Renormalons in heavy quark physics and lattice: the pole mass and the gluon condensate

Antonio Pineda

Universitat Autònoma de Barcelona

Confinement XI, Sant Petersburg, 8-12 September 2014

Renormalons

Originally (Lautrup, 't Hooft).
Renormalon: summation of "bubbles": $\beta_0 = -\frac{4}{3} T_F n_l \rightarrow \beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_F n_l$
naive non-abelianization. Running of α.

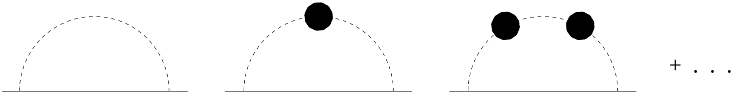


Figure: Sum of the bubbles in the quark propagator.

Pole mass (Bigi, Shifman, Uraltsev, Vainshtein; Beneke, Braun)

$$m_{\rm OS} = m_{\overline{\rm MS}}(1 + B_1\alpha_s + B_2\alpha_s^2 + \cdots) \qquad B_n \sim n!$$

Beyond bubbles $\rightarrow$ Parisi; Beneke, ...
NP OPE (Novikov, Shifman, Vainshtein, Zakharov))

Renormalons

Originally (Lautrup, 't Hooft).
Renormalon: summation of "bubbles": $\beta_0 = -\frac{4}{3} T_F n_l \to \beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_F n_l$
naive non-abelianization. Running of α.

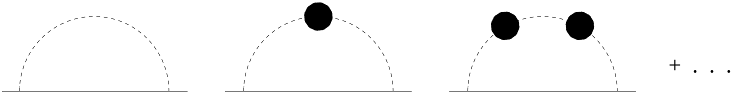


Figure: Sum of the bubbles in the quark propagator.

Pole mass (Bigi, Shifman, Uraltsev, Vainshtein; Beneke, Braun)

$$m_{\rm OS} = m_{\overline{\rm MS}}(1 + B_1\alpha_s + B_2\alpha_s^2 + \cdots) \qquad B_n \sim n!$$

Beyond bubbles $\to$ Parisi; Beneke, ...
NP OPE (Novikov, Shifman, Vainshtein, Zakharov))

Modern view: Effective field theories/Factorization

$$\mathcal{L} = \sum_n \frac{1}{m^n} c_n O_n \qquad c(\nu) = \bar{c} + \sum_{n=0}^{\infty} c_n \alpha_s^{n+1}.$$

The Wilson coefficients are believed to be asymptotic: $c_n \sim n!$
IF SO such behavior should comply with the Operator Product Expansion.

Effective-field-theory/factorization definition of renormalon: Asymptotic behavior of the perturbative expansion such that the associated ambiguity in the summation of the perturbative series can be absorbed into a higher order operator.
Example:

$$M_B = m_{\rm OS} + \bar{\Lambda}_B + \mathcal{O}(1/m_{\rm OS})\,, \quad m_{\tilde{G}} = m_{\tilde{g},{\rm OS}} + \Lambda_H + \mathcal{O}\left(1/m_{\tilde{g},{\rm OS}}\right)$$

M_B is renormalon free. Therefore $m_{\rm OS}$ suffers from renormalon ambiguities:

$$m_{\rm OS} = m_{\overline{\rm MS}}(1 + B_1\alpha_s + B_2\alpha_s^2 + \cdots)$$

with $B_n \sim n!$. In other words

$$\delta_{np}^{\rm (pert.)} m_{\rm OS} = \delta_{np}^{\rm (pert.)} m_{\overline{\rm MS}}(1 + B_1\alpha_s + B_2\alpha_s^2 + \cdots) \sim \Lambda_{\rm QCD}!$$

Modern view: Effective field theories/Factorization

$$\mathcal{L} = \sum_n \frac{1}{m^n} c_n O_n \qquad c(\nu) = \bar{c} + \sum_{n=0}^{\infty} c_n \alpha_s^{n+1}.$$

The Wilson coefficients are believed to be asymptotic: $c_n \sim n!$
IF SO such behavior should comply with the Operator Product Expansion.

Effective-field-theory/factorization definition of renormalon: Asymptotic behavior of the perturbative expansion such that the associated ambiguity in the summation of the perturbative series can be absorbed into a higher order operator.
Example:

$$M_B = m_{\rm OS} + \bar{\Lambda}_B + \mathcal{O}(1/m_{\rm OS}) \,, \quad m_{\tilde{G}} = m_{\tilde{g},\rm OS} + \Lambda_H + \mathcal{O}\left(1/m_{\tilde{g},\rm OS}\right)$$

M_B is renormalon free. Therefore $m_{\rm OS}$ suffers from renormalon ambiguities:

$$m_{\rm OS} = m_{\overline{\rm MS}}(1 + B_1\alpha_s + B_2\alpha_s^2 + \cdots)$$

with $B_n \sim n!$. In other words

$$\delta_{np}^{(\rm pert.)} m_{\rm OS} = \delta_{np}^{(\rm pert.)} m_{\overline{\rm MS}}(1 + B_1\alpha_s + B_2\alpha_s^2 + \cdots) \sim \Lambda_{\rm QCD}!$$

Modern view: Effective field theories/Factorization

$$\mathcal{L} = \sum_n \frac{1}{m^n} c_n O_n \qquad c(\nu) = \bar{c} + \sum_{n=0}^{\infty} c_n \alpha_s^{n+1} .$$

The Wilson coefficients are believed to be asymptotic: $c_n \sim n!$
IF SO such behavior should comply with the Operator Product Expansion.

Effective-field-theory/factorization definition of renormalon: Asymptotic behavior of the perturbative expansion such that the associated ambiguity in the summation of the perturbative series can be absorbed into a higher order operator.

Example:

$$M_B = m_{\rm OS} + \bar{\Lambda}_B + \mathcal{O}(1/m_{\rm OS}) \, , \quad m_{\tilde{G}} = m_{\tilde{g},{\rm OS}} + \Lambda_H + \mathcal{O}\left(1/m_{\tilde{g},{\rm OS}}\right)$$

M_B is renormalon free. Therefore $m_{\rm OS}$ suffers from renormalon ambiguities:

$$m_{\rm OS} = m_{\overline{\rm MS}}(1 + B_1\alpha_s + B_2\alpha_s^2 + \cdots)$$

with $B_n \sim n!$. In other words

$$\delta_{np}^{\rm (pert.)} m_{\rm OS} = \delta_{np}^{\rm (pert.)} m_{\overline{\rm MS}}(1 + B_1\alpha_s + B_2\alpha_s^2 + \cdots) \sim \Lambda_{\rm QCD}!$$

Modern view: Effective field theories/Factorization

$$\mathcal{L} = \sum_n \frac{1}{m^n} c_n O_n \qquad c(\nu) = \bar{c} + \sum_{n=0}^{\infty} c_n \alpha_s^{n+1}.$$

The Wilson coefficients are believed to be asymptotic: $c_n \sim n!$
IF SO such behavior should comply with the Operator Product Expansion.

Effective-field-theory/factorization definition of renormalon: Asymptotic behavior of the perturbative expansion such that the associated ambiguity in the summation of the perturbative series can be absorbed into a higher order operator.
Example:

$$M_B = m_{\rm OS} + \bar{\Lambda}_B + \mathcal{O}(1/m_{\rm OS}) \,, \quad m_{\tilde{G}} = m_{\tilde{g},\rm OS} + \Lambda_H + \mathcal{O}\left(1/m_{\tilde{g},\rm OS}\right)$$

M_B is renormalon free. Therefore $m_{\rm OS}$ suffers from renormalon ambiguities:

$$m_{\rm OS} = m_{\overline{\rm MS}}(1 + B_1\alpha_s + B_2\alpha_s^2 + \cdots)$$

with $B_n \sim n!$. In other words

$$\delta_{np}^{(\rm pert.)} m_{\rm OS} = \delta_{np}^{(\rm pert.)} m_{\overline{\rm MS}}(1 + B_1\alpha_s + B_2\alpha_s^2 + \cdots) \sim \Lambda_{\rm QCD}!$$

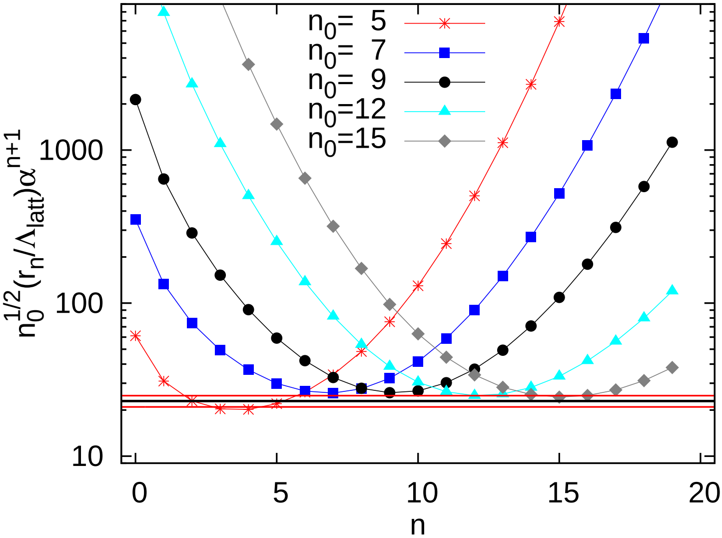


Figure: *c_n times $\sqrt{n_0}$, for five different values of the lattice scheme coupling constant α, ranging from $\alpha(\nu) \approx 0.096$ ($n_0 = 5$) to $\alpha(\nu) \approx 0.036$ ($n_0 = 15$). Bali, Bauer, AP, Torrero, 1303.3279.*

The maximal accuracy with which one can obtain the matching coefficients from a perturbative calculation is (roughly) of the order of

$$\delta c \sim r_{n^*} \alpha_s^{n^*} \,,$$

where $n^* \sim \frac{a}{\alpha_s}$. If a is positive c suffers from a non-perturbative ambiguity of order

$$\delta c \sim (\Lambda_{\rm QCD})^{\frac{|a|\beta_0}{2\pi}} \,.$$

The Borel transform of $c(\nu)$ reads

$$B[c](t) \equiv \sum_{n=0}^{\infty} c_n \frac{t^n}{n!},$$

and c is written in terms of its Borel transform as

$$c = \bar{c} + \int_0^\infty \mathrm{d}t \, e^{-t/\alpha_s} B[c](t).$$

The ambiguities in the matching coefficient ($c_n \sim n!$) reflects in poles in the Borel transform. If we take the one closest to the origin,

$$\delta B[c](t) \sim \frac{1}{a-t},$$

where a is a positive number.

The maximal accuracy with which one can obtain the matching coefficients from a perturbative calculation is (roughly) of the order of

$$\delta c \sim r_{n^*} \alpha_s^{n^*} ,$$

where $n^* \sim \frac{a}{\alpha_s}$. If a is positive c suffers from a non-perturbative ambiguity of order

$$\delta c \sim (\Lambda_{\rm QCD})^{\frac{|a|\beta_0}{2\pi}} .$$

The Borel transform of $c(\nu)$ reads

$$B[c](t) \equiv \sum_{n=0}^{\infty} c_n \frac{t^n}{n!},$$

and c is written in terms of its Borel transform as

$$c = \bar{c} + \int_0^\infty \mathrm{d}t \, e^{-t/\alpha_s} B[c](t).$$

The ambiguities in the matching coefficient ($c_n \sim n!$) reflects in poles in the Borel transform. If we take the one closest to the origin,

$$\delta B[c](t) \sim \frac{1}{a-t},$$

where a is a positive number.

Where do we expect renormalon effects to be more important?

Current-current correlator ($c_{n*}\alpha_s^{n*+1} \sim \Lambda_{\rm QCD}^4/Q^4 \to c_n \sim n!$):

$$\int d^4x e^{iqx} \langle vac|J(x)J(0)|vac\rangle = (Pert.\ th.) + \frac{\Lambda_{\rm QCD}^4}{Q^4} + \cdots$$

Plaquette:

$$\langle P\rangle = (Pert.\ th.) + a^4\Lambda_{\rm QCD}^4 + \cdots$$

DIS:

$$(Pert.\ th.) + \frac{\Lambda_{\rm QCD}^2}{Q^2} + \cdots$$

Heavy quark physics:

$$(Pert.\ th.) + \frac{\Lambda_{\rm QCD}}{m_Q} + \cdots$$

The natural place to look for these effects.
AP: hep-ph/0105008, hep-ph/0208031, hep-lat/0509022
Bali, AP: hep-ph/0310130
Ayala, Cvetic, AP, 1407.2128.

Where do we expect renormalon effects to be more important?

Current-current correlator ($c_{n*}\,\alpha_s^{n*+1} \sim \Lambda_{\rm QCD}^4/Q^4 \rightarrow c_n \sim n!$):

$$\int d^4x e^{iqx} \langle vac|J(x)J(0)|vac\rangle = (Pert.\ th.) + \frac{\Lambda_{\rm QCD}^4}{Q^4} + \cdots$$

Plaquette:

$$\langle P \rangle = (Pert.\ th.) + a^4 \Lambda_{\rm QCD}^4 + \cdots$$

DIS:

$$(Pert.\ th.) + \frac{\Lambda_{\rm QCD}^2}{Q^2} + \cdots$$

Heavy quark physics:

$$(Pert.\ th.) + \frac{\Lambda_{\rm QCD}}{m_Q} + \cdots$$

The natural place to look for these effects.
AP: hep-ph/0105008, hep-ph/0208031, hep-lat/0509022
Bali, AP: hep-ph/0310130
Ayala, Cvetic, AP, 1407.2128.

Where do we expect renormalon effects to be more important?

