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A Covariant Variational Approach to Yang-Mills Theory

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Quark Confinement and the Hadron Spectrum
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Overview

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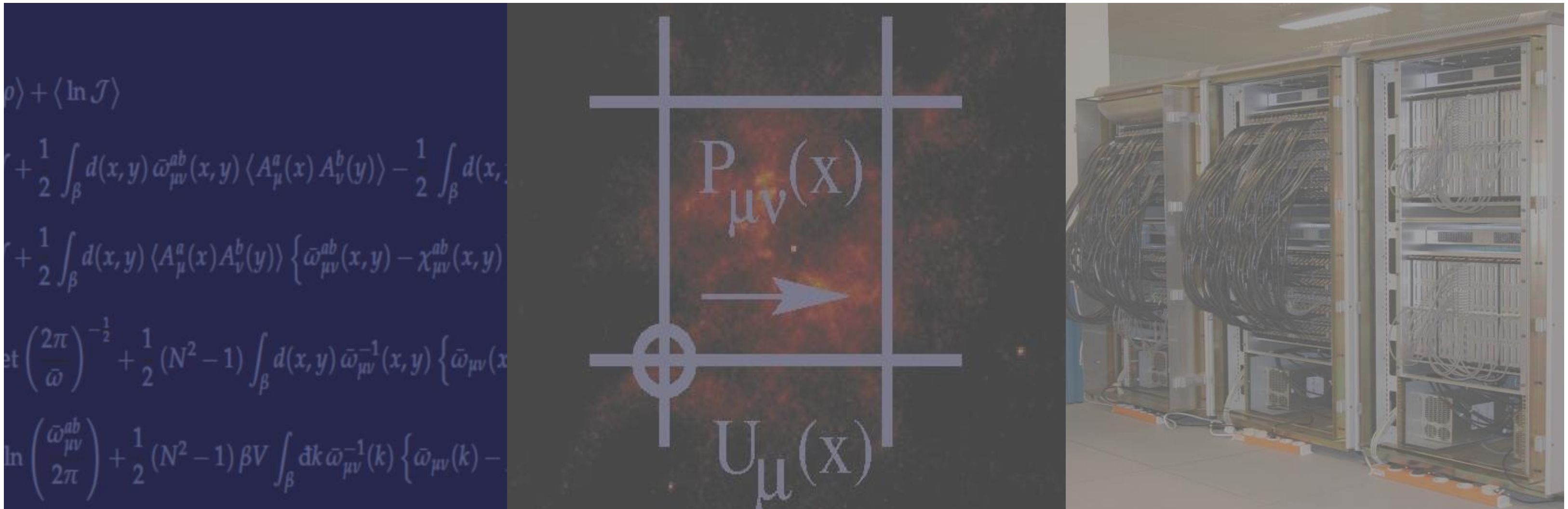


Effective Action Principle

Effective Action Principle

- Variation principle for functional probability measure

$$\langle\, \hat{A}(x_1) \cdots \hat{A}(x_n)\, \rangle = \int d\mu(A)\, A(x_1) \cdots A(x_n)$$

- Entropy = available space for quantum fluctuations

$$d\mu = dA \cdot \rho(A) \quad \Longrightarrow \quad \mathcal{W}(\mu) = -\langle\, \ln \rho\, \rangle$$

- Free action

$$F(\mu) = \langle\, S(A)\, \rangle_\mu - \hbar\, \mathcal{W}(\mu)$$

- Variation principle I

$$F(\mu) \stackrel{!}{=} \min \quad \Longrightarrow \quad d\mu = d\mu_0$$

$$d\mu_0(A) = Z^{-1} \exp\left[\hbar^{-1} S(A)\right] dA$$ Gibbs measure

$$\langle\, A(x_1) \cdots A(x_n)\, \rangle_{\mu_0}$$ Schwinger functions

- Variation principle II (Quantum effective action)

$$\Gamma(\omega) = \min_{\mu} \left\{ F(\mu) \,\middle|\, \langle\, \Omega\, \rangle_{\mu} = \omega \right\} \quad \Longrightarrow \quad d\mu = d\mu_{\omega}$$

$$\Gamma(\omega) = F(\mu_{\omega}) \stackrel{!}{=} \min$$

Note: Usually $\Omega = A \quad (\omega = \mathcal{A})$ and proper functions $\dfrac{\delta\Gamma(\mathcal{A})}{\delta\mathcal{A}(x_1)\cdots\delta\mathcal{A}(x_n)}$

