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Dispersive approach to QCD: τ lepton decay and vacuum polarization function

A.V. Nesterenko

Bogoliubov Laboratory of Theoretical Physics

Joint Institute for Nuclear Research, Dubna, Russian Federation

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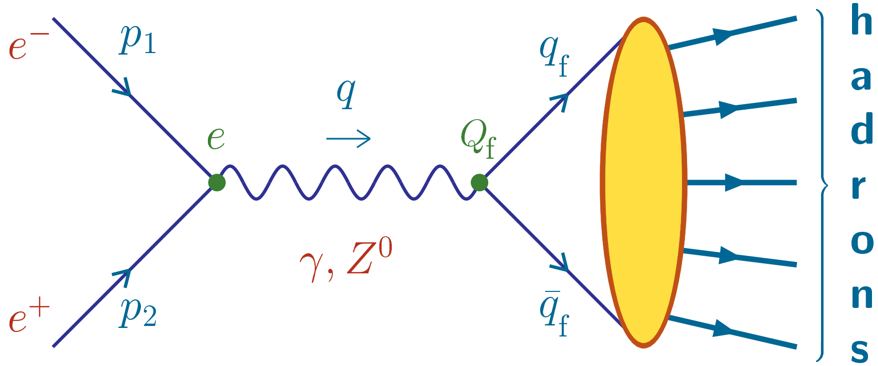
INTRODUCTION

Hadronic vacuum polarization function $\Pi(q^2)$ plays a central role in various issues of QCD and Standard Model. In particular, the theoretical description of some strong interaction processes and of hadronic contributions to electroweak observables is inherently based on $\Pi(q^2)$:

- electron–positron annihilation into hadrons
- inclusive τ lepton hadronic decay
- muon anomalous magnetic moment
- running of the electromagnetic coupling

GENERAL DISPERSION RELATIONS

Cross–section of $e^+e^- \to$ hadrons:



$$\sigma = 4\pi^2 \frac{2\alpha^2}{s^3} L^{\mu\nu} \Delta_{\mu\nu},$$

where $s = q^2 = (p_1 + p_2)^2 > 0,$

$$L_{\mu\nu} = \frac{1}{2}\Big[q_\mu q_\nu - g_{\mu\nu} q^2 - (p_1 - p_2)_\mu (p_1 - p_2)_\nu\Big],$$

$$\Delta_{\mu\nu} = (2\pi)^4 \sum_\Gamma \delta(p_1 + p_2 - p_\Gamma) \langle 0 | J_\mu(-q) | \Gamma \rangle \langle \Gamma | J_\nu(q) | 0 \rangle,$$

and $J_\mu = \sum_f Q_f :\bar{q}\,\gamma_\mu\, q:$ **is the electromagnetic quark current.**

Kinematic restriction: the hadronic tensor $\Delta_{\mu\nu}(q^2)$ assumes non–zero values only for $q^2 \geq m^2$, since otherwise no hadron state Γ could be excited

■ **Feynman (1972); Adler (1974).**

The hadronic tensor can be represented as $\Delta_{\mu\nu} = 2\,\mathrm{Im}\,\Pi_{\mu\nu}$,

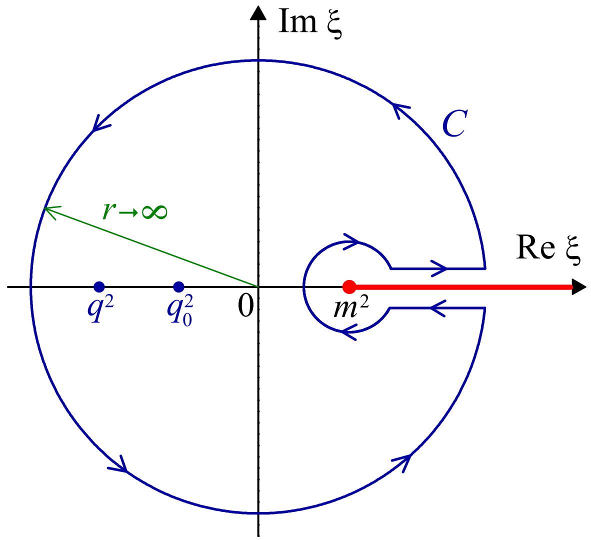
$$\Pi_{\mu\nu}(q^2) = i\int e^{iqx}\langle 0|\,T\{J_\mu(x)\,J_\nu(0)\}\,|0\rangle\, d^4x = i(q_\mu q_\nu - g_{\mu\nu}q^2)\frac{\Pi(q^2)}{12\pi^2}.$$