Current-current correlator ($c_{n*}\alpha_s^{n*+1} \sim \Lambda_{\rm QCD}^4/Q^4 \to c_n \sim n!$):

$$\int d^4x e^{iqx} \langle vac|J(x)J(0)|vac\rangle = (Pert.\ th.) + \frac{\Lambda_{\rm QCD}^4}{Q^4} + \cdots$$

Plaquette:

$$\langle P\rangle = (Pert.\ th.) + a^4\Lambda_{\rm QCD}^4 + \cdots$$

DIS:

$$(Pert.\ th.) + \frac{\Lambda_{\rm QCD}^2}{Q^2} + \cdots$$

Heavy quark physics:

$$(Pert.\ th.) + \frac{\Lambda_{\rm QCD}}{m_Q} + \cdots$$

The natural place to look for these effects.
AP: hep-ph/0105008, hep-ph/0208031, hep-lat/0509022
Bali, AP: hep-ph/0310130
Ayala, Cvetic, AP, 1407.2128.

Where do we expect renormalon effects to be more important?

Current-current correlator ($c_{n*}\,\alpha_s^{n*+1} \sim \Lambda_{\rm QCD}^4/Q^4 \rightarrow c_n \sim n!$):

$$\int d^4x e^{iqx} \langle vac|J(x)J(0)|vac\rangle = (Pert.\ th.) + \frac{\Lambda_{\rm QCD}^4}{Q^4} + \cdots$$

Plaquette:

$$\langle P\rangle = (Pert.\ th.) + a^4\Lambda_{\rm QCD}^4 + \cdots$$

DIS:

$$(Pert.\ th.) + \frac{\Lambda_{\rm QCD}^2}{Q^2} + \cdots$$

Heavy quark physics:

$$(Pert.\ th.) + \frac{\Lambda_{\rm QCD}}{m_Q} + \cdots$$

The natural place to look for these effects.
AP: hep-ph/0105008, hep-ph/0208031, hep-lat/0509022
Bali, AP: hep-ph/0310130
Ayala, Cvetic, AP, 1407.2128.

Where do we expect renormalon effects to be more important?

Current-current correlator ($c_{n*}\,\alpha_s^{n*+1} \sim \Lambda_{\rm QCD}^4/Q^4 \to c_n \sim n!$):

$$\int d^4x e^{iqx}\langle vac|J(x)J(0)|vac\rangle = (Pert.\ th.) + \frac{\Lambda_{\rm QCD}^4}{Q^4} + \cdots$$

Plaquette:

$$\langle P\rangle = (Pert.\ th.) + a^4\Lambda_{\rm QCD}^4 + \cdots$$

DIS:

$$(Pert.\ th.) + \frac{\Lambda_{\rm QCD}^2}{Q^2} + \cdots$$

Heavy quark physics:

$$(Pert.\ th.) + \frac{\Lambda_{\rm QCD}}{m_Q} + \cdots$$

The natural place to look for these effects.
AP: hep-ph/0105008, hep-ph/0208031, hep-lat/0509022
Bali, AP: hep-ph/0310130
Ayala, Cvetic, AP, 1407.2128.

Pole mass

$$m_{\rm OS} = m_{\overline{\rm MS}} + \sum_{n=0}^{\infty} r_n \alpha_s^{n+1} \,,$$

$$m_{\rm OS} = m_{\overline{\rm MS}} + \int_0^\infty {\rm d}t\, e^{-t/\alpha_s} B[m_{\rm OS}](t) \,, \qquad B[m_{\rm OS}](t) \equiv \sum_{n=0}^{\infty} r_n \frac{t^n}{n!} \,.$$

The behavior of the perturbative expansion at large orders is dictated by the closest singularity to the origin of its Borel transform ($u = \frac{\beta_0 t}{4\pi}$).

$$B[m_{\rm OS}](t) = N_m \nu \frac{1}{(1-2u)^{1+b}} \left(1 + c_1(1-2u) + c_2(1-2u)^2 + \cdots \right) + (\text{analytic term}),$$

Next renormalon at $u = 1$.

$$r_n^{asym} = N_m\, \nu \left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1 + \frac{b}{(n+b)} c_1 + \frac{b(b-1)}{(n+b)(n+b-1)} c_2 + \cdots \right) .$$

$$b = \frac{\beta_1}{2\beta_0^2} \,, \qquad c_1 = \frac{1}{4\, b \beta_0^3} \left(\frac{\beta_1^2}{\beta_0} - \beta_2 \right) \,, \qquad \cdots$$

Determination of N_m

$$N_m = \frac{r_n}{(r_n^{asym}/N_m)}$$

$\nu \sim m$

Large β_0 analysis

$$m\left(\frac{\nu}{m}\right)^{2u} \simeq \nu\{1 + (2u-1)\ln\frac{\nu}{m} + \cdots\}.$$

Therefore, the underlying assumption is that we are in a regime where (besides $2u - 1 \ll 1$)

$$(2u-1)\ln\frac{\nu}{m} \ll 1\,.$$

Determination of N_m

$$N_m = \frac{r_n}{(r_n^{asym}/N_m)}$$

$\nu \sim m$

Large β_0 analysis

$$m\left(\frac{\nu}{m}\right)^{2u} \simeq \nu\{1+(2u-1)\ln\frac{\nu}{m}+\cdots\}.$$

Therefore, the underlying assumption is that we are in a regime where (besides $2u-1 \ll 1$)

$$(2u-1)\ln\frac{\nu}{m} \ll 1\,.$$

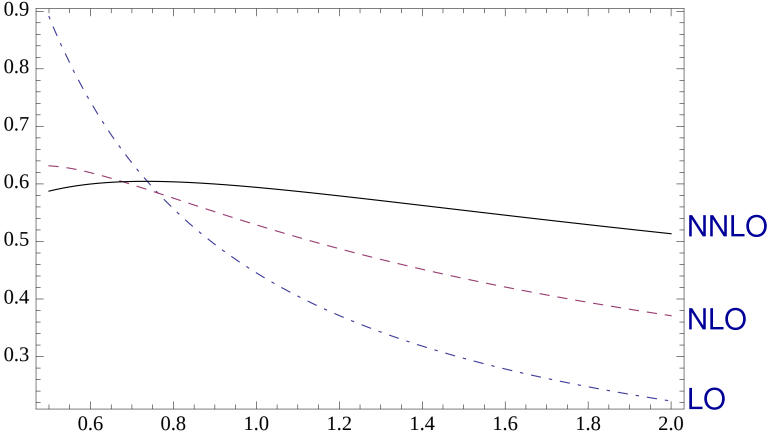


Figure: N_m for $n_l = 3$, as a function of $x \equiv \mu/m_b$, obtained from r_n/r_n^{asym} with r_n^{asym} truncated at $\mathcal{O}(1/n^3)$. We name the different lines as NLO (dashed-dotted), NLO (dashed) and NNLO (solid) for $n = 0, 1, 2$, respectively.

The static potential

$$V(r;\nu_{us}) = \sum_{n=0}^{\infty} V_n \alpha_s^{n+1},$$

$$V_n^{asym} = N_V\, \nu \left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1 + \frac{b}{(n+b)} c_1 + \frac{b(b-1)}{(n+b)(n+b-1)} c_2 + \cdots\right)$$

$2m_{\rm OS} + V_s$ can be understood as an observable up to $O(r^2\Lambda^3_{\rm QCD}, \Lambda^2_{\rm QCD}/m)$ contributions $\rightarrow 2N_m + N_V = 0$

$$N_V = \frac{V_n}{(V_n^{asym}/N_V)}$$

The static potential

$$V(r;\nu_{us}) = \sum_{n=0}^{\infty} V_n \alpha_s^{n+1},$$

$$V_n^{asym} = N_V\,\nu \left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)}\left(1+\frac{b}{(n+b)}c_1+\frac{b(b-1)}{(n+b)(n+b-1)}c_2+\cdots\right)$$

$2m_{\rm OS}+V_s$ can be understood as an observable up to $O(r^2\Lambda_{\rm QCD}^3, \Lambda_{\rm QCD}^2/m)$ contributions $\rightarrow 2N_m + N_V = 0$

$$N_V = \frac{V_n}{(V_n^{asym}/N_V)}$$

The static potential

$$V(r;\nu_{us}) = \sum_{n=0}^{\infty} V_n \alpha_s^{n+1},$$

$$V_n^{asym} = N_V\,\nu \left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)}\left(1+\frac{b}{(n+b)}c_1+\frac{b(b-1)}{(n+b)(n+b-1)}c_2+\cdots\right)$$

$2m_{\rm OS}+V_s$ can be understood as an observable up to $O(r^2\Lambda^3_{\rm QCD},\Lambda^2_{\rm QCD}/m)$ contributions $\rightarrow 2N_m + N_V = 0$

$$N_V = \frac{V_n}{(V_n^{asym}/N_V)}$$

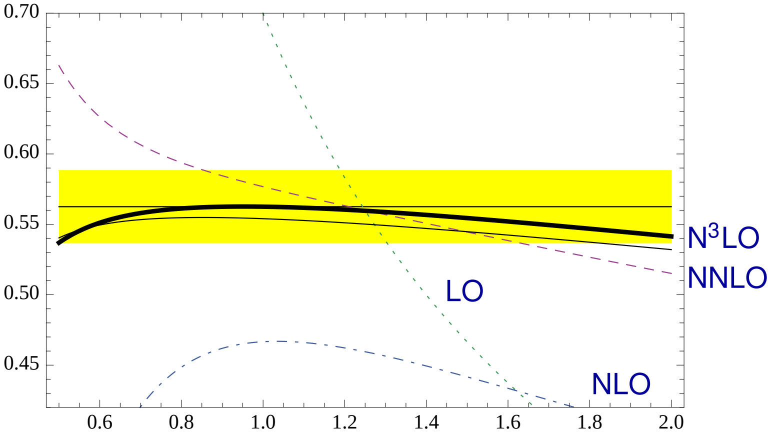


Figure: $-N_V/2 = N_m$ for $n_l = 3$, as a function of $x \equiv \nu r$, obtained from $-(N_V/2)v_n/v_n^{asym}$. v_n^{asym} is truncated at $\mathcal{O}(1/n^3)$.

$$N_m(n_l = 0) = 0.600(29)\,, \qquad N_m(n_l = 3) = 0.563(26)\,.$$

Yet...

- Not possible to compute using known semiclassical analysis.
- Based on few orders in perturbation theory ($\sim 3, 4$)
- Against renormalon existence (Suslov), or against renormalon relevance (Zakharov and followers).

We would like to have a proof (at the same level of existing proofs of a linear potential at long distances), beyond any reasonable doubt, of the existence of the renormalon in QCD.

Bauer, Bali, Pineda: arXiv:1111.3946
Bali, Bauer, Pineda, Torrero: arXiv:1303.3279
Bali, Bauer, Pineda: arXiv:1311.0114

Yet...

- Not possible to compute using known semiclassical analysis.
- Based on few orders in perturbation theory ($\sim 3, 4$)
- Against renormalon existence (Suslov), or against renormalon relevance (Zakharov and followers).

We would like to have a proof (at the same level of existing proofs of a linear potential at long distances), beyond any reasonable doubt, of the existence of the renormalon in QCD.

Bauer, Bali, Pineda: arXiv:1111.3946
Bali, Bauer, Pineda, Torrero: arXiv:1303.3279
Bali, Bauer, Pineda: arXiv:1311.0114

Yet...

- Not possible to compute using known semiclassical analysis.
- Based on few orders in perturbation theory ($\sim 3, 4$)
- Against renormalon existence (Suslov), or against renormalon relevance (Zakharov and followers).

We would like to have a proof (at the same level of existing proofs of a linear potential at long distances), beyond any reasonable doubt, of the existence of the renormalon in QCD.

Bauer, Bali, Pineda: arXiv:1111.3946
Bali, Bauer, Pineda, Torrero: arXiv:1303.3279
Bali, Bauer, Pineda: arXiv:1311.0114

Yet...

- Not possible to compute using known semiclassical analysis.
- Based on few orders in perturbation theory ($\sim 3, 4$)
- Against renormalon existence (Suslov), or against renormalon relevance (Zakharov and followers).

We would like to have a proof (at the same level of existing proofs of a linear potential at long distances), beyond any reasonable doubt, of the existence of the renormalon in QCD.

Bauer, Bali, Pineda: arXiv:1111.3946
Bali, Bauer, Pineda, Torrero: arXiv:1303.3279
Bali, Bauer, Pineda: arXiv:1311.0114

Yet...

- Not possible to compute using known semiclassical analysis.
- Based on few orders in perturbation theory ($\sim 3, 4$)
- Against renormalon existence (Suslov), or against renormalon relevance (Zakharov and followers).

We would like to have a proof (at the same level of existing proofs of a linear potential at long distances), beyond any reasonable doubt, of the existence of the renormalon in QCD.

Bauer, Bali, Pineda: arXiv:1111.3946
Bali, Bauer, Pineda, Torrero: arXiv:1303.3279
Bali, Bauer, Pineda: arXiv:1311.0114

Yet...

- Not possible to compute using known semiclassical analysis.
- Based on few orders in perturbation theory ($\sim 3, 4$)
- Against renormalon existence (Suslov), or against renormalon relevance (Zakharov and followers).

We would like to have a proof (at the same level of existing proofs of a linear potential at long distances), beyond any reasonable doubt, of the existence of the renormalon in QCD.