Modification for gf. Yang-Mills Theory

$$d\mu_0(A) = dA\,\mathcal{J}(A)\,\exp\left[-\hbar^{-1}\,S_{gf}(A)\right]$$

replace entropy by relative entropy

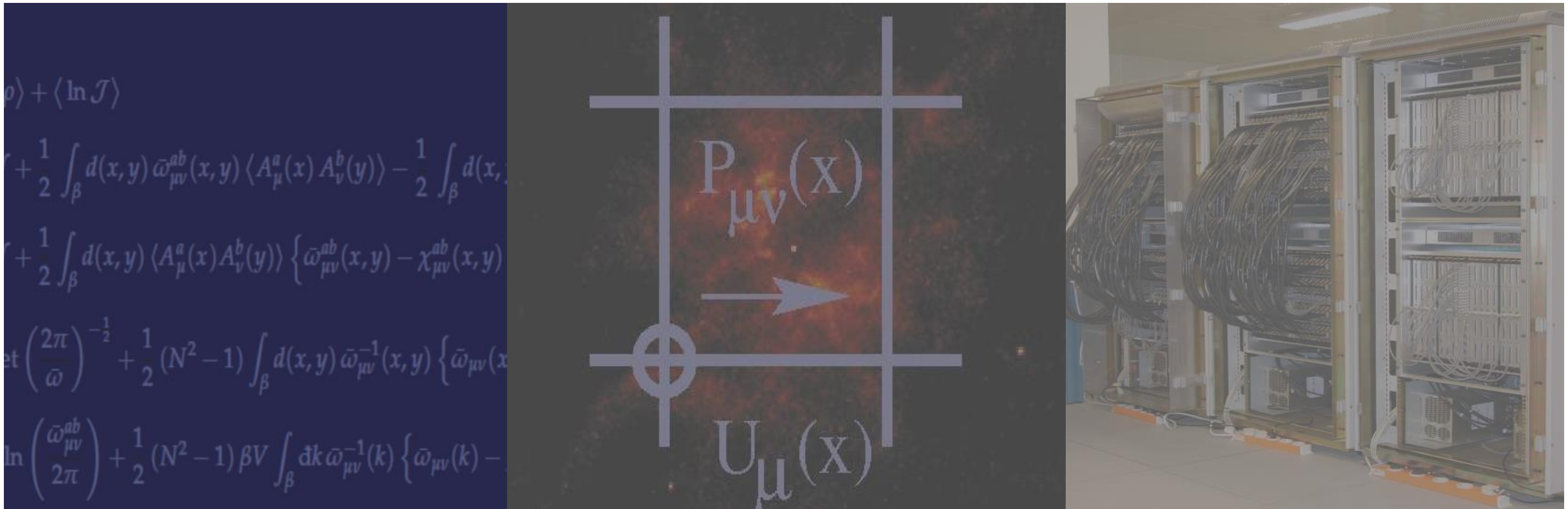
$$\bar{\mathcal{W}}(\mu) = \mathcal{W}(\mu) + \langle\,\ln \mathcal{J}\,\rangle = -\langle\,\ln(\rho/\mathcal{J})\,\rangle$$

(available quantum fluctuations in curved field space)

1. Take (restricted) measure space with parameters K
2. Minimize $\Gamma(\mathcal{A},K) \implies K_{\mathcal{A}}$ [gap equation]
3. Quantum effective action $\Gamma(\mathcal{A}) = \Gamma(\mathcal{A},K_{\mathcal{A}})$
4. Proper functions = derivatives of $\Gamma(\mathcal{A})$

Comparision

- Pro
  - non-perturbative corrections to arbitrary proper functions
  - closed system of integral equations
  - renormalization through standard counter terms
  - can discriminate competing solutions
  - can be optimized to many physical questions (choice of Ω)
- Cons
  - vertex corrections not necessarily reliable
  - Confinement ?
  - Strong phase transitions ?
  - BRST ? [similar to Coulomb gauge]



Gaussian Trial Measure

- Physical picture in Landau gauge

  - UV : gluons weakly interacting
  - IR : configurations near Gribov horizon dominant
    self-interaction in such configs sub-dominant

⇒ Gaussian trial measure for effective (constitutent) gluon

$$d\mu(A) = \mathcal{N}_\alpha(\omega) \cdot \mathcal{J}(A)^{1-2\alpha} \cdot \exp\left[-\frac{1}{2}\int d(x,y)\, A_\mu(c)\, \delta^{ab}\, \omega_{\mu\nu}(x,y)\, A^b_\nu(y)\right]$$

- variational parameters α, ω
- Lorentz structure in Landau gauge:

$$\omega_{\mu\nu}(x,y) = \int \frac{d^4k}{(2\pi)^4}\, e^{i\,k(x-y)} \left[\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\,(1-\zeta^{-1})\right] \omega(k)$$

Curvature Approximation

$$\ln \mathcal{J}(A) = -\frac{1}{2}\int d(x,y)\, A_\mu(x)\, \chi^{ab}_{\mu\nu}(x,y)\, A^b_\nu(y) + \cdots$$

$$\chi^{ab}_{\mu\nu}(x,y) = -\left\langle \frac{\delta^2 \ln \mathcal{J}}{\delta A^a_\mu(x)\, \delta A^b_\nu(y)} \right\rangle \longrightarrow \delta^{ab}\, t_{\mu\nu}(k)\, \chi(k)$$

curvature

- trial measure

$$d\mu(A) = \det\left(\frac{\bar{\omega}}{2\pi}\right)^{\frac{1}{2}} \cdot \exp\left[-\frac{1}{2}\int (A - \mathcal{A})\, \bar{\omega}\, (A - \mathcal{A})\right] dA$$

$$\bar{\omega} = \omega + (1 - 2\alpha)\, \chi$$

⇒ curvature gone, reappears in entropy (natural)

Remark

- put classical field $\mathcal{A} = 0$ if interested in propagator

$$\Gamma(\mathcal{A}) = \frac{1}{2} \int \mathcal{A} \bar{\omega}_{\mathcal{A}} \mathcal{A} = \frac{1}{2} \int \mathcal{A} \cdot \bar{\omega}_0 \cdot \mathcal{A} + \cdots$$

- alternative: effective action for gluon propagator

$$\Gamma(\mathcal{A} = 0, \bar{\omega}) = \left\{ F(\mu) \,\middle|\, \langle A(1)\, A(2) \rangle = \bar{\omega}^{-1}(1,2) \right\} \equiv \Gamma(\bar{\omega})$$

⇒ gap equation $\delta\Gamma / \delta\bar{\omega}^{-1} = 0$ gives best gluon propagator that is available within the ansatz

Free action

- classical action: Wick‘s theorem

$\langle S_{\mathrm{gf}} \rangle_\mu$ involves
$$\langle A(1)\, A(2) \rangle \sim \bar{\omega}^{-1}(1,2)$$
$$\langle A(1) \cdots A(4) \rangle \sim \bar{\omega}^{-1}(1,2)\, \bar{\omega}^{-1}(3,4) + \mathrm{perm}$$