Kinematic restriction: $\Pi(q^2)$ **has the only cut** $q^2 \geq m^2$

Dispersion relation for $\Pi(q^2)$**:**

$$\Delta\Pi(q^2,\, q_0^2) = \frac{1}{2\pi i}(q^2 - q_0^2)\oint_C \frac{\Pi(\xi)}{(\xi - q^2)(\xi - q_0^2)}\, d\xi$$

$$= (q^2 - q_0^2)\int_{m^2}^{\infty} \frac{R(s)}{(s - q^2)(s - q_0^2)}\, ds,$$



where $\Delta\Pi(q^2,\, q_0^2) = \Pi(q^2) - \Pi(q_0^2)$ **and** $R(s)$ **denotes the measurable ratio of two cross–sections**

$$R(s) = \frac{1}{2\pi i}\lim_{\varepsilon\to 0_+}\Big[\Pi(s+i\varepsilon) - \Pi(s-i\varepsilon)\Big] = \frac{\sigma(e^+e^- \to \text{hadrons};\, s)}{\sigma(e^+e^- \to \mu^+\mu^-;\, s)}.$$

Kinematic restriction: $R(s) = 0$ **for** $s < m^2$

For practical purposes it proves to be convenient to deal with the Adler function ($Q^2 = -q^2 \geq 0$)

$$D(Q^2) = -\frac{d\,\Pi(-Q^2)}{d\ln Q^2}, \qquad D(Q^2) = Q^2 \int_{m^2}^{\infty} \frac{R(s)}{(s+Q^2)^2}\, ds$$

■ Adler (1974); De Rujula, Georgi (1976); Bjorken (1989).

This dispersion relation provides a link between experimentally measurable and theoretically computable quantities.

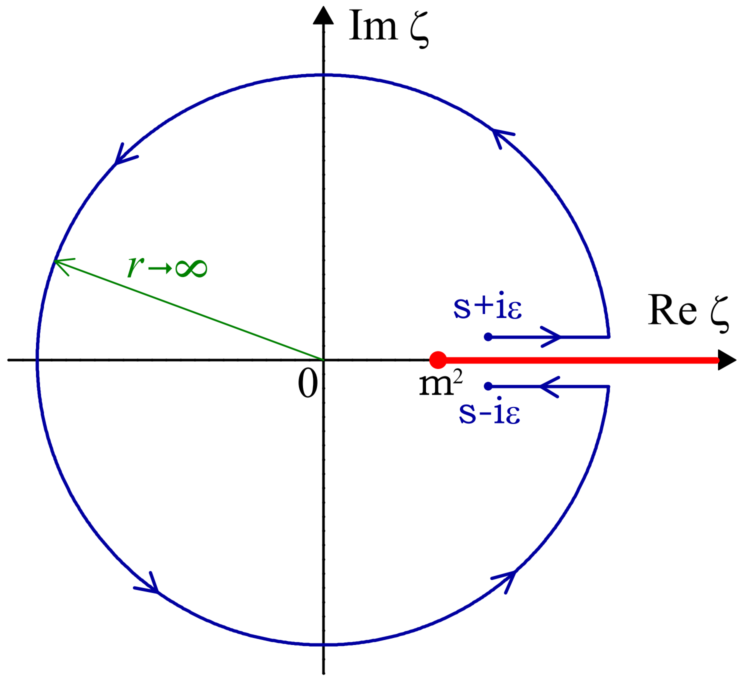
The inverse relations between the functions on hand read

$$R(s) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \int_{s+i\varepsilon}^{s-i\varepsilon} D(-\zeta)\, \frac{d\zeta}{\zeta},$$

■ Radyushkin (1982); Krasnikov, Pivovarov (1982)

$$\Delta\Pi(-Q^2, -Q_0^2) = -\int_{Q_0^2}^{Q^2} D(\sigma)\, \frac{d\sigma}{\sigma}$$

■ Nesterenko (2013).



The complete set of relations between $\Pi(q^2)$, $R(s)$, and $D(Q^2)$:

$$\Delta\Pi(q^2,\, q_0^2) = (q^2 - q_0^2) \int_{m^2}^{\infty} \frac{R(\sigma)}{(\sigma - q^2)(\sigma - q_0^2)}\, d\sigma = - \int_{-q_0^2}^{-q^2} D(\sigma)\, \frac{d\sigma}{\sigma},$$

$$R(s) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \Big[\Pi(s + i\varepsilon) - \Pi(s - i\varepsilon)\Big] = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \int_{s+i\varepsilon}^{s-i\varepsilon} D(-\zeta)\frac{d\zeta}{\zeta},$$

$$D(Q^2) = -\frac{d\,\Pi(-Q^2)}{d\,\ln Q^2} = Q^2 \int_{m^2}^{\infty} \frac{R(\sigma)}{(\sigma + Q^2)^2}\, d\sigma.$$

Derivation of these relations requires only the location of cut of $\Pi(q^2)$ and its UV asymptotic. Neither additional approximations nor phenomenological assumptions are involved.