Bauer, Bali, Pineda: arXiv:1111.3946
Bali, Bauer, Pineda, Torrero: arXiv:1303.3279
Bali, Bauer, Pineda: arXiv:1311.0114

POLYAKOV LOOP versus δm (and m)

Possible to compute the energy of an static source in the lattice: δm of HQET.
We use Numerical Stochastic Perturbation Theory (Di Renzo et al.).

$$L^{(R)}(N_S, N_T) = \frac{1}{N_S^3} \sum_{\mathbf{n}} \frac{1}{d_R} \mathrm{tr} \left[\prod_{n_4=0}^{N_T-1} U_4^R(n) \right] \qquad U_\mu^R(n) \approx e^{iA_\mu^R[(n+1/2)a]}$$

We implement triplet and octet representations R ($d_R = 3, 8$).

$$L^{(R)}(N_S, N_T) \overset{N_T \to \infty}{\sim} e^{-N_T a \delta m^{(R)}(N_S)}$$

$$\begin{aligned} \delta m^{(R)}(N_S) &= -\lim_{N_T \to \infty} \frac{\ln \langle L^{(R,\rho)}(N_S, N_T) \rangle}{a N_T} = \lim_{N_T \to \infty} \sum_{n=0}^{\infty} c_n^{(R,\rho)}(N_S, N_T) \alpha^{n+1} \\ &= \frac{1}{a} \sum_{n=0}^{\infty} c_n^{(R,\rho)}(N_S) \alpha^{n+1}(1/a) \end{aligned}$$

POLYAKOV LOOP versus δm (and m)

Possible to compute the energy of an static source in the lattice: δm of HQET.
We use Numerical Stochastic Perturbation Theory (Di Renzo et al.).

$$L^{(R)}(N_S, N_T) = \frac{1}{N_S^3} \sum_{\mathbf{n}} \frac{1}{d_R} \mathrm{tr} \left[\prod_{n_4=0}^{N_T-1} U_4^R(n) \right] \qquad U_\mu^R(n) \approx e^{iA_\mu^R[(n+1/2)a]}$$

We implement triplet and octet representations R ($d_R = 3, 8$).

$$L^{(R)}(N_S, N_T) \overset{N_T \to \infty}{\sim} e^{-N_T a \delta m^{(R)}(N_S)}$$

$$\begin{aligned} \delta m^{(R)}(N_S) &= -\lim_{N_T \to \infty} \frac{\ln \langle L^{(R,\rho)}(N_S, N_T) \rangle}{a N_T} = \lim_{N_T \to \infty} \sum_{n=0}^{\infty} c_n^{(R,\rho)}(N_S, N_T) \alpha^{n+1} \\ &= \frac{1}{a} \sum_{n=0}^{\infty} c_n^{(R,\rho)}(N_S) \alpha^{n+1}(1/a) \end{aligned}$$

POLYAKOV LOOP versus δm (and m)

Possible to compute the energy of an static source in the lattice: δm of HQET.
We use Numerical Stochastic Perturbation Theory (Di Renzo et al.).

$$L^{(R)}(N_S, N_T) = \frac{1}{N_S^3} \sum_{\mathbf{n}} \frac{1}{d_R} \mathrm{tr} \left[\prod_{n_4=0}^{N_T-1} U_4^R(n) \right] \qquad U_\mu^R(n) \approx e^{iA_\mu^R[(n+1/2)a]}$$

We implement triplet and octet representations R ($d_R = 3, 8$).

$$L^{(R)}(N_S, N_T) \overset{N_T \to \infty}{\sim} e^{-N_T a \delta m^{(R)}(N_S)}$$

$$\delta m^{(R)}(N_S) = -\lim_{N_T \to \infty} \frac{\ln \langle L^{(R,\rho)}(N_S, N_T) \rangle}{a N_T} = \lim_{N_T \to \infty} \sum_{n=0}^{\infty} c_n^{(R,\rho)}(N_S, N_T) \alpha^{n+1}$$

$$= \frac{1}{a} \sum_{n=0}^{\infty} c_n^{(R,\rho)}(N_S) \alpha^{n+1}(1/a)$$

POLYAKOV LOOP versus δm (and m)

Possible to compute the energy of an static source in the lattice: δm of HQET. We use Numerical Stochastic Perturbation Theory (Di Renzo et al.).

$$L^{(R)}(N_S, N_T) = \frac{1}{N_S^3} \sum_{\mathbf{n}} \frac{1}{d_R} \mathrm{tr} \left[\prod_{n_4=0}^{N_T-1} U_4^R(n) \right] \qquad U_\mu^R(n) \approx e^{iA_\mu^R[(n+1/2)a]}$$

We implement triplet and octet representations R ($d_R = 3, 8$).

$$L^{(R)}(N_S, N_T) \overset{N_T \to \infty}{\sim} e^{-N_T a \delta m^{(R)}(N_S)}$$

$$\begin{aligned} \delta m^{(R)}(N_S) &= -\lim_{N_T \to \infty} \frac{\ln \langle L^{(R,\rho)}(N_S, N_T) \rangle}{a N_T} = \lim_{N_T \to \infty} \sum_{n=0}^{\infty} c_n^{(R,\rho)}(N_S, N_T) \alpha^{n+1} \\ &= \frac{1}{a} \sum_{n=0}^{\infty} c_n^{(R,\rho)}(N_S) \alpha^{n+1}(1/a) \end{aligned}$$

Perturbative OPE (Zimmermann) at finite volume ($N_S \to \infty$)

$$\delta m = \lim_{N_S\to\infty} \delta m(N_S) \quad c_n = \lim_{N_S\to\infty} c_n(N_S) \quad \left(\lim_{n\to\infty} c_n^{(3,\rho)} = r_n(\nu)/\nu\right).$$

For large N_S, we write (OPE: $\frac{1}{a} \gg \frac{1}{N_S a}$)

$$\delta m(N_S) = \frac{1}{a}\sum_{n=0}^{\infty} c_n \alpha^{n+1}\left(a^{-1}\right) - \frac{1}{aN_S}\sum_{n=0}^{\infty} f_n \alpha^{n+1}\left((aN_S)^{-1}\right) + \mathcal{O}\left(\frac{1}{N_S^2}\right).$$

Taylor expansion of $\alpha\left((aN_S)^{-1}\right)$ in powers of $\alpha(a^{-1})$:

$$c_n(N_S) = c_n - \frac{f_n(N_S)}{N_S} + \mathcal{O}\left(\frac{1}{N_S^2}\right); \qquad f_n(N_S) = \sum_{i=0}^{n} f_n^{(i)} \ln^i(N_S),$$

$f_n^{(0)} = f_n$. The coefficients $f_n^{(i)}$ for $i > 0$ are determined by f_m and β_j.

$$f_1(N_S) = f_1 + f_0 \frac{\beta_0}{2\pi} \ln(N_S),$$

$$f_2(N_S) = f_2 + \left[2f_1 \frac{\beta_0}{2\pi} + f_0 \frac{\beta_1}{8\pi^2}\right] \ln(N_S) + f_0 \left(\frac{\beta_0}{2\pi}\right)^2 \ln^2(N_S),$$

and so on.

Perturbative OPE (Zimmermann) at finite volume ($N_S \to \infty$)

$$\delta m = \lim_{N_S \to \infty} \delta m(N_S) \qquad c_n = \lim_{N_S \to \infty} c_n(N_S) \qquad \left(\lim_{n \to \infty} c_n^{(3,\rho)} = r_n(\nu)/\nu \right).$$

For large N_S, we write (OPE: $\frac{1}{a} \gg \frac{1}{N_S a}$)

$$\delta m(N_S) = \frac{1}{a} \sum_{n=0}^{\infty} c_n \alpha^{n+1}\left(a^{-1}\right) - \frac{1}{aN_S} \sum_{n=0}^{\infty} f_n \alpha^{n+1}\left((aN_S)^{-1}\right) + \mathcal{O}\left(\frac{1}{N_S^2}\right).$$

Taylor expansion of $\alpha\left((aN_S)^{-1}\right)$ in powers of $\alpha(a^{-1})$:

$$c_n(N_S) = c_n - \frac{f_n(N_S)}{N_S} + \mathcal{O}\left(\frac{1}{N_S^2}\right); \qquad f_n(N_S) = \sum_{i=0}^{n} f_n^{(i)} \ln^i(N_S),$$

$f_n^{(0)} = f_n$. The coefficients $f_n^{(i)}$ for $i > 0$ are determined by f_m and β_j.

$$f_1(N_S) = f_1 + f_0 \frac{\beta_0}{2\pi} \ln(N_S),$$

$$f_2(N_S) = f_2 + \left[2f_1 \frac{\beta_0}{2\pi} + f_0 \frac{\beta_1}{8\pi^2}\right] \ln(N_S) + f_0 \left(\frac{\beta_0}{2\pi}\right)^2 \ln^2(N_S),$$

and so on.

Perturbative OPE (Zimmermann) at finite volume ($N_S \to \infty$)

$$\delta m = \lim_{N_S\to\infty} \delta m(N_S) \quad c_n = \lim_{N_S\to\infty} c_n(N_S) \quad \left(\lim_{n\to\infty} c_n^{(3,\rho)} = r_n(\nu)/\nu\right).$$

For large N_S, we write (OPE: $\frac{1}{a} \gg \frac{1}{N_S a}$)

$$\delta m(N_S) = \frac{1}{a}\sum_{n=0}^{\infty} c_n \alpha^{n+1}\left(a^{-1}\right) - \frac{1}{aN_S}\sum_{n=0}^{\infty} f_n \alpha^{n+1}\left((aN_S)^{-1}\right) + \mathcal{O}\left(\frac{1}{N_S^2}\right).$$

Taylor expansion of $\alpha\left((aN_S)^{-1}\right)$ in powers of $\alpha(a^{-1})$:

$$c_n(N_S) = c_n - \frac{f_n(N_S)}{N_S} + \mathcal{O}\left(\frac{1}{N_S^2}\right); \qquad f_n(N_S) = \sum_{i=0}^{n} f_n^{(i)} \ln^i(N_S),$$

$f_n^{(0)} = f_n$. The coefficients $f_n^{(i)}$ for $i > 0$ are determined by f_m and β_j.

$$f_1(N_S) = f_1 + f_0 \frac{\beta_0}{2\pi} \ln(N_S),$$

$$f_2(N_S) = f_2 + \left[2f_1 \frac{\beta_0}{2\pi} + f_0 \frac{\beta_1}{8\pi^2}\right] \ln(N_S) + f_0 \left(\frac{\beta_0}{2\pi}\right)^2 \ln^2(N_S),$$

and so on.

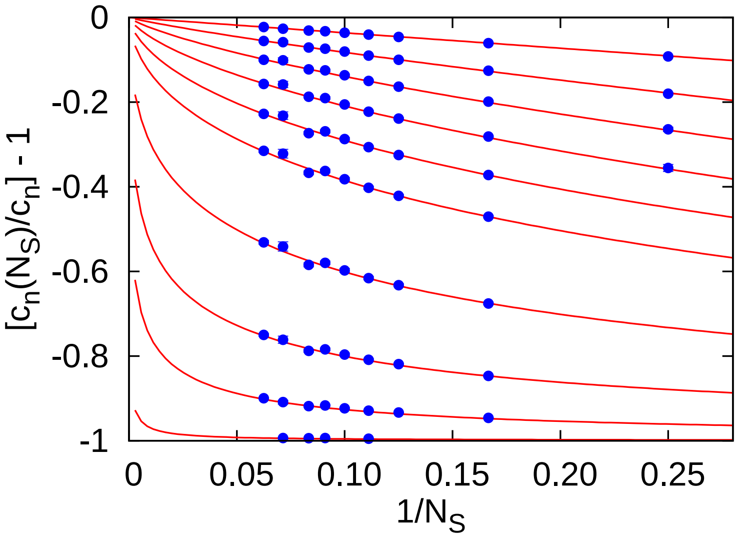


Figure: $c_n^{(3,0)}(N_S)/c_n^{(3,0)} - 1$ *for* $n \in \{0, 1, 2, 3, 4, 5, 7, 9, 11, 15\}$ *(top to bottom). For each value of* N_S *we have plotted the data point with the maximum value of* N_T*. The curves represent the global fit.* $-(1/N_S)f_{0,\mathrm{DLPT}}^{(3,0)}/c_{0,\mathrm{DLPT}}^{(3,0)}$ *is shown for* $n = 0$.

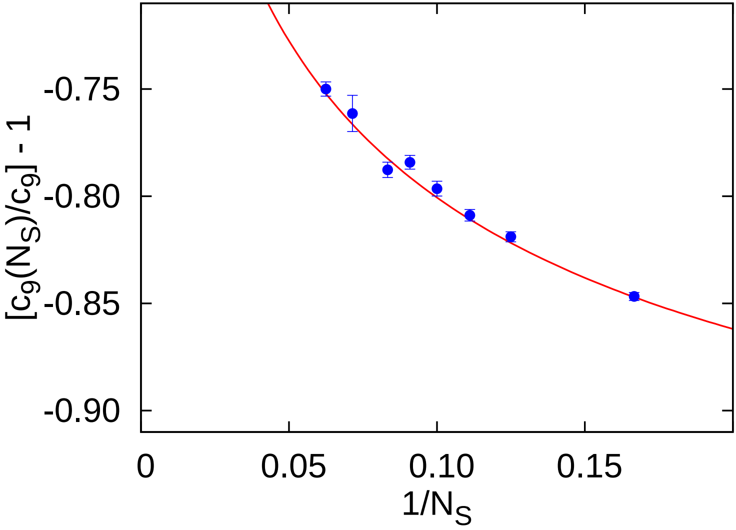


Figure: *Zoom of previous Figure for $n = 9$.*

	$c_n^{(3,0)}$	$c_n^{(3,1/6)}$	$c_n^{(8,0)} C_F/C_A$	$c_n^{(8,1/6)} C_F/C_A$
c_0	2.117274357	0.72181(99)	2.117274357	0.72181(99)
c_1	11.136(11)	6.385(10)	11.140(12)	6.387(10)
$c_2/10$	8.610(13)	8.124(12)	8.587(14)	8.129(12)
$c_3/10^2$	7.945(16)	7.670(13)	7.917(20)	7.682(15)
$c_4/10^3$	8.215(34)	8.017(33)	8.197(42)	8.017(36)
$c_5/10^4$	9.322(59)	9.160(59)	9.295(76)	9.139(64)
$c_6/10^6$	1.153(11)	1.138(11)	1.144(13)	1.134(12)
$c_7/10^7$	1.558(21)	1.541(22)	1.533(25)	1.535(22)
$c_8/10^8$	2.304(43)	2.284(45)	2.254(51)	2.275(45)
$c_9/10^9$	3.747(95)	3.717(97)	3.64(11)	3.703(98)
$c_{10}/10^{10}$	6.70(22)	6.65(22)	6.49(25)	6.63(22)
$c_{11}/10^{12}$	1.316(52)	1.306(53)	1.269(59)	1.303(53)
$c_{12}/10^{13}$	2.81(13)	2.79(13)	2.71(14)	2.78(13)
$c_{13}/10^{14}$	6.51(35)	6.46(35)	6.29(37)	6.45(35)
$c_{14}/10^{16}$	1.628(96)	1.613(97)	1.57(10)	1.614(97)
$c_{15}/10^{17}$	4.36(28)	4.32(28)	4.22(29)	4.33(28)
$c_{16}/10^{19}$	1.247(86)	1.235(86)	1.206(89)	1.236(86)
$c_{17}/10^{20}$	3.78(28)	3.75(28)	3.66(28)	3.75(28)
$c_{18}/10^{22}$	1.215(93)	1.204(94)	1.176(95)	1.205(94)
$c_{19}/10^{23}$	4.12(33)	4.08(33)	3.99(34)	4.08(33)