⇒ *lengthy expression*

- relative entropy

$$\bar{\mathcal{W}} = \langle - \ln \rho \rangle_\mu + \langle \ln \mathcal{J} \rangle_\mu$$
$$= -\ln \mathcal{N}_\alpha(\bar{\omega}) + \frac{1}{2} \int \bar{\omega}(1,2)\, \langle A(1)\, A(2) \rangle_\mu - \frac{1}{2} \int \chi(1,2)\, \langle A(1)\, A(2) \rangle_\mu$$
$$= -\frac{1}{2}\, (N^2 - 1) \cdot 3 \cdot V_4 \int \frac{d^4 k}{(2\pi)^4} \left\{ \ln \bar{\omega}(k) + \frac{\chi(k)}{\bar{\omega}(k)} \right\}$$

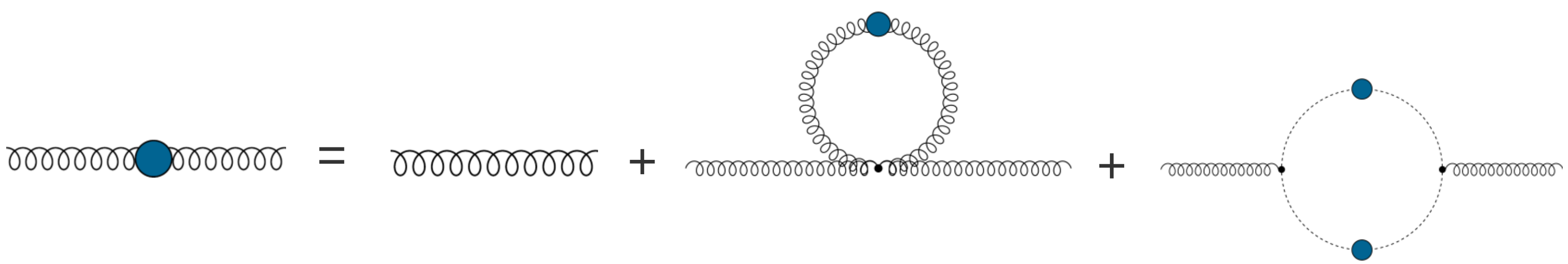
Gap Equation

$$\frac{\delta}{\delta\bar{\omega}^{-1}(k)}\left\{\langle S_{\mathrm{gf}}\rangle_{\mu}-\bar{\mathcal{W}}(\mu)\right\}=0$$

graph

Use O(4) symmetry to perform angular integrations

$$\bar{\omega}(k)=k^2+M^2+\chi(k)$$



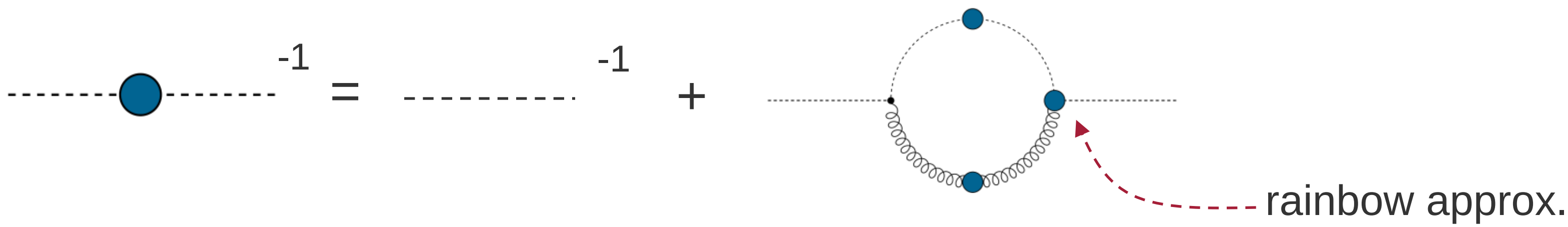
Ghost sector

Use resolvent identity on FP operator $G^{-1} = \partial_\mu \hat{D}^\mu = G_0^{-1} + h$

$$G_0 \langle G^{-1} \rangle = 1 + G_0 \langle hG \rangle \langle G^{-1} \rangle$$

in terms of ghost form factor $G(k) = \frac{\eta(k)}{k^2}$

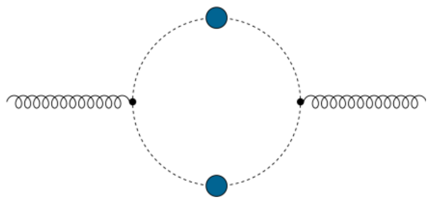
$$\eta(k)^{-1} = 1 - Ng^2 \int \frac{d^4q}{(2\pi)^4} \frac{\eta(k-q)}{(k-q)^2} \frac{1 - (\hat{k} \cdot \hat{q})}{\bar{\omega}(q)}$$



