Bound states in Minkowski space in 2+1 dimensions

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Abstract: The Nakanishi perturbative integral representation of the Bethe-Salpeter (BS) amplitude in three-dimensions (2+1) is investigated in order to derive a workable framework for bound states in Minkowski space. The projection onto the null-plane of the three-dimensional homogeneous BS amplitude is used to derive an equation for the Nakanishi weight function. The formal development is illustrated in detail and applied to the bound system composed by two massive scalars interacting through the exchange of a massive scalar. The explicit forms of the integral equations are obtained in ladder approximation.

Introduction

The study of relativistic bound-states in hadronic physics is a topic of constant research. One of the approaches to study this problem is the covariant homogeneous BS equation [1]. It was recognized from the very beginning that, when formulated in Minkowski space, the BS equation has singularities due to the free propagators of the constituent particles. To overcome this difficulty, Wick [2] formulated the BS equation in the Euclidean space. However, the original BS amplitude is lost and the rotated one can no longer be used in calculating other physical observables,like, for instance, form factors.

On the other hand the Nakanishi perturbative integral representation PTIR of the BS amplitude can be used to obtain solutions of the BS equation in the Minkowski space [3], [4]. The method introduced by Karmanov and Carbonell [5] for bound state problems in 3+1 dimensions allied to the Nakanishi PTIR of the BS amplitude, has a starting point the projection of the homogeneous BS equation onto the light-front. Considerable simplifications in the algebraic manipulations for bound and scattering states are found [6]. Furthermore, the singularities of the BS equation present in Minkowski space are absent. This process lead to equation for the Nakanishi weight function, which can be solved numerically.

In our study we are motivated to study the solution of the Bethe-Salpeter equation in 2+1 dimensions, because it may be applied to bound states of electron-hole pairs on graphene sheets with defects and their properties must be described within a the relativistic framework. We start with the simpler problem of the bound state of two spinless bosons with the interaction mediated by a spinless massive boson, and using the ladder approximation for the interaction kernel. However, this formalist must be extend to the study of fermion bound state, which describe more interesting problems.

The Bethe-Salpeter equation in 2+1 dimensions and PTIR

The BS equation in 2+1 dimensions for two interacting bosons of mass m, total momentum p, relative momentum k, exchanging a boson mass μ in the ladder approximation, is given by

$$\begin{aligned}\Phi(k,p) =& \frac{i}{(k+\frac{p}{2})^2-m^2+i\epsilon}\frac{i}{(k-\frac{p}{2})^2-m^2+i\epsilon} \\ &\times(ig)^2\int\frac{d^3k'}{(2\pi)^3}\frac{i}{(k'-k)^2-\mu^2+i\varepsilon}\Phi(k',p).\end{aligned} \tag{1}$$

The BS amplitude for S-wave state within PTIR is

$$\Phi(k,p) = -i\int_{-1}^{1}dz'\int_0^\infty d\gamma'\frac{g_B^{(n)}(\gamma',z';\kappa^2)}{(\gamma'+m^2-\frac{p^2}{4}-k^2-p\cdot kz'-i\epsilon)^{n+2}}, \tag{2}$$

where $g_B^{(n)}$ is the Nakanishi weight function. The bound state mass $M=\sqrt{p^2}$ is introduced by means of

$$\kappa^2 = m^2-\frac{p^2}{4}(>0 \text{ for the bound state}) \tag{3}$$

In order to obtain an equation for $g_B^{(n)}$, we introduce the Eq.(2) in the both sides of Eq.(1). Then the projection onto the light-front is performed. It is equivalent to integrate over $k^-=k^0+k^3$ on both sides of the BS equation.

PTIR and Light-Front projection

The projection onto the light-front of the BS amplitude gives the valence wave function as

$$\begin{aligned}\Psi_v(\gamma,z) =& i\int_{-1}^{1}dz'\int_0^\infty d\gamma' g_B^{(n)}(\gamma',z';\kappa^2) \\ &\times\int_{-\infty}^{\infty}\frac{dk^-}{2\pi}\frac{1}{(\gamma+\kappa^2-k^2-p\cdot kz'-i\epsilon)^{n+2}}\end{aligned} \tag{4}$$

where we introduce $\gamma=k_\perp^2$ and $z=-2k^+/M$, in the rest frame of the bound state. The right-hand side of the Eq.(1) after to the light front projection is

$$\begin{aligned}B(\gamma,z) =& (-1)^n(ig)^2\int_{-1}^{1}dz'\int_0^\infty d\gamma' g_B^{(n)}(\gamma',z';\kappa^2) \\ &\times\int_{-\infty}^{\infty}\frac{dk^-}{2\pi}\frac{1}{(\gamma+\kappa^2-k^2-p\cdot kz'-i\epsilon)^{n+2}} \\ &\times\frac{d^3k'}{(2\pi)^3}\frac{1}{(k-k')^2-\mu^2+i\epsilon}\frac{1}{(k'^2+p\cdot k'z'-\gamma'-\kappa^2+i\varepsilon)^{n+2}}\end{aligned} \tag{5}$$