	$f_n^{(3,0)}$	$f_n^{(3,1/6)}$	$f_n^{(8,0)} C_F/C_A$	$f_n^{(8,1/6)} C_F/C_A$
f_0	0.7696256328	0.7810(59)	0.7696256328	0.7810(69)
f_1	6.075(78)	6.046(58)	6.124(87)	6.063(68)
$f_2/10$	5.628(91)	5.644(62)	5.60(11)	5.691(78)
$f_3/10^2$	5.87(11)	5.858(76)	6.00(18)	5.946(91)
$f_4/10^3$	6.33(22)	6.29(17)	6.57(40)	6.26(23)
$f_5/10^4$	7.73(35)	7.71(26)	7.67(66)	7.78(42)
$f_6/10^5$	9.86(53)	9.80(42)	9.68(99)	9.79(69)
$f_7/10^7$	1.388(81)	1.378(71)	1.35(15)	1.38(11)
$f_8/10^8$	2.12(12)	2.11(12)	2.06(22)	2.10(17)
$f_9/10^9$	3.54(20)	3.52(20)	3.40(37)	3.51(27)
$f_{10}/10^{10}$	6.49(33)	6.44(34)	6.23(67)	6.44(43)
$f_{11}/10^{12}$	1.296(64)	1.286(66)	1.24(13)	1.286(74)
$f_{12}/10^{13}$	2.68(19)	2.64(18)	2.65(33)	2.65(21)
$f_{13}/10^{14}$	6.70(54)	6.68(52)	6.36(90)	6.66(57)
$f_{14}/10^{16}$	1.58(14)	1.56(14)	1.55(22)	1.57(15)
$f_{15}/10^{17}$	4.41(34)	4.37(33)	4.24(47)	4.37(35)
$f_{16}/10^{19}$	1.241(92)	1.230(91)	1.20(11)	1.231(94)
$f_{17}/10^{20}$	3.79(28)	3.75(28)	3.67(30)	3.76(28)
$f_{18}/10^{22}$	1.215(94)	1.204(94)	1.176(97)	1.205(94)
$f_{19}/10^{23}$	4.12(33)	4.08(33)	3.99(34)	4.08(33)

Ratios

$$\frac{c_n^{(3,\rho)}}{c_{n-1}^{(3,\rho)}}\frac{1}{n} = \frac{c_n^{(8,\rho)}}{c_{n-1}^{(8,\rho)}}\frac{1}{n}$$
$$= \frac{\beta_0}{2\pi}\left\{1 + \frac{b}{n} - \frac{bs_1}{n^2} + \frac{1}{n^3}\left[b^2 s_1^2 + b(b-1)(s_1 - 2s_2)\right] + \mathcal{O}\left(\frac{1}{n^4}\right)\right\} \; .$$

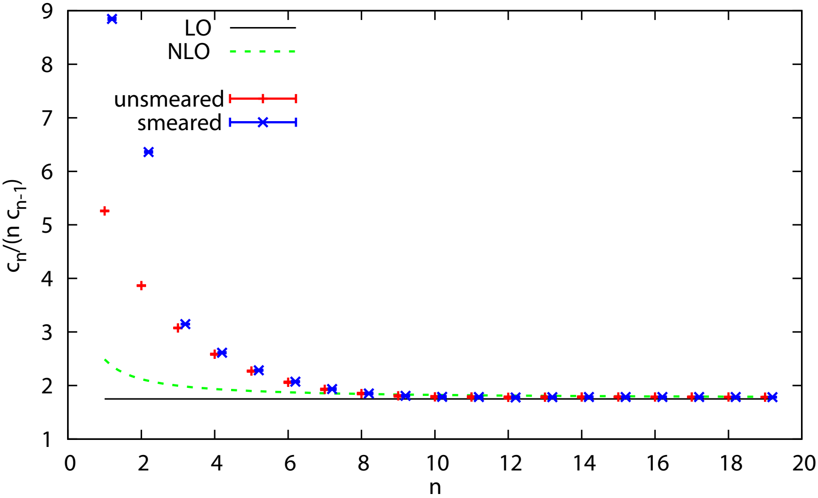


Figure: Ratios $c_n/(nc_{n-1})$ of the smeared (blue) and unsmeared (red) triplet static self-energy coefficients c_n in comparison to the theoretical prediction at different orders in the $1/n$ expansion.

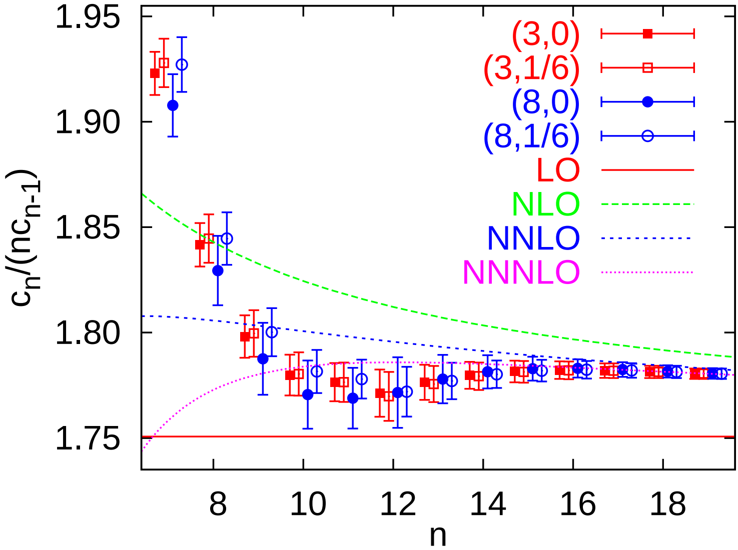


Figure: The ratios $c_n/(nc_{n-1})$ for the smeared and unsmeared, triplet and octet static self-energies, compared to the prediction for the LO, next-to-leading order (NLO), NNLO and NNNLO of the $1/n$ expansion.

N_m

$$c_n^{fitted} = N_m \left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1 + \frac{b}{(n+b)}c_1 + \frac{b(b-1)}{(n+b)(n+b-1)}c_2 + \cdots\right).$$

$$f_n^{fitted} = N_m \left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1 + \frac{b}{(n+b)}c_1 + \frac{b(b-1)}{(n+b)(n+b-1)}c_2 + \cdots\right).$$

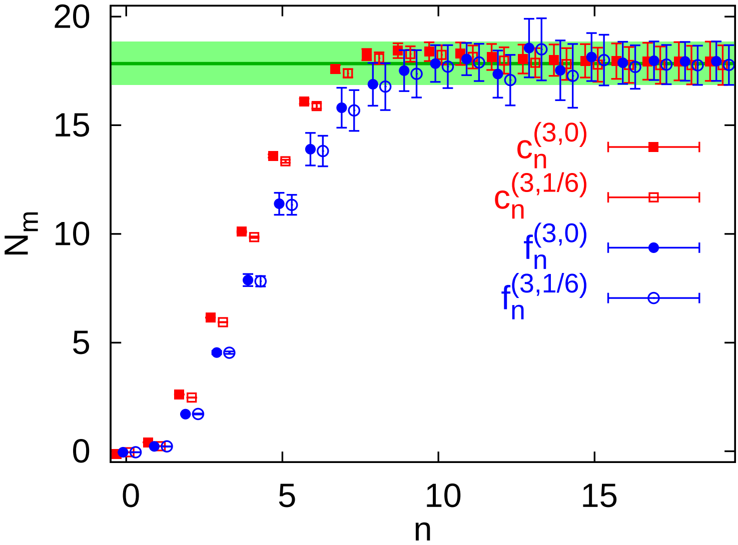


Figure: N_m^{latt} determined via r_n truncated at NNNLO, from the coefficients $c_n^{(3,0)}$, $c_n^{(3,1/6)}$, $f_n^{(3,0)}$ and $f_n^{(3,1/6)}$. The horizontal band is our final result: $N_m = 17.9(1.0)$.

From lattice to $\overline{\text{MS}}$ scheme

$$\alpha_{\overline{\text{MS}}}(\mu) = \alpha_{\text{latt}}(\mu)\left(1 + d_1\alpha_{\text{latt}}(\mu) + d_2\alpha^2_{\text{latt}}(\mu) + d_3\alpha^3_{\text{latt}}(\mu) + \mathcal{O}(\alpha^4_{\text{latt}})\right) \,,$$

$$N^{\overline{\text{MS}}}_{m,m_{\tilde{g}}} = N^{\text{latt}}_{m,m_{\tilde{g}}}\Lambda_{\text{latt}}/\Lambda_{\overline{\text{MS}}}\,, \quad \text{where} \quad \Lambda_{\overline{\text{MS}}} = e^{\frac{2\pi d_1}{\beta_0}}\Lambda_{\text{latt}} \approx 28.809338139488\,\Lambda_{\text{latt}}\,.$$

This yields the numerical values

$$N^{\overline{\text{MS}}}_m = 0.620(35)\,, \quad C_F/C_A\,N^{\overline{\text{MS}}}_{m_{\tilde{g}}} = -C_F/C_A\,N^{\overline{\text{MS}}}_{\Lambda} = 0.610(41)\,.$$

Before $N^{\overline{\text{MS}}}_m = 0.600(29)$. Combined $N^{\overline{\text{MS}}}_m = 0.608(22)$.

From lattice to $\overline{\rm MS}$ scheme

$$\alpha_{\overline{\rm MS}}(\mu) = \alpha_{\rm latt}(\mu)\left(1 + d_1\alpha_{\rm latt}(\mu) + d_2\alpha^2_{\rm latt}(\mu) + d_3\alpha^3_{\rm latt}(\mu) + \mathcal{O}(\alpha^4_{\rm latt})\right)\,,$$

$$N^{\overline{\rm MS}}_{m,m_{\tilde g}} = N^{\rm latt}_{m,m_{\tilde g}}\Lambda_{\rm latt}/\Lambda_{\overline{\rm MS}}\,, \quad \text{where} \quad \Lambda_{\overline{\rm MS}} = e^{\frac{2\pi d_1}{\beta_0}}\Lambda_{\rm latt} \approx 28.809338139488\,\Lambda_{\rm latt}\,.$$

This yields the numerical values

$$N^{\overline{\rm MS}}_m = 0.620(35)\,, \quad C_F/C_A\,N^{\overline{\rm MS}}_{m_{\tilde g}} = -C_F/C_A\,N^{\overline{\rm MS}}_{\Lambda} = 0.610(41)\,.$$

Before $N^{\overline{\rm MS}}_m = 0.600(29)$. Combined $N^{\overline{\rm MS}}_m = 0.608(22)$.

From lattice to $\overline{\rm MS}$ scheme

$$\alpha_{\overline{\rm MS}}(\mu) = \alpha_{\rm latt}(\mu)\left(1 + d_1\alpha_{\rm latt}(\mu) + d_2\alpha^2_{\rm latt}(\mu) + d_3\alpha^3_{\rm latt}(\mu) + \mathcal{O}(\alpha^4_{\rm latt})\right) \,,$$

$$N^{\overline{\rm MS}}_{m,m_{\tilde g}} = N^{\rm latt}_{m,m_{\tilde g}}\Lambda_{\rm latt}/\Lambda_{\overline{\rm MS}}\,, \quad \text{where} \quad \Lambda_{\overline{\rm MS}} = e^{\frac{2\pi d_1}{\beta_0}}\Lambda_{\rm latt} \approx 28.809338139488\,\Lambda_{\rm latt}\,.$$

This yields the numerical values

$$N^{\overline{\rm MS}}_m = 0.620(35)\,, \quad C_F/C_A\,N^{\overline{\rm MS}}_{m_{\tilde g}} = -C_F/C_A\,N^{\overline{\rm MS}}_{\Lambda} = 0.610(41)\,.$$

Before $N^{\overline{\rm MS}}_m = 0.600(29)$. Combined $N^{\overline{\rm MS}}_m = 0.608(22)$.

