0044
χ^2_{dof}	2.84	2.85

Table 2: Fit results for $\langle n_h(Q_0^2)\rangle_g$ and $\langle n_h(Q_0^2)\rangle_q$ at $Q_0 = 50$ GeV and for $\alpha_s^{(5)}(m_Z^2)$ with 90% CL errors and minimum values of χ^2_{dof} achieved in the $\mathrm{N}^3\mathrm{LO}_{\mathrm{approx+NNLL}}$ and $\mathrm{N}^3\mathrm{LO}_{\mathrm{approx+NLO+NNLL}}$ approximations.

We fit to the same experimental data as before and again put $Q_0 = 50$ GeV. The fit results are summarized in Table 2.

We observe from Table 2 that the results of the $\mathrm{N^3LO_{approx+NNLL}}$ **and** $\mathrm{N^3LO_{approx+NLO+NNLL}}$ **fits for** $\langle n_h(Q_0^2)\rangle_g$ **and** $\langle n_h(Q_0^2)\rangle_q$ **are mutually consistent. They are also consistent with the respective fit results in Table 1.** As expected, the values of χ^2_{dof} are reduced by relasing $\alpha_s^{(5)}(m_Z^2)$ in the fits, from 3.71 to 2.84 in the $\mathrm{N^3LO_{approx+NNLL}}$ approximation and from 2.95 to 2.85 in the $\mathrm{N^3LO_{approx+NLO+NNLL}}$ one. The three-parameter fits strongly confine $\alpha_s^{(5)}(m_Z^2)$, within an error of 3.7% at 90% CL in both approximations. **The inclusion of the** $r_-(Q^2)$ **term has the beneficial effect of shifting** $\alpha_s^{(5)}(m_Z^2)$ **closer to the world average,** 0.1184 ± 0.0007 **(J.Beringer et al. [Particle Data Group Collab.], 2012)**

4. Conclusion

- Prior to our analysis, experimental data on the average gluon and quark jet multiplicities could not be simultaneously described in a satisfactory way mainly because the theoretical formalism failed to account for the difference in hadronic contents between gluon and quark jets, although the convergence of perturbation theory seemed to be well under control.

- This problem may be solved by including the minus components governed by $\hat{T}_{-}^{\rm res}(0, Q^2, Q_0^2)$ The quark-singlet minus component comes with an arbitrary normalization and has a slow Q^2 dependence. Consequently, its numerical contribution may be approximately mimicked by a constant introduced to the average quark jet multiplicity as in (P.Abreu et al. [DELPHI Collab.], 1998).

- Motivated by the goodness of our $\mathrm{N^3LO_{approx+NNLL}}$ and $\mathrm{N^3LO_{approx+NLO+N}}$ fits with fixed value of $\alpha_s^{(5)}(m_Z^2)$ here, we then included $\alpha_s^{(5)}(m_Z^2)$ among the fit parameters, which yielded a further reduction of $\chi^2_{\rm dof}$. The obtained value $\alpha_s^{(5)}(m_Z^2) = 0.1199 \pm 0.0026$ is close to the world average one.

- We have some problems with a proper resummation of the non-diagonal terms in the Q^2-evolution. We hope to improve this part of our analysis in our future studies.