Now the BS equation takes the form $\Psi_v(\gamma,z)=B(\gamma,z)$. The equation for the Nakanishi weight function for $n=1$ can be written as

$$\begin{aligned}&\int_0^\infty d\gamma'\frac{g_B(\gamma',z';\kappa^2)}{[\gamma'+\gamma+z^2m^2+(1-z^2)\kappa^2]^2} \\ &= \int_{-1}^{1}dz'\int_0^\infty d\gamma' V(z,z',\gamma,\gamma')g_B(\gamma',z';\kappa^2)\end{aligned} \tag{6}$$

This integral equation for g_B is solved, with the potential term

$$V(z,z',\gamma,\gamma') = \int_0^\infty\frac{d^3k^-}{2\pi}\frac{g^2P(p,z',k,\gamma',\kappa^2)}{[(\frac{p}{2}+k)^2-m^2+i\epsilon][(\frac{p}{2}-k)^2-m^2+i\epsilon]} \tag{7}$$

and

$$\begin{aligned}P(p,z',k,\gamma',\kappa^2) =& \int\frac{d^3k'}{(2\pi)^3}\frac{1}{(k-k')^2-\mu^2+i\epsilon} \\ &\times\frac{1}{(k'^2+p\cdot k'z'-\gamma'-\kappa^2+i\epsilon)^3}.\end{aligned} \tag{8}$$

$P(p,z',k,\gamma',\kappa^2)$ is obtained as an integral over Feynman parameters and after the integration over k^- is done analytically .

$$\begin{aligned}V(z,z',\gamma,\gamma') =& \frac{3g^2}{64\pi}\frac{1}{[\gamma+(1-z^2)\kappa^2+z^2m^2]}\times \\ &\left[\left(\frac{1+z}{1+z'}\right)^{\frac{5}{2}}\theta(z'-z)F(z,z,\gamma,\gamma')+\left(\frac{1-z}{1-z'}\right)^{\frac{5}{2}}\theta(z'-z)F(z,z,\gamma,\gamma')\right]\end{aligned} \tag{9}$$

with

$$\begin{aligned}F(z,z',\gamma,\gamma') =& \frac{2[8a^2b+4a(3b^2+3bc+2c^2)+b^2(3b+2c)]}{3(a+b+c)^{\frac{3}{2}}(b^2-4ac)^2} \\ &-\frac{16\sqrt{a}b}{3(b^2-4ac)^2}\end{aligned} \tag{10}$$

where a, b, c are the following expressions

$$\begin{aligned}a &= \left(\frac{1+z}{1+z'}\right) \qquad b=\gamma+m^2z^2+\kappa^2(1-z^2)+\frac{1+z}{1+z'}(\gamma'-\mu^2) \\ c &= \left(z'^2\frac{M^2}{4}+\kappa^2\right)-(\gamma+m^2z^2+\kappa^2(1-z^2))\end{aligned} \tag{11}$$

Euclidean BS amplitude in PTIR

The Euclidean BS is found from turning $k^0\to ik_E^0$ in Eq.(2) and in the rest frame one obtains

$$\Phi(k_E,p) = -i\int_{-1}^{1}dz'\int_0^\infty d\gamma'\frac{g_B^{(n)}(\gamma',z';\kappa^2)}{(\gamma'+m^2-\frac{M^2}{4}+k_E^2-iMk_E^0z')^{n+2}} \tag{12}$$

with $k_E^2=(k_E^0)^2+\vec{k}^2$. The symmetry property of Nakanishi weight function $g_B^{(n)}(\gamma,z;\kappa^2)=g_B^{(n)}(\gamma,-z;\kappa^2)$ leads to

$$\begin{aligned}\Phi(k_E,p) =& -i\int_{-1}^{1}dz'\int_0^\infty d\gamma' g_B^{(n)}(\gamma',z';\kappa^2) \\ &\times\Re\left[\frac{1}{(\gamma'+m^2-\frac{M^2}{4}+k_E^2-iMk_E^0z')^{n+2}}\right]\end{aligned} \tag{13}$$

Euclidean BS equation

By performing the Wick-rotation to the Minkowski form Eq.(1),

$$\begin{aligned}\Phi(k_0,\vec{k}) =& \frac{1}{(\frac{M^2}{4}-\vec{k}^2-k_0^2-m^2)^2+M^2k_0^2} \\ &\times g^2\int\frac{dk_0'd^2k'}{(2\pi)^3}\frac{1}{(k'-k)^2+\mu^2}\Phi(k_0',\vec{k'}).\end{aligned} \tag{14}$$