Plaquette (Bali, Bauer, AP: 1401.7999, 1403.6477)

$$\langle P\rangle_{\rm pert}(N) \equiv \frac{1}{Z}\int [dU_{x,\mu}]\, e^{-S[U]}\, P[U]\Big|_{\rm NSPT} = \sum_{n\geq 0} p_n(N)\alpha^{n+1}$$

Perturbative OPE

$$\frac{1}{a} \gg \frac{1}{Na} \to \langle P\rangle_{\rm pert}(N) = P_{\rm pert}(\alpha)\langle 1\rangle + \frac{\pi^2}{36}C_{\rm G}(\alpha)\, a^4\langle G^2\rangle_{\rm soft} + \mathcal{O}\left(\frac{1}{N^6}\right),$$

where

$$P_{\rm pert} = \sum_{n\geq 0} p_n\alpha^{n+1},\ C_{\rm G} = 1+\sum_{k\geq 0} c_k\alpha^{k+1},\ \frac{\pi^2}{36}\, a^4\langle G^2\rangle_{\rm soft} = -\frac{1}{N^4}\sum_{n\geq 0} f_n\alpha^{n+1}((Na)^{-1})$$

$$\langle P\rangle_{\rm pert}(N) = \sum_{n\geq 0}\left[p_n - \frac{f_n(N)}{N^4}\right]\alpha^{n+1}$$

$$= \sum_{n\geq 0} p_n\alpha^{n+1} - \frac{1}{N^4}\left(1+\sum_{k\geq 0} c_k\alpha^{k+1}(a^{-1})\right)\times \sum_{n\geq 0} f_n\alpha^{n+1}((Na)^{-1}) + \mathcal{O}\left(\frac{1}{N^6}\right),$$

$$\left(\delta m(N_S) = \frac{1}{a}\sum_{n=0}^{\infty} c_n\alpha^{n+1}\left(a^{-1}\right) - \frac{1}{aN_S}\sum_{n=0}^{\infty} f_n\alpha^{n+1}\left((aN_S)^{-1}\right) + \mathcal{O}\left(\frac{1}{N_S^2}\right)\right)$$

Plaquette (Bali, Bauer, AP: 1401.7999, 1403.6477)

$$\langle P\rangle_{\rm pert}(N) \equiv \frac{1}{Z}\int [dU_{x,\mu}]\, e^{-S[U]} P[U]\Big|_{\rm NSPT} = \sum_{n\geq 0} p_n(N)\alpha^{n+1}$$

Perturbative OPE

$$\frac{1}{a} \gg \frac{1}{Na} \to \langle P\rangle_{\rm pert}(N) = P_{\rm pert}(\alpha)\langle 1\rangle + \frac{\pi^2}{36} C_{\rm G}(\alpha)\, a^4\langle G^2\rangle_{\rm soft} + \mathcal{O}\left(\frac{1}{N^6}\right),$$

where

$$P_{\rm pert} = \sum_{n\geq 0} p_n\alpha^{n+1},\ C_{\rm G} = 1+\sum_{k\geq 0} c_k\alpha^{k+1},\ \frac{\pi^2}{36}\, a^4\langle G^2\rangle_{\rm soft} = -\frac{1}{N^4}\sum_{n\geq 0} f_n\alpha^{n+1}((Na)^{-1})$$

$$\langle P\rangle_{\rm pert}(N) = \sum_{n\geq 0}\left[p_n - \frac{f_n(N)}{N^4}\right]\alpha^{n+1}$$

$$= \sum_{n\geq 0} p_n\alpha^{n+1} - \frac{1}{N^4}\left(1+\sum_{k\geq 0} c_k\alpha^{k+1}(a^{-1})\right)\times\sum_{n\geq 0} f_n\alpha^{n+1}((Na)^{-1}) + \mathcal{O}\left(\frac{1}{N^6}\right),$$

$$\left(\delta m(N_S) = \frac{1}{a}\sum_{n=0}^{\infty} c_n\alpha^{n+1}\left(a^{-1}\right) - \frac{1}{aN_S}\sum_{n=0}^{\infty} f_n\alpha^{n+1}\left((aN_S)^{-1}\right) + \mathcal{O}\left(\frac{1}{N_S^2}\right)\right)$$

Plaquette (Bali, Bauer, AP: 1401.7999, 1403.6477)

$$\langle P\rangle_{\rm pert}(N) \equiv \frac{1}{Z}\int [dU_{x,\mu}]\, e^{-S[U]} P[U]\Big|_{\rm NSPT} = \sum_{n\geq 0} p_n(N)\alpha^{n+1}$$

Perturbative OPE

$$\frac{1}{a} \gg \frac{1}{Na} \to \langle P\rangle_{\rm pert}(N) = P_{\rm pert}(\alpha)\langle 1\rangle + \frac{\pi^2}{36} C_{\rm G}(\alpha)\, a^4 \langle G^2\rangle_{\rm soft} + \mathcal{O}\left(\frac{1}{N^6}\right),$$

where

$$P_{\rm pert} = \sum_{n\geq 0} p_n \alpha^{n+1}, \; C_{\rm G} = 1 + \sum_{k\geq 0} c_k \alpha^{k+1}, \; \frac{\pi^2}{36} a^4 \langle G^2\rangle_{\rm soft} = -\frac{1}{N^4}\sum_{n\geq 0} f_n \alpha^{n+1}((Na)^{-1})$$

$$\langle P\rangle_{\rm pert}(N) = \sum_{n\geq 0}\left[p_n - \frac{f_n(N)}{N^4}\right]\alpha^{n+1}$$

$$= \sum_{n\geq 0} p_n \alpha^{n+1} - \frac{1}{N^4}\left(1 + \sum_{k\geq 0} c_k \alpha^{k+1}(a^{-1})\right) \times \sum_{n\geq 0} f_n \alpha^{n+1}((Na)^{-1}) + \mathcal{O}\left(\frac{1}{N^6}\right),$$

$$\left(\delta m(N_S) = \frac{1}{a}\sum_{n=0}^{\infty} c_n \alpha^{n+1}\left(a^{-1}\right) - \frac{1}{aN_S}\sum_{n=0}^{\infty} f_n \alpha^{n+1}\left((aN_S)^{-1}\right) + \mathcal{O}\left(\frac{1}{N_S^2}\right)\right)$$

$$\langle P\rangle = \sum_{n=0}^{N} p_n \alpha^{n+1}(a^{-1}) + a^4 \frac{\pi^2}{36}\langle G^2\rangle + \cdots$$

$$d = 1(n_0 \sim 7) \longrightarrow d = 4(n_0 \sim 28)$$

$$N + 1 = 35$$

(before Di Renzo et al. N+1=8; Horsley et al. N+1=20)
Renormalon expectations:

$$p_n^{\rm latt} \overset{n\rightarrow\infty}{=} N_P^{\rm latt}\left(\frac{\beta_0}{2\pi d}\right)^n \frac{\Gamma(n+1+db)}{\Gamma(1+db)} \times \left\{1 + \frac{20.08931\ldots}{n+db} + \frac{505\pm 33}{(n+db)(n+db-1)} + \mathcal{O}\left(\frac{1}{n^3}\right)\right\}.$$

$$\frac{p_n}{np_{n-1}} = \frac{\beta_0}{2\pi d}\left\{1 + \frac{db}{n} + \frac{db(1-ds_1)}{n^2} + \frac{db\left[1-3ds_1+d^2b(s_1+2s_2)\right]}{n^3} + \mathcal{O}\left(\frac{1}{n^4}\right)\right\}.$$

$$\langle P \rangle = \sum_{n=0}^{N} p_n \alpha^{n+1}(a^{-1}) + a^4 \frac{\pi^2}{36} \langle G^2 \rangle + \cdots$$

$$d = 1 (n_0 \sim 7) \longrightarrow d = 4 (n_0 \sim 28)$$

$$N + 1 = 35$$

(before Di Renzo et al. N+1=8; Horsley et al. N+1=20)

Renormalon expectations:

$$p_n^{\text{latt}} \stackrel{n \to \infty}{=} N_P^{\text{latt}} \left(\frac{\beta_0}{2\pi d} \right)^n \frac{\Gamma(n+1+db)}{\Gamma(1+db)} \times \left\{ 1 + \frac{20.08931\ldots}{n+db} + \frac{505 \pm 33}{(n+db)(n+db-1)} + \mathcal{O}\left(\frac{1}{n^3}\right) \right\} .$$

$$\frac{p_n}{n p_{n-1}} = \frac{\beta_0}{2\pi d} \left\{ 1 + \frac{db}{n} + \frac{db(1-ds_1)}{n^2} + \frac{db\left[1 - 3ds_1 + d^2 b(s_1 + 2s_2)\right]}{n^3} + \mathcal{O}\left(\frac{1}{n^4}\right) \right\} .$$

$$\langle P\rangle = \sum_{n=0}^{N} p_n \alpha^{n+1}(a^{-1}) + a^4 \frac{\pi^2}{36}\langle G^2\rangle + \cdots$$

$$d = 1 (n_0 \sim 7) \longrightarrow d = 4 (n_0 \sim 28)$$

$$N + 1 = 35$$

(before Di Renzo et al. N+1=8; Horsley et al. N+1=20)
Renormalon expectations:

$$p_n^{\rm latt} \overset{n\to\infty}{=} N_P^{\rm latt} \left(\frac{\beta_0}{2\pi d}\right)^n \frac{\Gamma(n+1+db)}{\Gamma(1+db)} \times \left\{1 + \frac{20.08931\ldots}{n+db} + \frac{505 \pm 33}{(n+db)(n+db-1)} + \mathcal{O}\left(\frac{1}{n^3}\right)\right\}.$$

$$\frac{p_n}{n p_{n-1}} = \frac{\beta_0}{2\pi d}\left\{1 + \frac{db}{n} + \frac{db(1-ds_1)}{n^2} + \frac{db\left[1 - 3ds_1 + d^2 b(s_1 + 2s_2)\right]}{n^3} + \mathcal{O}\left(\frac{1}{n^4}\right)\right\}.$$

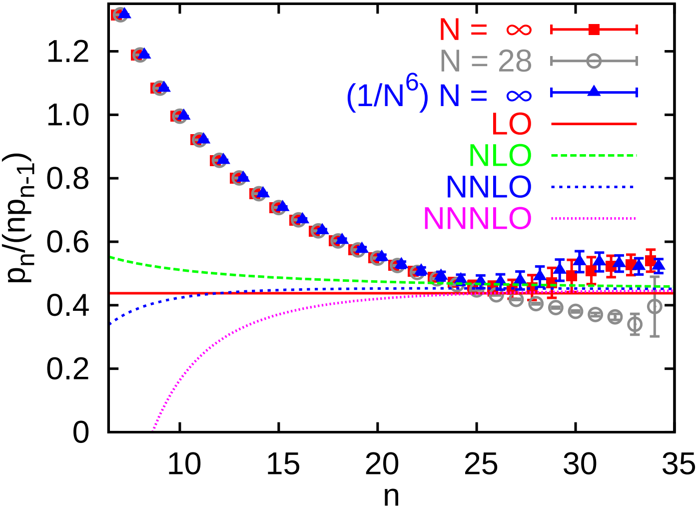


Figure: Ratios $p_n/(np_{n-1})$ of the plaquette coefficients p_n ($N = \infty$, $N = 28$) in comparison to the theoretical prediction at different orders in the $1/n$ expansion.

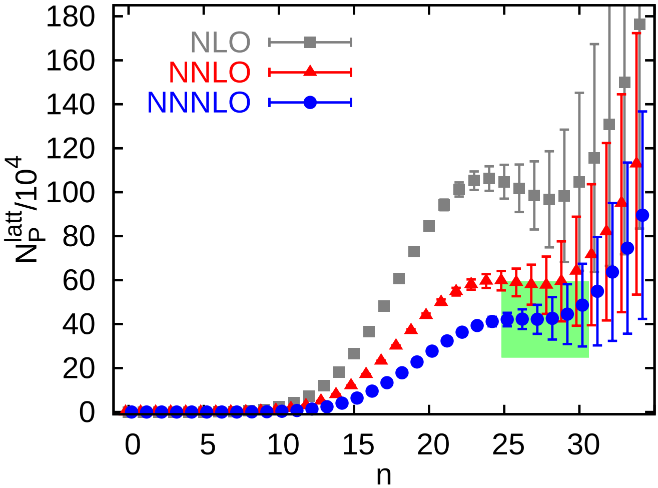


Figure: N_P, determined from the coefficients p_n truncated at NLO, NNLO and NNNLO. The green box marks our final result.

$$N_P^{\overline{\text{MS}}} = 0.61(25) \qquad N_G^{\overline{\text{MS}}} = \frac{36}{\pi^2} N_P^{\overline{\text{MS}}} = 2.24(92)\,.$$

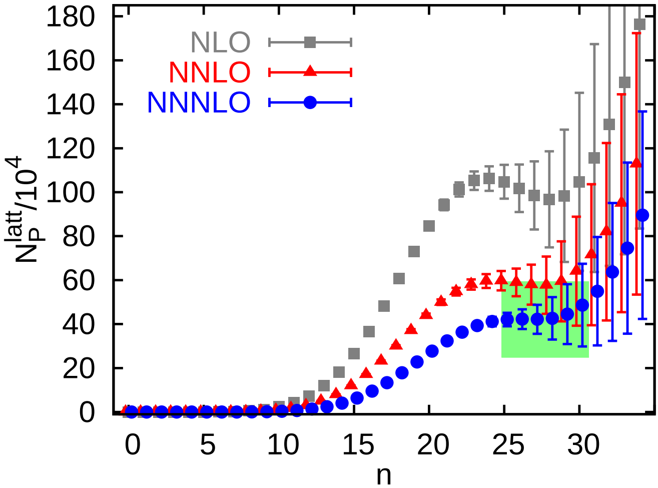


Figure: N_P, determined from the coefficients p_n truncated at NLO, NNLO and NNNLO. The green box marks our final result.

$$N_P^{\overline{\text{MS}}} = 0.61(25) \qquad N_G^{\overline{\text{MS}}} = \frac{36}{\pi^2} N_P^{\overline{\text{MS}}} = 2.24(92)\,.$$

Beyond perturbation theory

$$\langle P \rangle_{\rm pert} = \frac{1}{Z} \int [dU_{x,\mu}]\, e^{-S[U]} P[U] \Big|_{\rm NSPT} = P_{\rm pert}(\alpha)\langle 1 \rangle + \frac{\pi^2}{36} C_{\rm G}(\alpha)\, a^4 \langle O_{\rm G} \rangle_{\rm soft} + \mathcal{O}\left(a^6\right) .$$

$$\frac{1}{a} \gg \frac{1}{Na}$$

Beyond perturbation theory

$$\langle P\rangle_{\rm MC} = \frac{1}{Z}\int [dU_{x,\mu}]\, e^{-S[U]} P[U]\Big|_{\rm MC} = P_{\rm pert}(\alpha)\langle 1\rangle + \frac{\pi^2}{36} C_{\rm G}(\alpha)\, a^4 \langle G^2\rangle_{\rm MC} + \mathcal{O}\left(a^6\right) .$$

$$\frac{1}{a} \gg \frac{1}{Na} \gg \Lambda_{\rm QCD} \quad \to \quad \langle G^2\rangle_{\rm MC} = \langle G^2\rangle_{\rm soft}\left[1 + \mathcal{O}(\Lambda_{\rm QCD}^2 (Na)^2)\right]$$

$$\frac{1}{a} \gg \Lambda_{\rm QCD} \gg \frac{1}{Na} \quad \to \quad \langle G^2\rangle_{\rm MC} = \langle G^2\rangle_{\rm NP}\left[1 + \mathcal{O}\left(\frac{1}{\Lambda_{\rm QCD}^2 (Na)^2}\right)\right] ,$$

where $\langle G^2\rangle_{\rm NP} \sim \Lambda_{\rm QCD}^4$ is the NP gluon condensate (Vainshtein, Zakharov, Shifman).

$$\langle G^2\rangle_{\rm NP} = \frac{36 C_{\rm G}^{-1}(\alpha)}{\pi^2 a^4(\alpha)} \left[\langle P\rangle_{\rm MC}(\alpha) - S_P(\alpha)\right] + \mathcal{O}(a^2 \Lambda_{\rm QCD}^2) .$$

$$S_P(\alpha) \equiv S_{n_0}(\alpha) , \quad \text{where} \quad S_n(\alpha) = \sum_{j=0}^{n} p_j \alpha^{j+1} .$$

$n_0 \equiv n_0(\alpha)$ is the order for which $p_{n_0}\alpha^{n_0+1}$ is minimal.