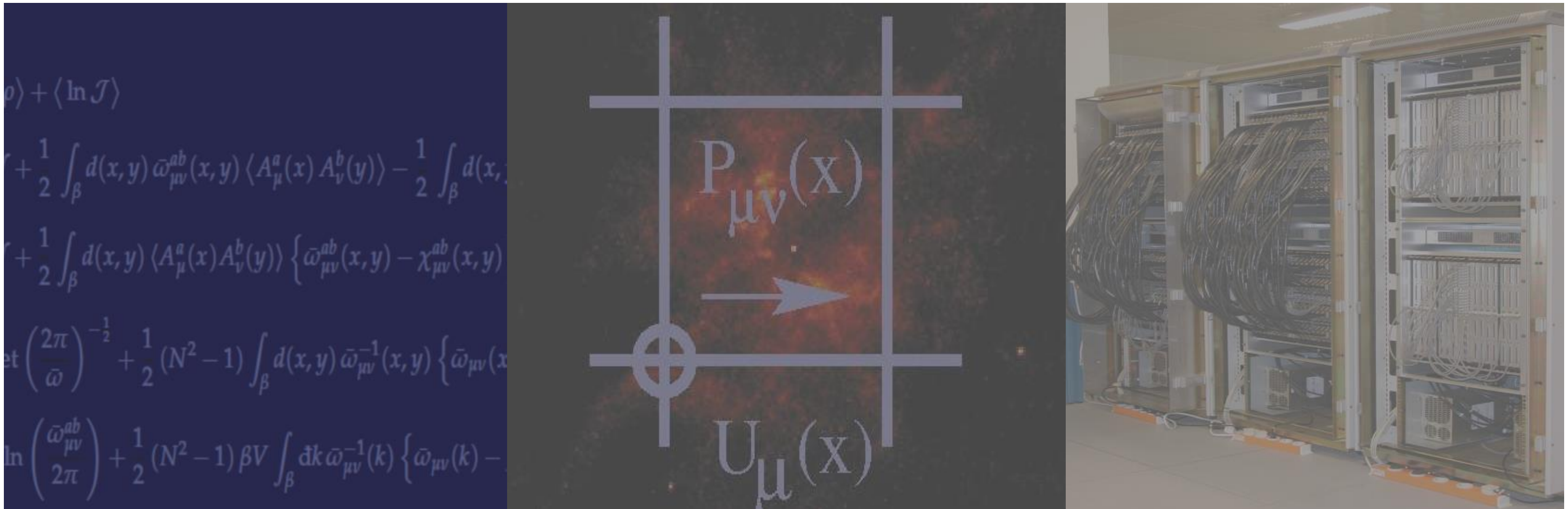
Curvature Equation

To given loop order

$$\chi(k) = \mathrm{Tr}\left\langle G\,\frac{\delta(-\partial D)}{\delta A(2)}\,G\,\frac{\delta(-\partial D)}{\delta A(1)}\right\rangle \approx \mathrm{Tr}\langle G\rangle\,\Gamma_0\,\langle G\rangle\,\Gamma_0$$

$=$ 

$$\chi(k) = \frac{1}{3}\,N g^2 \int \frac{d^4 q}{(2\pi)^4}\,\frac{\eta(k-q)\,\eta(q)}{(k-q)^2}\,\left[1 - (\hat{k}\cdot\hat{q})\right]$$



Renormalization

Counterterms

$$\mathcal{L}_{\text{ct}} = \delta Z_A \cdot \frac{1}{4} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 + \delta M^2 \cdot \frac{1}{2} \left(A_\mu^a\right)^2 + \delta Z_c \cdot \partial_\mu \bar{c}\, \partial^\mu c$$

gluon field — gluon mass — ghost field

❗ Added to exponent of trial measure, **not** S_{YM}

Renormalization conditions (3 scales $0 \le \mu_c \le \mu_0 \ll \mu$)

- fix $\eta^{-1}(\mu_c) = \cdots$ — scaling/decoupling
- fix $\bar{\omega}(\mu) = Z\,\mu^2$
- fix $\bar{\omega}(\mu_0) = Z\,M_A^2$ — constitutent mass at $\mu_0 \to 0$

- All countertems have definite values, e.g.