After partial wave projection to L=0, we have

$$\begin{aligned}\Phi(k_0,\vec{k}) =& \frac{2\pi}{(\frac{M^2}{4}-\vec{k}^2-k_0^2-m^2)^2+M^2k_0^2}g^2 \\ &\times\int_{-\infty}^{\infty}\frac{dk_0'}{(2\pi)^3}\int_0^\infty dk'k'\frac{\Phi(k_0,\vec{k})}{[((k_0-k_0')^2+k'^2+k^2+\mu^2)^2-(2k'k)^2]^{\frac{1}{2}}}.\end{aligned} \tag{15}$$

This equation can be solved numerically by Gauss-Legendre quadrature and it will be an eigenvalue equation with the eigenvalue g^2 for a given binding energy.

Numerical results

To solve the integral equations numerically, we added a small parameter (here $\epsilon=10^{-6}$) to the integral operator to achieve good stability

$$\frac{1}{\alpha}\Phi=B(M)\Phi \qquad B_{ij}\to B_{ij}+\epsilon\delta_{ij}. \tag{16}$$

For the solution of the Eq.(6), we choose a proper basis, which allows us to expand the weight function taking into account the symmetry property of $g_B^{(n)}(\gamma,z;\kappa^2)$ in respect to z, the constrain $g_B^{(n)}(\gamma,z=\pm1;\kappa^2)=0$ and the fall-off in γ:

$$g_B^{(n)}(\gamma,z;\kappa^2)=\sum_{\ell=0}^{N_z}\sum_{j=0}^{N_g}A_{\ell j}G_\ell(z)\mathcal{L}_j(\gamma) \tag{17}$$

The $G_\ell(z)$ are given in terms of the even Gegenbauer polynomials, $C_{2\ell}^{\frac{5}{2}}(z)$, for the z-dependence

$$G_\ell(z)=4\,(1-z^2)\,\Gamma(5/2)\sqrt{\frac{(2\ell+5/2)\,(2\ell)!}{\pi\Gamma(2\ell+5)}}\,C_{2\ell}^{(5/2)}(z) \tag{18}$$

while the functions $\mathcal{L}_j(\gamma)$ are expressed in terms of Laguerre polynomials adopted for the γ-dependence

$$\mathcal{L}_j(\gamma)=\sqrt{a}\,L_j(a\gamma)\,e^{-a\gamma/2}. \tag{19}$$

In order to speed up the convergence, the parameter values a=24.0 and a=2.0 have been adopted for the cases μ=0.1 and μ=0.5 respectively. Also, the variable γ has been rescaled according to $\gamma\to2\gamma/a_0$ with a_0=12. The Eq.(6) can be written symbolically as

$$\frac{1}{\alpha}D(M)g^L=A^L(M)g^L. \tag{20}$$

To solve this equation as the eigenvalue equation, one relies in the existence of the inverse of the integral operator $D(M)$.

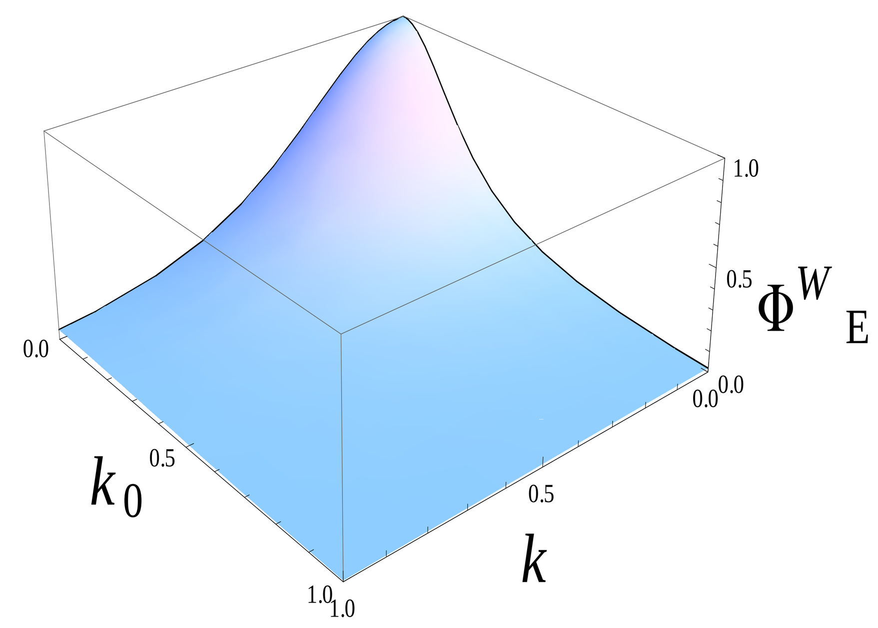
The kernel contain a highly non-linear dependence upon the mass M, but a linear dependence upon the coupling constant α, given the adopted ladder approximation. Then, we choose the value of the binding energy in the interval

$$0<\frac{B}{m}=2-\frac{M}{m}\leq2. \tag{21}$$

After that, we should look for the minimal value of the coupling constant that allows a binding energy. In order to compare the results of the two methods, in the next Table we show a comparison between the eigenvalues obtained from the Eqs.(6) and (15).