Beyond perturbation theory

$$\langle P\rangle_{\rm MC} = \frac{1}{Z}\int [dU_{x,\mu}]\, e^{-S[U]} P[U]\Big|_{\rm MC} = P_{\rm pert}(\alpha)\langle 1\rangle + \frac{\pi^2}{36} C_{\rm G}(\alpha)\, a^4 \langle G^2\rangle_{\rm MC} + \mathcal{O}\left(a^6\right) .$$

$$\frac{1}{a} \gg \frac{1}{Na} \gg \Lambda_{\rm QCD} \quad \rightarrow \quad \langle G^2\rangle_{\rm MC} = \langle G^2\rangle_{\rm soft}\left[1 + \mathcal{O}(\Lambda^2_{\rm QCD}(Na)^2)\right]$$

$$\frac{1}{a} \gg \Lambda_{\rm QCD} \gg \frac{1}{Na} \quad \rightarrow \quad \langle G^2\rangle_{\rm MC} = \langle G^2\rangle_{\rm NP}\left[1 + \mathcal{O}\left(\frac{1}{\Lambda^2_{\rm QCD}(Na)^2}\right)\right] ,$$

where $\langle G^2\rangle_{\rm NP} \sim \Lambda^4_{\rm QCD}$ is the NP gluon condensate (Vainshtein, Zakharov, Shifman).

$$\langle G^2\rangle_{\rm NP} = \frac{36 C_{\rm G}^{-1}(\alpha)}{\pi^2 a^4(\alpha)}\left[\langle P\rangle_{\rm MC}(\alpha) - S_P(\alpha)\right] + \mathcal{O}(a^2\Lambda^2_{\rm QCD}) .$$

$$S_P(\alpha) \equiv S_{n_0}(\alpha) , \quad \text{where} \quad S_n(\alpha) = \sum_{j=0}^{n} p_j \alpha^{j+1} .$$

$n_0 \equiv n_0(\alpha)$ is the order for which $p_{n_0}\alpha^{n_0+1}$ is minimal.

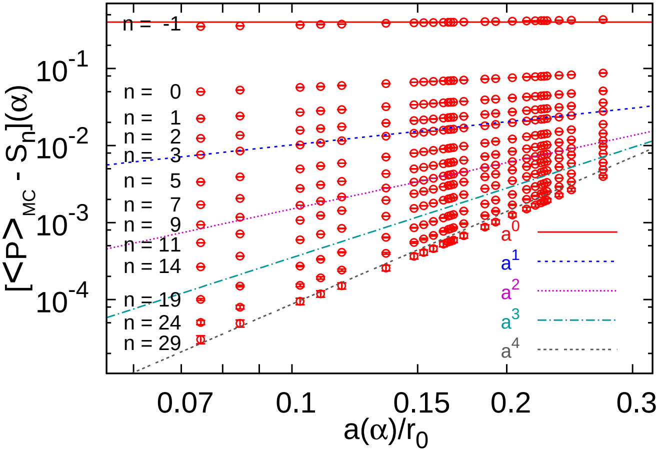


Figure: $\langle P\rangle_{\rm MC}(\alpha) - S_n(\alpha)$ between MC data and sums truncated at orders α^{n+1} ($S_{-1} = 0$) vs. $a(\alpha)/r_0$. The lines $\propto a^i$ are drawn to guide the eye.

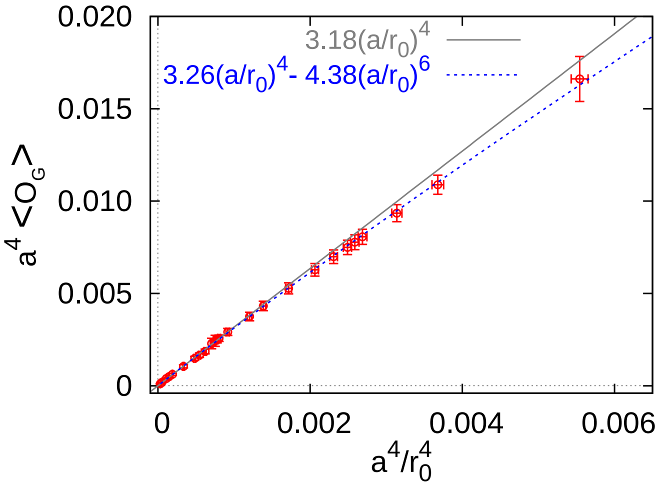


Figure: $\langle P\rangle_{\rm MC}(\alpha) - S_P(\alpha)$. The linear fit is to $a^4 < 0.0013\, r_0^4$ points only.

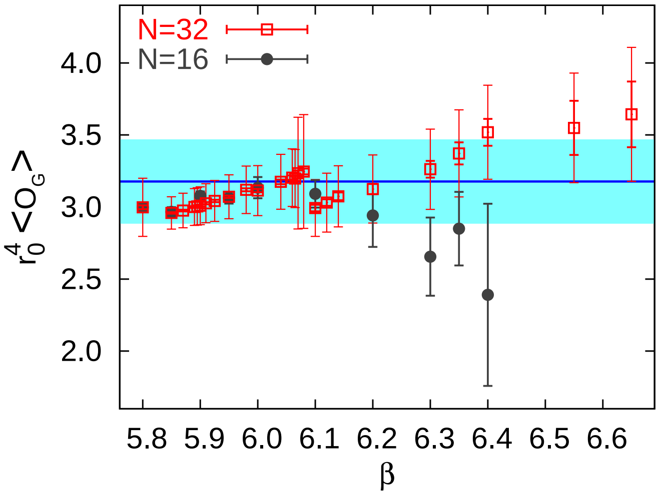


Figure: $\langle G^2 \rangle$ evaluated using the $N = 16$ and $N = 32$ MC data of Boyd et al. The error band is our prediction for $\langle G^2 \rangle$.

$$\langle G^2 \rangle = 3.18(29) r_0^{-4} = 24.2(8.0) \Lambda_{\overline{\text{MS}}}^4 \simeq 0.077\, \text{GeV}^4 \,.$$

Uncertainty of the sum due to the truncation

$$\delta S_P = \sqrt{n_0}\, p_{n_0} \alpha^{n_0+1} \approx \frac{(2\pi)^{3/2} d^{1+db}}{2^{db}\beta_0\, \Gamma(1+db)} N_P(\Lambda a)^4 \approx 12.06\, N_P(\Lambda a)^4 \,.$$

This object is scheme- and scale-independent (to $1/n$-precision)

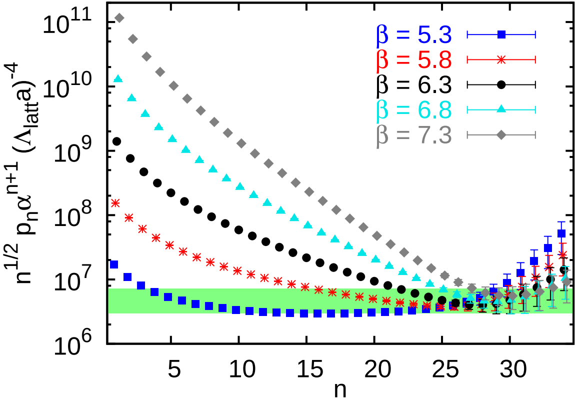


Figure: The combination $\sqrt{n}\, p_n \alpha^{n+1}/(\Lambda_{\rm latt} a)^4$, as a function of n for $\beta = 5.3, 5.8, 6.3, 6.8$ and 7.3. The error band corresponds to the theoretical expectation $12.06\, N_P = 5.1(2.1) \times 10^6$.

$$\sqrt{n_0}\frac{|r_{n_0}|}{\Lambda_{\rm latt}}\alpha^{n_0+1}(\nu) = \frac{2^{3/2-b}\pi^{3/2}}{\beta_0\Gamma(1+b)}|N_m| \approx 1.206\,|N_m|\,,$$

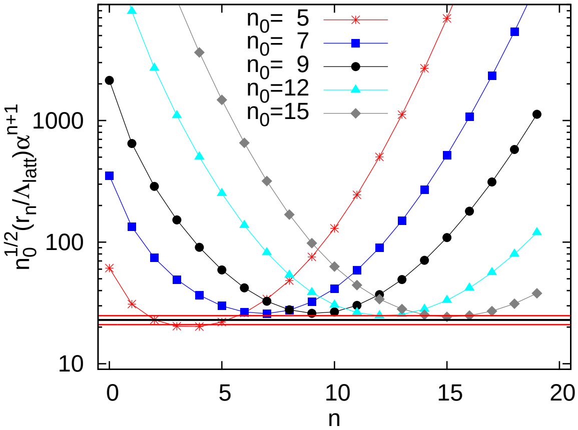


Figure: c_n times $\sqrt{n_0}$, for five different values of the lattice scheme coupling constant α, ranging from $\alpha(\nu) \approx 0.096$ ($n_0 = 5$) to $\alpha(\nu) \approx 0.036$ ($n_0 = 15$).

$$\delta\langle G^2\rangle_{\rm NP} \simeq \frac{(2\pi)^{3/2} d^{1+db}}{2^{db}\beta_0\,\Gamma(1+db)} N_{\rm G}^{\overline{\rm MS}}\Big|_{n_l=0} \Lambda_{\overline{\rm MS}}^4 = 27(11)\,\Lambda_{\overline{\rm MS}}^4 \sim 0.087\,{\rm GeV}^4\,.$$

$$\langle G^2\rangle = 3.18(29) r_0^{-4} = 24.2(8.0)\Lambda_{\overline{\rm MS}}^4 \simeq 0.077\,{\rm GeV}^4\,.$$

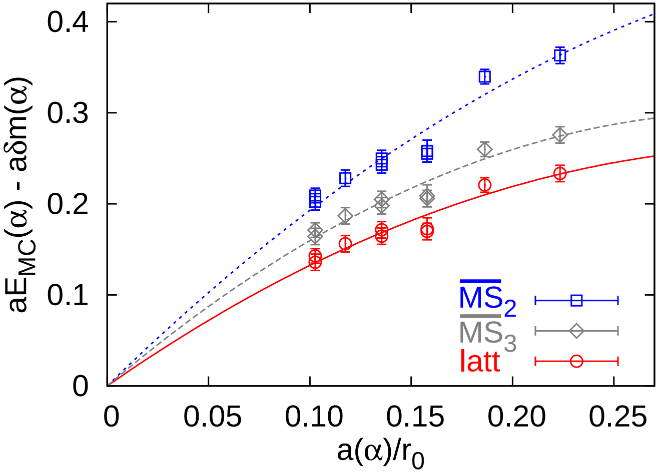


Figure: $aE_{\rm MC} - a\delta m$ vs. a/r_0. The expansion of $a\delta m$ was also converted into the $\overline{\rm MS}$ scheme at two ($\overline{\rm MS}_2$) and three ($\overline{\rm MS}_3$) loops. The curves are fits to $\overline{\Lambda}a + ca^2$.

CONCLUSIONS

Renormalons go beyond large-β_0 analysis: $\rightarrow$ (NP)OPE

Strong evidence of renormalon dominance in heavy quark physics from $\mathcal{O}(\alpha^{3/4})$ $\overline{\rm MS}$-like computations: Pole mass, static potential, $\cdots$

$$N_m^{\overline{\rm MS}}(n_l=0)=0.600(29)\,,\quad N_m^{\overline{\rm MS}}(n_l=3)=0.563(26)\,.$$

Lattice: For the first time it was possible to follow the factorial growth of the coefficients over many orders, from around α^9 up to α^{20}, vastly increasing the credibility of the prediction.

$$N_m^{\overline{\rm MS}}(n_l=0)=0.620(35)\,,\quad C_F/C_A\,N_\Lambda^{\overline{\rm MS}}(n_l=0)=-0.610(41)\,.$$

Two independent determinations with very different systematics.
We have (numerically) proven, beyond any reasonable doubt ($\sim$ 20 standard deviations!), the existence of the renormalon in QCD.