→ Important for finite-T calculation

$$\delta Z_c = 1 - N g^2 \, I_\eta(\mu_c) - \eta(\mu_c)^{-1}$$

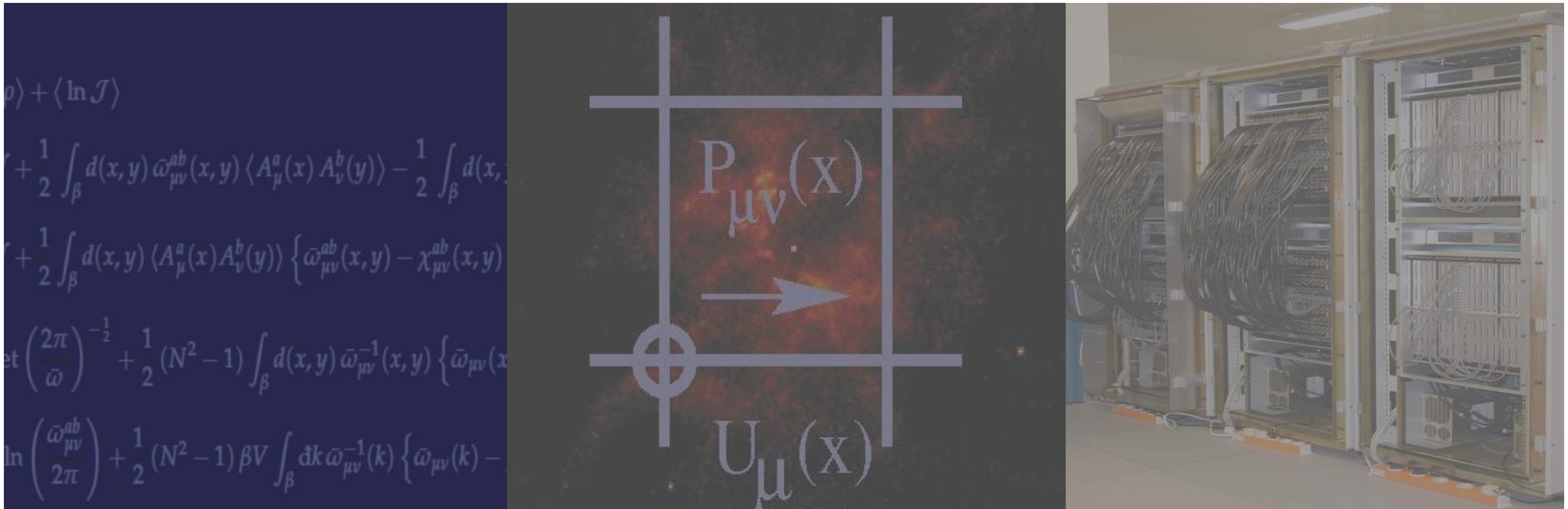
$\delta M^2, \ \delta Z_A$ similarly

- Final system

$$\eta_0(k)^{-1} = \eta_0(\mu_c)^{-1} - N g^2 \left[I_\eta^{(0)}(k) - I_\eta^{(0)}(\mu_c) \right]$$

$$\bar{\omega}(k) = Z \, \frac{\mu^2 - M_A^2}{\mu^2 - \mu_0^2} \, k^2 + Z \, \frac{M_A^2 - \mu_0^2}{\mu^2 - \mu_0^2} \, \mu^2 + \frac{N g^2}{\mu^2 - \mu_0^2} \Big[\mu^2 \big(I_\chi^{(0)}(k) - I_\chi^{(0)}(\mu_0) \big) - \\ - k^2 \big(I_\chi^{(0)}(\mu) - I_\chi^{(0)}(\mu_0) \big) - \mu_0^2 \big(I_\chi^{(0)}(k) - I_\chi^{(0)}(\mu) \big) \Big]$$

curvature: quadratic + subleading log. divergence subtracted !



Numerical Results

Infrared Analysis

Assume power law in the deep IR

$$\bar{\omega}^{-1}(k) \sim (k^2)^{\alpha}, \qquad \eta(k) \sim (k^2)^{-\beta}$$

IR exponents

$$\Rightarrow \quad -\alpha + 2\beta = \frac{d}{2} - 1$$

Non-renormalization of ghost-gluon vertex

- scaling solution: $\eta^{-1}(0) = 0 \Rightarrow \beta = \frac{1}{98}\left(93 \mp \sqrt{1201}\right) \approx \{\, 0.5954, \; 1.3025 \,\}$ (✓ 0.5954, ✗ 1.3025)

  Gluon prop IR vanishing, $\alpha > 0$ constituent mass M_A^2 irrelevant

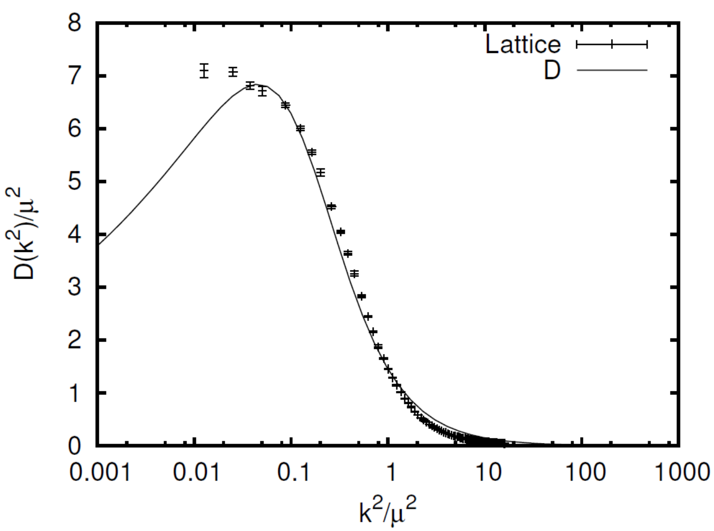
- decoupling solution: $\eta^{-1}(0) > 0 \Rightarrow \beta = 0$

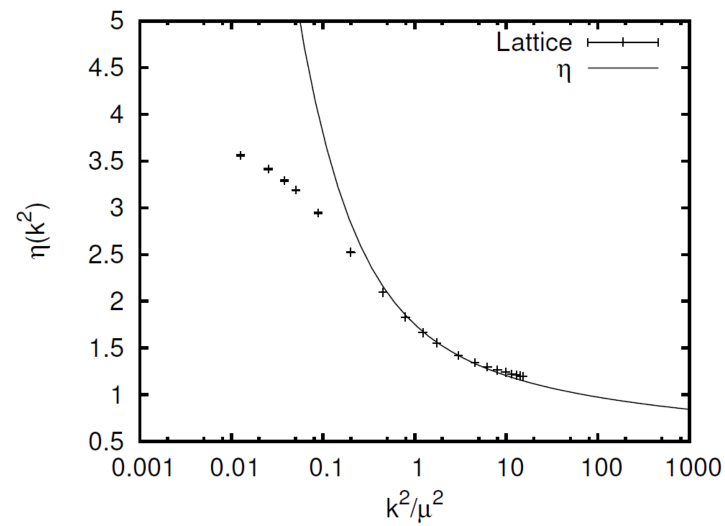
  Gluon prop. IR finite, $\alpha = 0$ constituent mass M_A^2 dominant ✓

  Gluon prop. IR diverging, $\alpha = -1$ constituent mass $M_A^2 = 0$ ✗

Scaling Solution

Lattice data from Bogolubsky et al., Phys. Lett. **B676** 69 (2009)