Table 1: Values of $\frac{g^2}{m^3}$ calculated by two methods for different binding energies B and exchange boson mass.

B/m ($\mu=0.1$)	**PTIR**	**EUCL**	B/m ($\mu=0.5$)	**PTIR**	**EUCL**
0.01	0.82	0.79	0.01	5.33	5.31
0.1	4.26	4.26	0.1	14.88	14.88
0.2	8.07	8.06	0.2	22.67	22.67
0.5	19.5	19.51	0.5	42.33	42.33
1	36.06	36.03	1	67.38	67.39



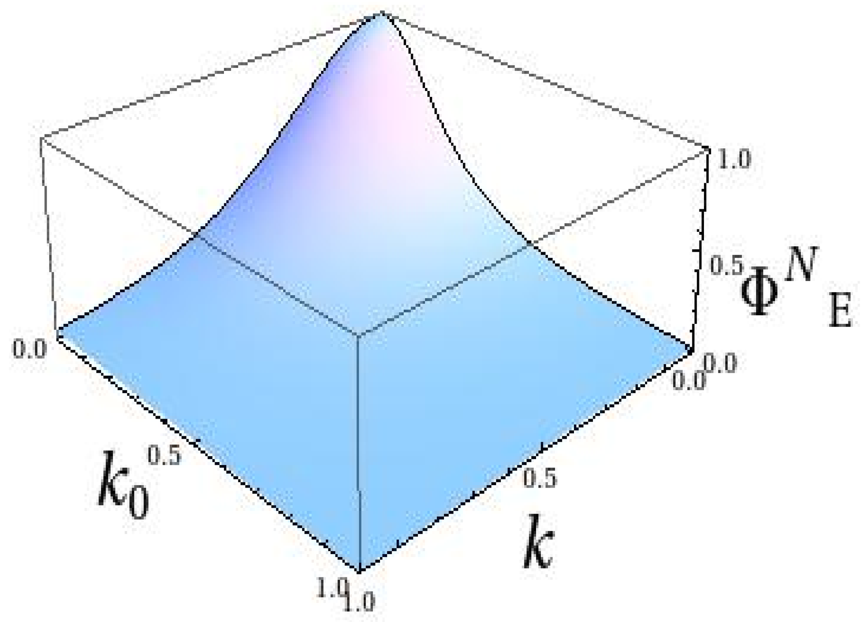


Figure 1: Euclidean BS amplitude calculated by means of Eqs. (15) (upper frame) and (13) (lower frame), for the values $\mu=0.5$ and $B=0.5$

For the calculated wave function, we have the following

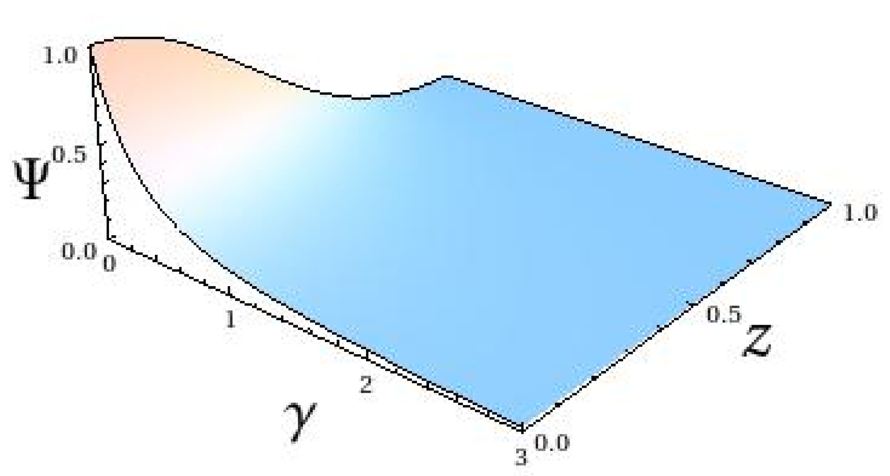


Figure 2: Wave function calculated by means of the Eq.(4) for the values $\mu=0.5$ and $B=0.5$

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