RENORMALON OK

CONCLUSIONS

Renormalons go beyond large-β_0 analysis: $\rightarrow$ (NP)OPE

Strong evidence of renormalon dominance in heavy quark physics from $\mathcal{O}(\alpha^{3/4})$ $\overline{\text{MS}}$-like computations: Pole mass, static potential, $\cdots$

$$N_m^{\overline{\text{MS}}}(n_l = 0) = 0.600(29)\,, \quad N_m^{\overline{\text{MS}}}(n_l = 3) = 0.563(26)\,.$$

Lattice: For the first time it was possible to follow the factorial growth of the coefficients over many orders, from around α^9 up to α^{20}, vastly increasing the credibility of the prediction.

$$N_m^{\overline{\text{MS}}}(n_l = 0) = 0.620(35)\,, \quad C_F/C_A\, N_\Lambda^{\overline{\text{MS}}}(n_l = 0) = -0.610(41)\,.$$

Two independent determinations with very different systematics.
We have (numerically) proven, beyond any reasonable doubt ($\sim$ 20 standard deviations!), the existence of the renormalon in QCD.

RENORMALON OK

CONCLUSIONS

Renormalons go beyond large-β_0 analysis: $\rightarrow$ (NP)OPE

Strong evidence of renormalon dominance in heavy quark physics from $\mathcal{O}(\alpha^{3/4})$ $\overline{\text{MS}}$-like computations: Pole mass, static potential, $\cdots$

$$N_m^{\overline{\text{MS}}}(n_l = 0) = 0.600(29)\,, \quad N_m^{\overline{\text{MS}}}(n_l = 3) = 0.563(26)\,.$$

Lattice: For the first time it was possible to follow the factorial growth of the coefficients over many orders, from around α^9 up to α^{20}, vastly increasing the credibility of the prediction.

$$N_m^{\overline{\text{MS}}}(n_l = 0) = 0.620(35)\,, \quad C_F/C_A\, N_\Lambda^{\overline{\text{MS}}}(n_l = 0) = -0.610(41)\,.$$

Two independent determinations with very different systematics.
We have (numerically) proven, beyond any reasonable doubt ($\sim$ 20 standard deviations!), the existence of the renormalon in QCD.

RENORMALON OK

CONCLUSIONS

Renormalons go beyond large-β_0 analysis: $\rightarrow$ (NP)OPE

Strong evidence of renormalon dominance in heavy quark physics from $\mathcal{O}(\alpha^{3/4})$ $\overline{\rm MS}$-like computations: Pole mass, static potential, $\cdots$

$$N_m^{\overline{\rm MS}}(n_l = 0) = 0.600(29)\,, \quad N_m^{\overline{\rm MS}}(n_l = 3) = 0.563(26)\,.$$

Lattice: For the first time it was possible to follow the factorial growth of the coefficients over many orders, from around α^9 up to α^{20}, vastly increasing the credibility of the prediction.

$$N_m^{\overline{\rm MS}}(n_l = 0) = 0.620(35)\,, \quad C_F/C_A\, N_\Lambda^{\overline{\rm MS}}(n_l = 0) = -0.610(41)\,.$$

Two independent determinations with very different systematics.
We have (numerically) proven, beyond any reasonable doubt ($\sim$ 20 standard deviations!), the existence of the renormalon in QCD.

RENORMALON OK

CONCLUSIONS

Renormalons go beyond large-β_0 analysis: $\rightarrow$ (NP)OPE

Strong evidence of renormalon dominance in heavy quark physics from $\mathcal{O}(\alpha^{3/4})$ $\overline{\rm MS}$-like computations: Pole mass, static potential, $\cdots$

$$N_m^{\overline{\rm MS}}(n_l=0)=0.600(29)\,,\quad N_m^{\overline{\rm MS}}(n_l=3)=0.563(26)\,.$$

Lattice: For the first time it was possible to follow the factorial growth of the coefficients over many orders, from around α^9 up to α^{20}, vastly increasing the credibility of the prediction.

$$N_m^{\overline{\rm MS}}(n_l=0)=0.620(35)\,,\quad C_F/C_A\,N_\Lambda^{\overline{\rm MS}}(n_l=0)=-0.610(41)\,.$$

Two independent determinations with very different systematics.
We have (numerically) proven, beyond any reasonable doubt ($\sim$ 20 standard deviations!), the existence of the renormalon in QCD.

RENORMALON OK

CONCLUSIONS

Renormalons go beyond large-β_0 analysis: $\rightarrow$ (NP)OPE

Strong evidence of renormalon dominance in heavy quark physics from $\mathcal{O}(\alpha^{3/4})$ $\overline{\text{MS}}$-like computations: Pole mass, static potential, $\cdots$

$$N_m^{\overline{\text{MS}}}(n_l = 0) = 0.600(29)\,, \quad N_m^{\overline{\text{MS}}}(n_l = 3) = 0.563(26)\,.$$

Lattice: For the first time it was possible to follow the factorial growth of the coefficients over many orders, from around α^9 up to α^{20}, vastly increasing the credibility of the prediction.

$$N_m^{\overline{\text{MS}}}(n_l = 0) = 0.620(35)\,, \quad C_F/C_A\, N_\Lambda^{\overline{\text{MS}}}(n_l = 0) = -0.610(41)\,.$$

Two independent determinations with very different systematics.
We have (numerically) proven, beyond any reasonable doubt ($\sim$ 20 standard deviations!), the existence of the renormalon in QCD.

RENORMALON OK

CONCLUSIONS

Renormalons go beyond large-β_0 analysis: $\rightarrow$ (NP)OPE

Strong evidence of renormalon dominance in heavy quark physics from $\mathcal{O}(\alpha^{3/4})$ $\overline{\text{MS}}$-like computations: Pole mass, static potential, $\cdots$

$$N_m^{\overline{\text{MS}}}(n_l = 0) = 0.600(29)\,, \quad N_m^{\overline{\text{MS}}}(n_l = 3) = 0.563(26)\,.$$

Lattice: For the first time it was possible to follow the factorial growth of the coefficients over many orders, from around α^9 up to α^{20}, vastly increasing the credibility of the prediction.

$$N_m^{\overline{\text{MS}}}(n_l = 0) = 0.620(35)\,, \quad C_F/C_A\, N_\Lambda^{\overline{\text{MS}}}(n_l = 0) = -0.610(41)\,.$$

Two independent determinations with very different systematics.
We have (numerically) proven, beyond any reasonable doubt ($\sim$ 20 standard deviations!), the existence of the renormalon in QCD.

RENORMALON OK

CONCLUSIONS: OPE and the plaquette

$$N_P^{\overline{\rm MS}} = 0.61(25) \qquad N_{\rm G}^{\overline{\rm MS}} = \frac{36}{\pi^2} N_P^{\overline{\rm MS}} = 2.24(92)\,.$$

Nonperturbative quantities ($\bar{\Lambda}$, Λ_H, $\langle G^2\rangle$, $\cdots$) can only be defined after subtracting the divergent perturbative series.

$$\delta\langle G^2\rangle_{\rm NP} = 27(11)\,\Lambda^4_{\overline{\rm MS}} \simeq 0.087\,{\rm GeV}^4\,. \qquad \langle G^2\rangle = 24.2(8.0)\Lambda^4_{\overline{\rm MS}} \simeq 0.077\,{\rm GeV}^4\,.$$

Non-perturbative OPE OK (for the plaquette)

Dimension two condensates: artifacts of incomplete subtractions

- unquantifible error due to the simplified parameterization of higher order perturbation theory
- short distance effect $\rightarrow$ process dependent

FUTURE:
Control of the subtraction-scheme dependence??
Problem for sum rules?
Lattice $\rightarrow$ Model independent/systematic procedure to get ALL condensates

CONCLUSIONS: OPE and the plaquette

$$N_P^{\overline{\rm MS}} = 0.61(25) \qquad N_{\rm G}^{\overline{\rm MS}} = \frac{36}{\pi^2} N_P^{\overline{\rm MS}} = 2.24(92)\,.$$

Nonperturbative quantities ($\bar{\Lambda}$, Λ_H, $\langle G^2\rangle$, $\cdots$) can only be defined after subtracting the divergent perturbative series.

$$\delta\langle G^2\rangle_{\rm NP} = 27(11)\,\Lambda^4_{\overline{\rm MS}} \simeq 0.087\,{\rm GeV}^4\,. \qquad \langle G^2\rangle = 24.2(8.0)\Lambda^4_{\overline{\rm MS}} \simeq 0.077\,{\rm GeV}^4\,.$$

Non-perturbative OPE OK (for the plaquette)

Dimension two condensates: artifacts of incomplete subtractions

- unquantifible error due to the simplified parameterization of higher order perturbation theory
- short distance effect $\rightarrow$ process dependent

FUTURE:
Control of the subtraction-scheme dependence??
Problem for sum rules?
Lattice $\rightarrow$ Model independent/systematic procedure to get ALL condensates

CONCLUSIONS: OPE and the plaquette

$$N_P^{\overline{\mathrm{MS}}} = 0.61(25) \qquad N_{\mathrm{G}}^{\overline{\mathrm{MS}}} = \frac{36}{\pi^2} N_P^{\overline{\mathrm{MS}}} = 2.24(92)\,.$$

Nonperturbative quantities ($\bar{\Lambda}$, Λ_H, $\langle G^2 \rangle$, $\cdots$) can only be defined after subtracting the divergent perturbative series.

$$\delta\langle G^2\rangle_{\mathrm{NP}} = 27(11)\,\Lambda_{\overline{\mathrm{MS}}}^4 \simeq 0.087\,\mathrm{GeV}^4\,. \qquad \langle G^2\rangle = 24.2(8.0)\Lambda_{\overline{\mathrm{MS}}}^4 \simeq 0.077\,\mathrm{GeV}^4\,.$$

Non-perturbative OPE OK (for the plaquette)

Dimension two condensates: artifacts of incomplete subtractions

- unquantifible error due to the simplified parameterization of higher order perturbation theory
- short distance effect $\rightarrow$ process dependent

FUTURE:
Control of the subtraction-scheme dependence??
Problem for sum rules?
Lattice $\rightarrow$ Model independent/systematic procedure to get ALL condensates

CONCLUSIONS: OPE and the plaquette

$$N_P^{\overline{\rm MS}} = 0.61(25) \qquad N_{\rm G}^{\overline{\rm MS}} = \frac{36}{\pi^2} N_P^{\overline{\rm MS}} = 2.24(92)\,.$$

Nonperturbative quantities ($\bar\Lambda$, Λ_H, $\langle G^2\rangle$, $\cdots$) can only be defined after subtracting the divergent perturbative series.

$$\delta\langle G^2\rangle_{\rm NP} = 27(11)\,\Lambda^4_{\overline{\rm MS}} \simeq 0.087\,{\rm GeV}^4\,. \qquad \langle G^2\rangle = 24.2(8.0)\Lambda^4_{\overline{\rm MS}} \simeq 0.077\,{\rm GeV}^4\,.$$

Non-perturbative OPE OK (for the plaquette)

Dimension two condensates: artifacts of incomplete subtractions

- unquantifible error due to the simplified parameterization of higher order perturbation theory
- short distance effect $\rightarrow$ process dependent

FUTURE:
Control of the subtraction-scheme dependence??
Problem for sum rules?
Lattice $\rightarrow$ Model independent/systematic procedure to get ALL condensates

CONCLUSIONS: OPE and the plaquette

$$N_P^{\overline{\rm MS}} = 0.61(25) \qquad N_{\rm G}^{\overline{\rm MS}} = \frac{36}{\pi^2} N_P^{\overline{\rm MS}} = 2.24(92) \,.$$

Nonperturbative quantities ($\bar\Lambda$, Λ_H, $\langle G^2 \rangle$, $\cdots$) can only be defined after subtracting the divergent perturbative series.

$$\delta \langle G^2 \rangle_{\rm NP} = 27(11)\, \Lambda^4_{\overline{\rm MS}} \simeq 0.087\, {\rm GeV}^4 \,. \qquad \langle G^2 \rangle = 24.2(8.0) \Lambda^4_{\overline{\rm MS}} \simeq 0.077\, {\rm GeV}^4 \,.$$

Non-perturbative OPE OK (for the plaquette)

Dimension two condensates: artifacts of incomplete subtractions

- unquantifible error due to the simplified parameterization of higher order perturbation theory
- short distance effect $\rightarrow$ process dependent

FUTURE:
Control of the subtraction-scheme dependence??
Problem for sum rules?
Lattice $\rightarrow$ Model independent/systematic procedure to get ALL condensates

Some applications

Over the years a lot of evidence in favour of the existence of the renormalon. Particularly important for heavy quark physics.