H. Reinhardt, J. Heffner, M.Q., Phys. Rev. **D89** 065037 (2014)

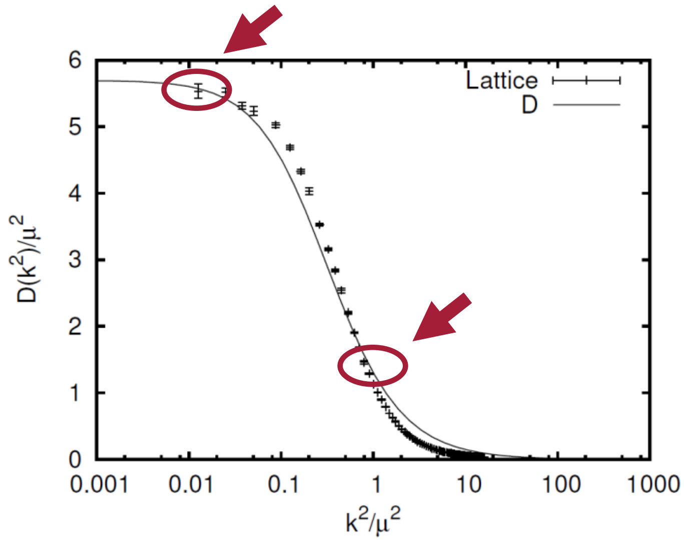
IR exponents: $\beta = 0.595(3)$ $\alpha = 0.191(1)$

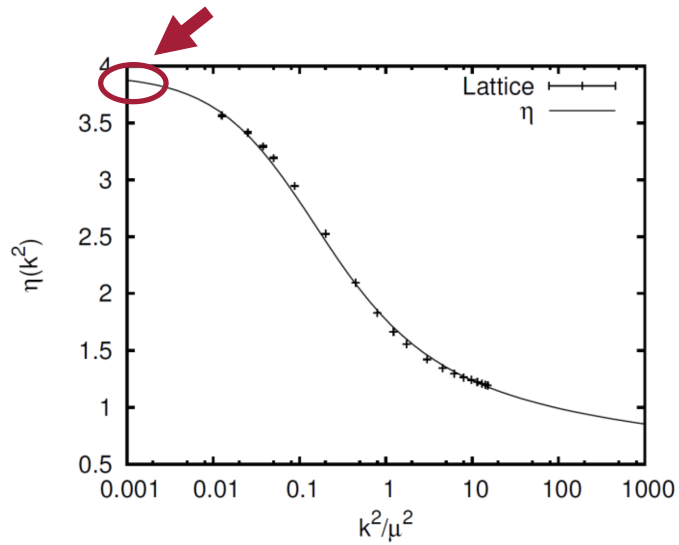
sum rule violation: $< 10^{-3}$

lattice data: ✘

Decoupling Solution

Lattice data from Bogolubsky et al., Phys. Lett. **B676** 69 (2009)





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lattice data: ✓

sum rule: ✘

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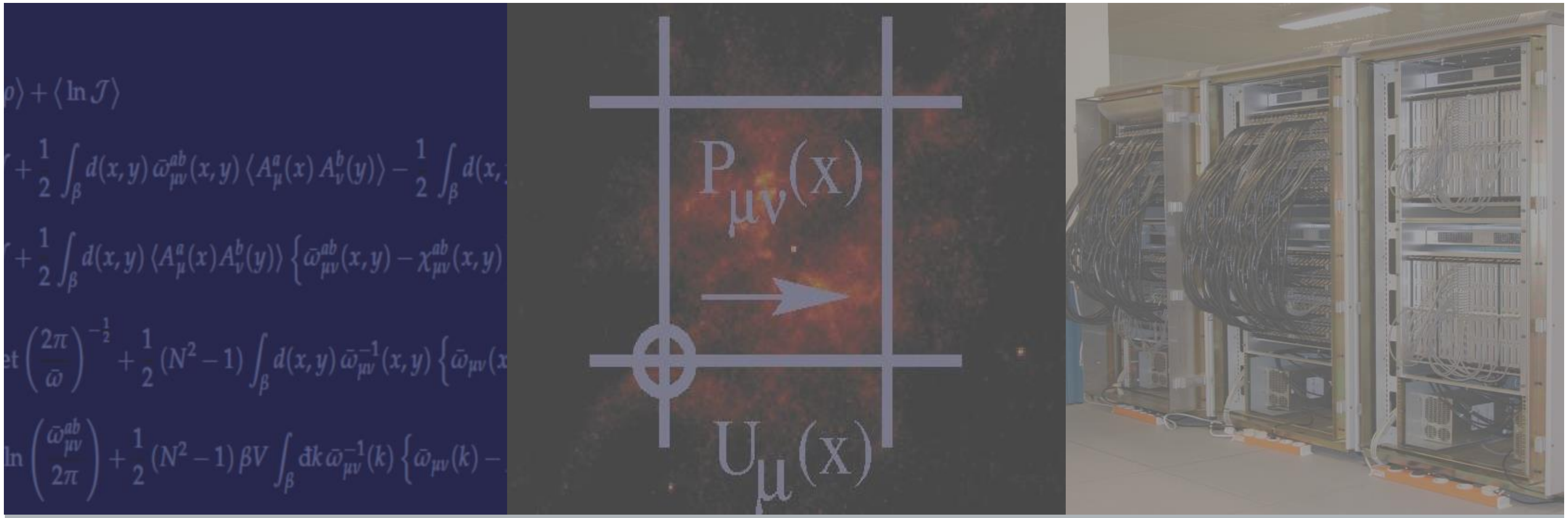
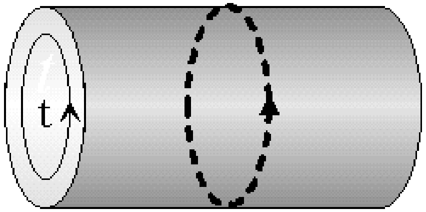


Finite Temperature

Extension to Finite Temperature

- imaginary time formalism



compactify euclidean time $t \in [0, \beta]$

periodic b.c. for gluons (up to center twists)

periodic b.c. for ghosts (even though fermions) !