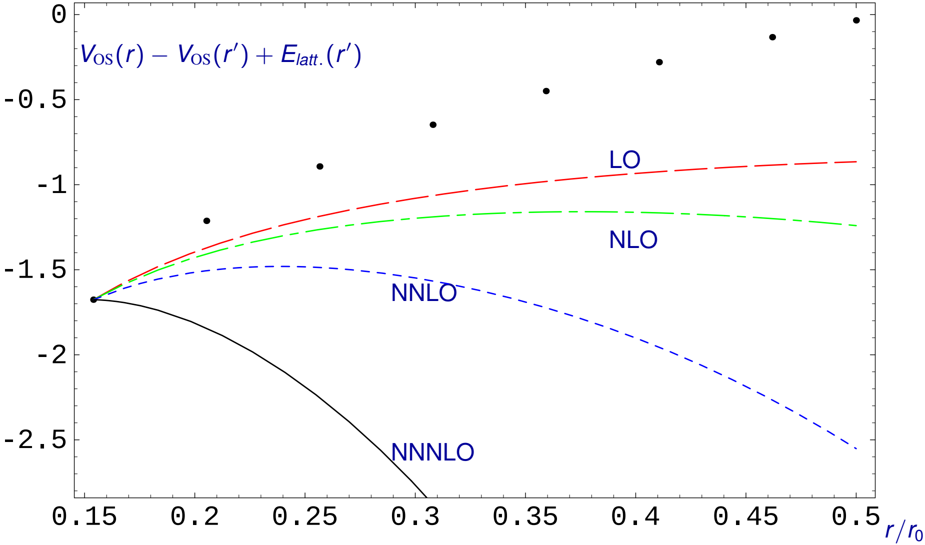
- $$2m_{\rm OS} + E_s(r) = 2m_{\rm OS} + V_s(r) + \mathcal{O}(r^2)$$

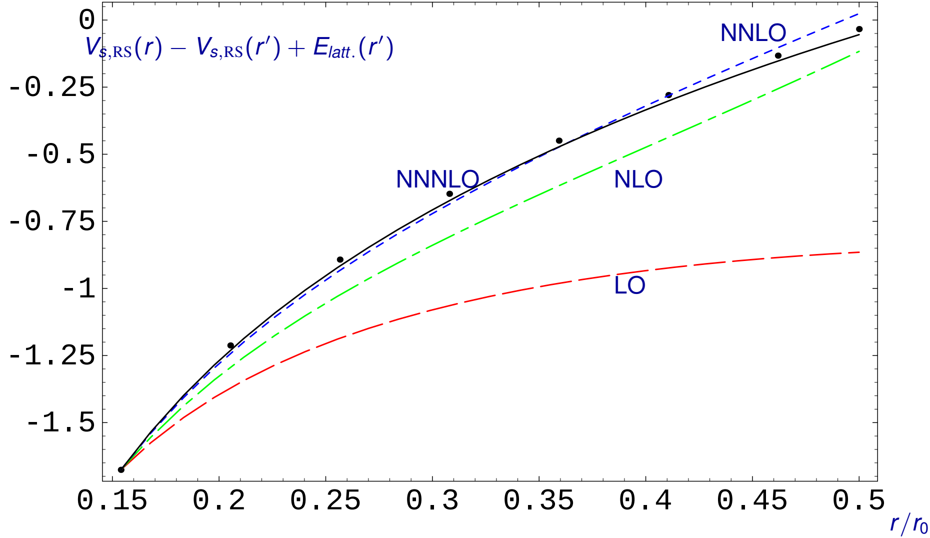
  $2m_{\rm OS} + V_s = 2m_{\rm RS} + V_{s,{\rm RS}}$ is renormalon free. Good description of the singlet static potential at short distances.

- $$2m_{\rm OS} + E_H(r) = 2m_{\rm OS} + V_o(r) + \Lambda_H + \mathcal{O}(r^2)$$

  $2m_{\rm OS} + V_o + \Lambda_H = 2m_{\rm RS} + V_{o,{\rm RS}} + \Lambda_{H,{\rm RS}}$ is renormalon free. Good description of the octet static potential at short distances.

- Good description of heavy quarkonium properties: low-lying bound states, non-relativistic sum rules,...
- ...


$V_{OS}(r) - V_{OS}(r') + E_{latt.}(r')$
LO
NLO
NNLO
NNNLO
0
-0.5
-1
-1.5
-2
-2.5
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
r/r_0



0
-0.25
-0.5
-0.75
-1
-1.25
-1.5
$V_{s,\rm RS}(r) - V_{s,\rm RS}(r') + E_{latt.}(r')$
NNLO
NNNLO
NLO
LO
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
r/r_0

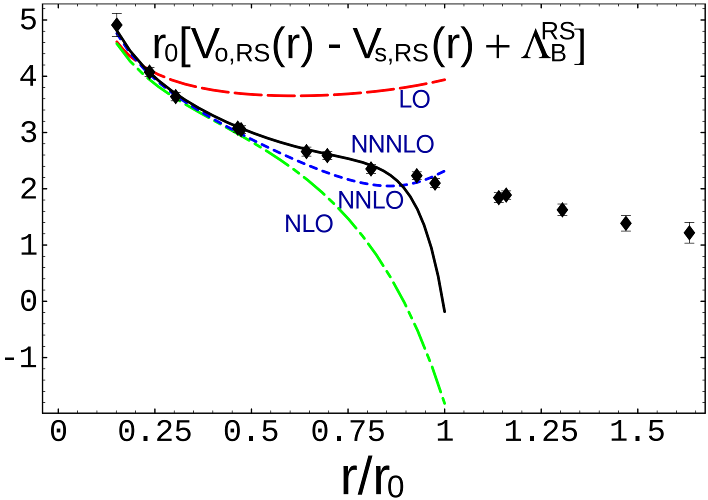


Figure: Splitting between the Π_u and the Σ_g^+ potentials and the comparison with the theoretical prediction.

$2m + V_o + \Lambda_B$ is renormalon free.

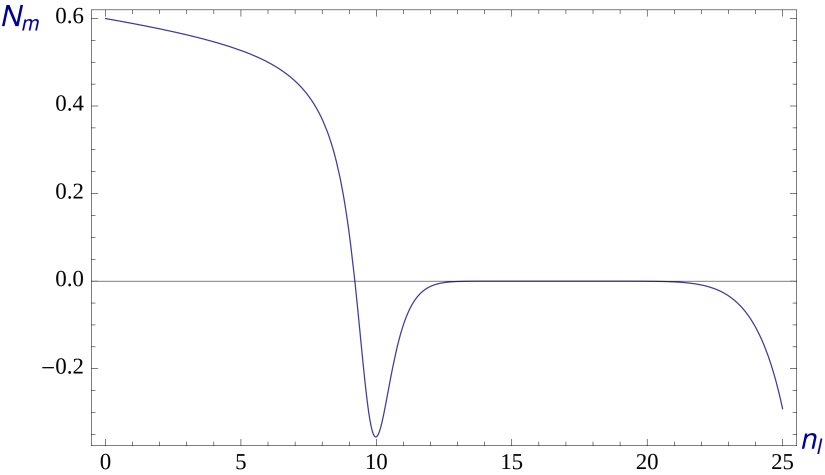
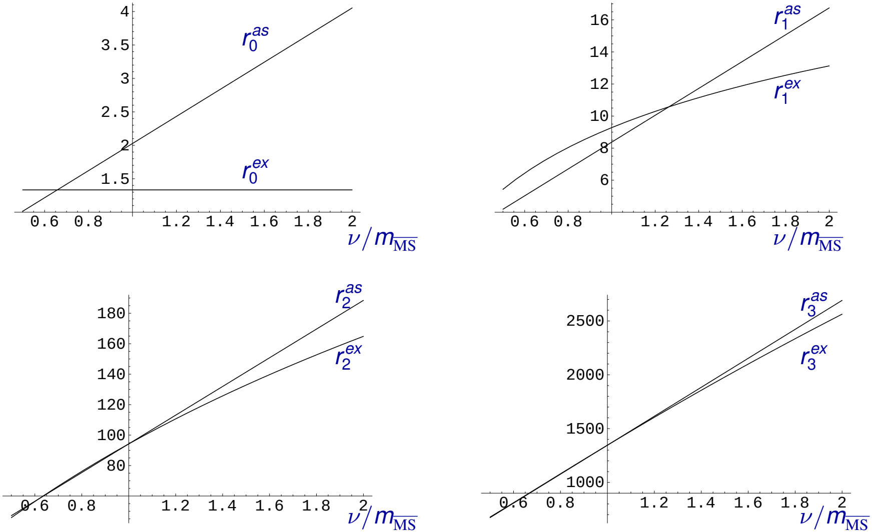


Figure: *$N_m(x=1)$ obtained from $-(N_V/2)v_n/v_n^{asym}$ ($n=3$) from the static potential (NNNLO) as a function of n_l.*

Check : $$r_n \overset{n\to\infty}{\sim} m_{\overline{\text{MS}}} \left(\frac{\beta_0}{2\pi}\right)^n n! N_m \sum_{s=0}^{n} \frac{\ln^s[\nu/m_{\overline{\text{MS}}}]}{s!} \sim \nu$$



Renormalon subtracted matching and power counting

Effective field theory with renormalon free parameters but preserving the power counting rules.
The renormalon is associated to the non-analytic behavior in $1-2u$. These terms also exist in the effective theory. Procedure: to explicitly subtract them from the matching coefficients (the mass).

$$B[m_{\rm RS}] \equiv B[m_{\rm OS}] - N_m \nu_f \frac{1}{(1-2u)^{1+b}} \left(1 + c_1(1-2u) + c_2(1-2u)^2 + \cdots\right),$$

$$m_{\rm RS}(\nu_f) = m_{\rm OS} - \sum_{n=0}^{\infty} N_m\, \nu_f \left(\frac{\beta_0}{2\pi}\right)^n \alpha_s^{n+1}(\nu_f) \sum_{k=0}^{\infty} c_k \frac{\Gamma(n+1+b-k)}{\Gamma(1+b-k)}.$$

Expansion in $\alpha_s(\nu)$

$$m_{\rm RS}(\nu_f) = m_{\overline{\rm MS}} + \sum_{n=0}^{\infty} r_n^{\rm RS} \alpha_s^{n+1},$$

where $r_n^{\rm RS} = r_n^{\rm RS}(m_{\overline{\rm MS}}, \nu, \nu_f)$. They are the ones expected to be of natural size. We now do not loose accuracy if we first obtain $m_{\rm RS}$ and later on $m_{\overline{\rm MS}}$.
Different scheme

$$B[m_{RS'}] \equiv B[m_{RS}] + N_m \nu_f \left(1 + c_1 + c_2 + \cdots\right).$$

	$\mathcal{O}(\alpha^4)$	$\mathcal{O}(\alpha^{20})$	$\mathcal{O}(\alpha^{32})$
$N_S(N_T)$	4(4)	8(8, 10, 12, 14)	4(8)

Table: *The first arrow states to which order in α the coefficients of $c_n^{(R)}(N_T, N_S)$ have been computed for each specific lattice volume for PBC.*

$\mathcal{O}(\alpha^3)$	$N_S(N_T)$	5(5, 6, 7, 8, 10)			
$\mathcal{O}(\alpha^4)$	$N_S(N_T)$	4(5, 6, 7, 8, 10, 12, 16, 20, 24)	12(16, 20)		
$\mathcal{O}(\alpha^{12})$	$N_S(N_T)$	6(6, 8, 10, 12, 16)	8(12, 16)		
$\mathcal{O}(\alpha^{12})$	$N_S(N_T)$	10(8, 12, 16, 20)	16(12, 16, 20)		
$\mathcal{O}(\alpha^{20})$	$N_S(N_T)$	7(7, 8)	8(8, 10)	9(12)	10(10)
$\mathcal{O}(\alpha^{20})$	$N_S(N_T)$	11(16)	12(12)	14(14)	

Table: *The first column states to which order in α the coefficients of $c_n^{(R)}(N_T, N_S)$ and the associated ratios have been computed for each specific lattice volume for TBC.*

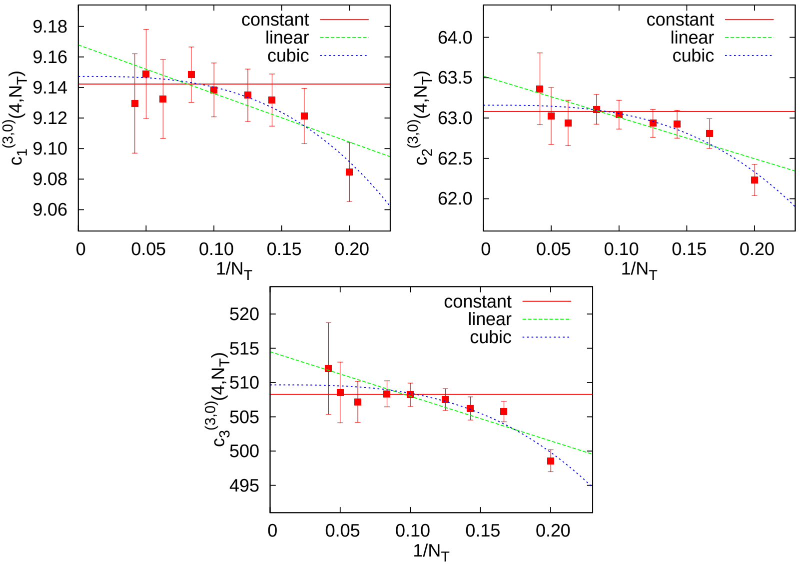


Figure: $c_{1,2,3}^{(3,0)}(4, N_T)$ *as a function of* $1/N_T$*, in comparison to a constant plus linear fit, a constant plus cubic fit, and a constant fitted only to the* $N_T > 10$ *points.*

"Physical interpretation"

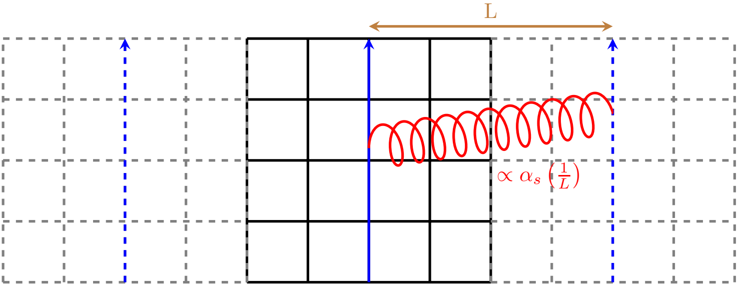


Figure: *Self-interactions with replicas producing* $1/L = 1/(aN_S)$ *Coulomb terms.*

$$P \propto \int_{1/(aN_S)}^{1/a} dk\, \alpha(k) \sim \frac{1}{a}\sum_n c_n \alpha^{n+1}\left(a^{-1}\right) - \frac{1}{aN_S}\sum_n c_n \alpha^{n+1}\left((aN_S)^{-1}\right),$$

$$c_n \simeq N_m \left(\frac{\beta_0}{2\pi}\right)^n n!\,, \qquad f_n^{(i)}(N_S) \simeq N_m \left(\frac{\beta_0}{2\pi}\right)^n \frac{n!}{i!}\,.$$