$$A(t, \mathbf{x}) = \beta^{-1} \sum_{n \in \mathbb{Z}} \int \frac{d^3k}{(2\pi)^3} e^{i(\nu_n t + \mathbf{k}\mathbf{x})} A_n(\mathbf{k})$$

$$\nu_n = \frac{2\pi}{\beta} n \qquad (n \in \mathbb{Z})$$

⇨ Extension to T>0 straightforward

$$\int \frac{d^4k}{(2\pi)^4} \longrightarrow \int_\beta đq \equiv \beta^{-1} \sum_{n \in \mathbb{Z}} \int \frac{d^3k}{(2\pi)^3}$$

- Lorentz structure of propagator

heat bath singles out restframe (1,0,0,0) ⟹ breaks Lorentz invariance

$$D_{\mu\nu}(k) = D_T(k)\, \mathcal{P}^T_{\mu\nu}(k) + D_L(k)\, \mathcal{P}^L_{\mu\nu}(k) + \frac{\zeta}{k^2}\frac{k_\mu k_\nu}{k^2}$$

two different 4-transversal projectors

!

$$\mathcal{P}^T_{\mu\nu}(k) = (1-\delta_{\mu 0})\,(1-\delta_{\nu 0})\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{\mathbf{k}^2}\right)$$ ← 3-transversal

$$\mathcal{P}^L_{\mu\nu}(k) = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) - \mathcal{P}^T_{\mu\nu}(k)$$ ← 3-longitudinal

⟹ Same Lorentz structure for Gaussian kernel and curvature

$$\omega_{\mu\nu}(k) = \omega(k)\cdot\mathcal{P}^T_{\mu\nu}(k) + \sigma(k)\cdot\mathcal{P}^L_{\mu\nu}(k) + \text{long.}$$

$$\chi_{\mu\nu}(k) = \chi(k)\cdot\mathcal{P}^T_{\mu\nu}(k) + \theta(k)\cdot\mathcal{P}^L_{\mu\nu}(k) + \text{long.}$$

Gap Equations

$$\bar{\omega}(k) = k_0^2 + \mathbf{k}^2 + \chi(k) + M^2(\beta)$$

$$\bar{\sigma}(k) = k_0^2 + \mathbf{k}^2 + \theta(k) + M^2(\beta) + \frac{\mathbf{k}^2}{k_0^2 + \mathbf{k}^2}\,\widetilde{M}^2(\beta)$$

induced gluon masses now temperature-dependent

must be finite (no counter term)

$$M^2(\beta) = \frac{1}{2}\,N g^2 \int_\beta đq \left[\frac{A}{\bar{\omega}(q)} + \frac{B(q)}{\bar{\sigma}(q)}\right]$$

$$\widetilde{M}^2(\beta) = \frac{1}{3}\,N g^2 \int_\beta đq \left[\frac{2}{\bar{\omega}(q)} + \left(\frac{q_0^2 - 3\mathbf{q}^2}{q_0^2 + \mathbf{q}^2}\right)\frac{1}{\bar{\sigma}(q)}\right] \to 0 \qquad (\beta \to \infty)$$

Renormalization

- counterterms must be fixed at T=0
- example: ghost equation

counter term δZ_c

$$\eta(k)^{-1} = 1 - Ng^2 I_\eta(k) - Ng^2 \frac{\mathbf{k}^2}{k^2} L_\eta(k) - \left[1 - Ng^2 I_\eta^{(0)}(k) - \eta_0(\mu_c)^{-1}\right]$$

$$= \eta_0(\mu_c)^{-1} - Ng^2 \left[I_\eta^{(0)}(k) - I_\eta^{(0)}(\mu_c)\right] - Ng^2 \left[I_\eta(k) - I_\eta^{(0)}(k)\right] - Ng^2 \frac{\mathbf{k}^2}{k^2} L_\eta(k)$$

$$\eta(k)^{-1} = \eta_0(k)^{-1} - Ng^2 \left[I_\eta(k) - I_\eta^{(0)}(k)\right] - Ng^2 \frac{\mathbf{k}^2}{k^2} L_\eta(k)$$

Ren. T=0 profile

finite T correction

finite T correction vanishes at T=0

Renormalized System at T>0

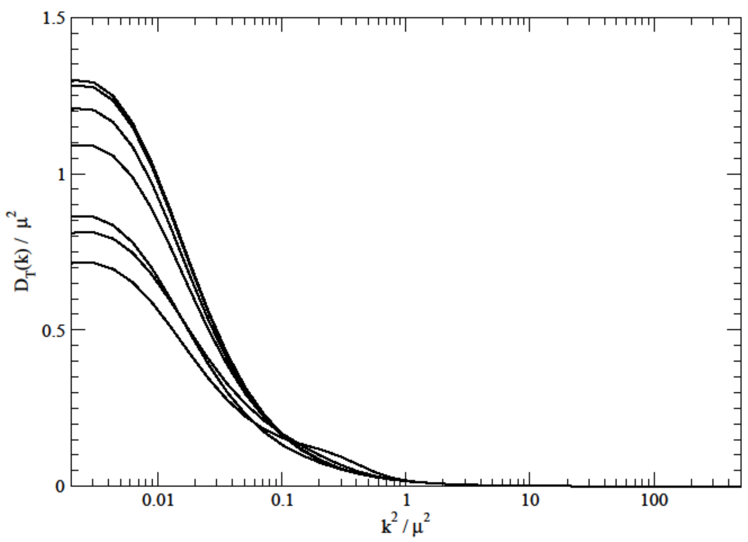
$$\bar{\eta}(k_0,|\mathbf{k}|)^{-1} = \bar{\eta}_0(k)^{-1} - \left[I_\eta(k_0,|\mathbf{k}|) - I_\eta^{(0)}(k)\right] \quad - \frac{\mathbf{k}^2}{k_0^2+\mathbf{k}^2}\left[L_\eta(k_0,|\mathbf{k}|) - L_\eta^{(0)}(k)\right]$$

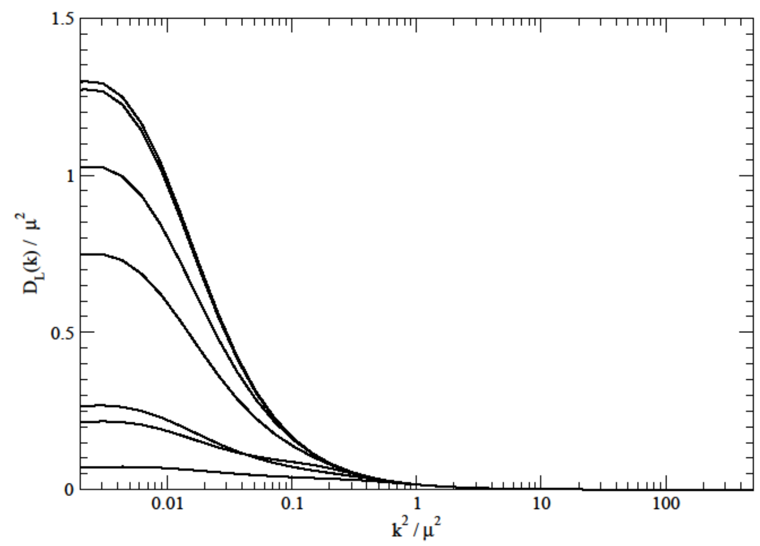
$$\bar{\omega}(k_0,|\mathbf{k}|) = \bar{\omega}_0(k) \quad + \left[\widetilde{I}_\chi(k_0,|\mathbf{k}|) - I_\chi^{(0)}(k)\right] \quad + \left[M^2(\beta) - M_0^2\right]$$

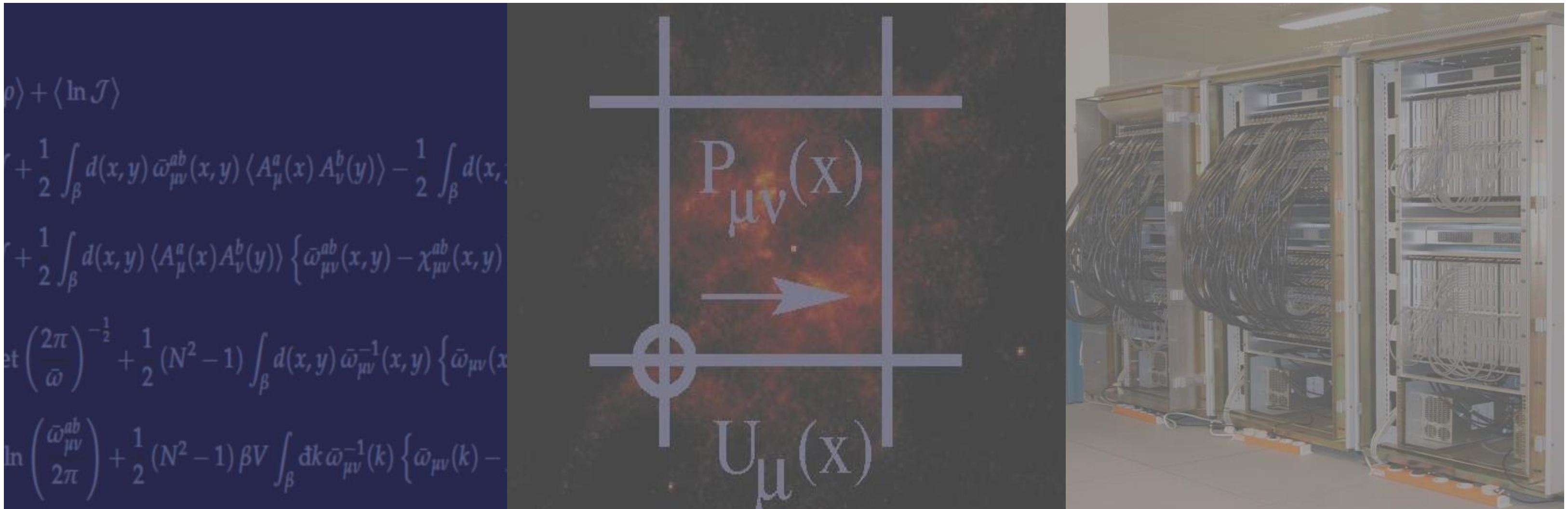
$$\bar{\sigma}(k_0,|\mathbf{k}|) = \bar{\omega}_0(k) \quad + 3\left[I_\chi(k_0,|\mathbf{k}|) - I_\chi^{(0)}(k)\right] - 2\left[\widetilde{I}_\chi(k_0,|\mathbf{k}|) - I_\chi^{(0)}(k)\right]$$
$$+ \left[M^2(\beta) - M_0^2\right] \qquad + \left[\widetilde{M}^2(\beta) - \widetilde{M}^2(0)\right].$$

- All finite temperature corrections have similar structure
- Iterative solution

Preliminary Results







Conclusion

Conclusion and Outlook

- Variational Principle for Effective Action + Gaussian Ansatz
  - yields optimal truncation
  - conventionally renormalizable
  - very good agreement with lattice data
  - Easily extensible to finite temperatures
  - BRST? Confinement?

- Best applied where effective 1-gluon exchange is relevant
  - inclusion of fermions
  - chemical potential