Nonperturbative constraints:

- $\Pi(q^2)$: **has the only cut** $q^2 \geq m^2$;
- $R(s)$: **embodies** π^2**–terms, vanishes for** $s < m^2$;
- $D(Q^2)$: **has the only cut** $Q^2 \leq -m^2$, **vanishes at** $Q^2 \to 0$.

DISPERSIVE APPROACH TO QCD

Functions on hand in terms of the common spectral density:

$$\Delta\Pi(q^2,\, q_0^2) = \Delta\Pi^{(0)}(q^2,\, q_0^2) + \int_{m^2}^{\infty} \rho(\sigma) \ln\left(\frac{\sigma - q^2}{\sigma - q_0^2}\,\frac{m^2 - q_0^2}{m^2 - q^2}\right) \frac{d\,\sigma}{\sigma},$$

$$R(s) = R^{(0)}(s) + \theta(s - m^2) \int_{s}^{\infty} \rho(\sigma)\, \frac{d\,\sigma}{\sigma},$$

$$D(Q^2) = D^{(0)}(Q^2) + \frac{Q^2}{Q^2 + m^2} \int_{m^2}^{\infty} \rho(\sigma)\, \frac{\sigma - m^2}{\sigma + Q^2}\, \frac{d\,\sigma}{\sigma},$$

$$\rho(\sigma) = \frac{1}{\pi}\frac{d}{d\,\ln\sigma}\, \mathrm{Im} \lim_{\varepsilon\to 0_+} p(\sigma - i\varepsilon) = -\frac{d\,r(\sigma)}{d\,\ln\sigma} = \frac{1}{\pi}\, \mathrm{Im} \lim_{\varepsilon\to 0_+} d(-\sigma - i\varepsilon),$$

where $\Delta\Pi^{(0)}(q^2,\, q_0^2)$, $R^{(0)}(s)$, $D^{(0)}(Q^2)$ denote the leading–order terms and $p(q^2)$, $r(s)$, $d(Q^2)$ stand for the strong corrections

■ Nesterenko, Papavassiliou (2005–2007); Nesterenko (2007–2014).

Derivation of obtained representations involves neither additional approximations nor model–dependent assumptions, with all the nonperturbative constraints being embodied.

The leading–order terms of the functions on hand read

$$\Delta\Pi^{(0)}(q^2,\, q_0^2) = 2\,\frac{\varphi - \tan\varphi}{\tan^3\varphi} - 2\,\frac{\varphi_0 - \tan\varphi_0}{\tan^3\varphi_0}, \qquad \sin^2\varphi = \frac{q^2}{m^2},$$

$$R^{(0)}(s) = \theta(s - m^2)\left(1 - \frac{m^2}{s}\right)^{3/2}, \qquad \sin^2\varphi_0 = \frac{q_0^2}{m^2},$$

$$D^{(0)}(Q^2) = 1 + \frac{3}{\xi}\left[1 - \sqrt{1+\xi^{-1}}\,\sinh^{-1}\!\left(\xi^{1/2}\right)\right], \qquad \xi = \frac{Q^2}{m^2}$$

■ Feynman (1972); Akhiezer, Berestetsky (1965).

Perturbative contribution to the spectral density:

$$\rho_{\text{pert}}(\sigma) = \frac{1}{\pi}\,\frac{d\,\mathrm{Im}\,p_{\text{pert}}(\sigma - i0_+)}{d\,\ln\sigma} = -\frac{d\,r_{\text{pert}}(\sigma)}{d\,\ln\sigma} = \frac{1}{\pi}\,\mathrm{Im}\,d_{\text{pert}}(-\sigma - i0_+),$$

namely, at the one–loop level $\rho^{(1)}_{\text{pert}}(\sigma) = (4/\beta_0)[\ln^2(\sigma/\Lambda^2) + \pi^2]^{-1}$**.**

Note on the massless limit

In the limit $m = 0$ the obtained integral representations read

$$\Delta\Pi(q^2,\, q_0^2) = -\ln\left(\frac{-q^2}{-q_0^2}\right) + \int_0^\infty \rho(\sigma) \ln\left[\frac{1-(\sigma/q^2)}{1-(\sigma/q_0^2)}\right]\frac{d\,\sigma}{\sigma},$$

$$R(s) = \theta(s)\left[1 + \int_s^\infty \rho(\sigma)\,\frac{d\,\sigma}{\sigma}\right],$$

$$D(Q^2) = 1 + \int_0^\infty \frac{\rho(\sigma)}{\sigma + Q^2}\,d\,\sigma.$$

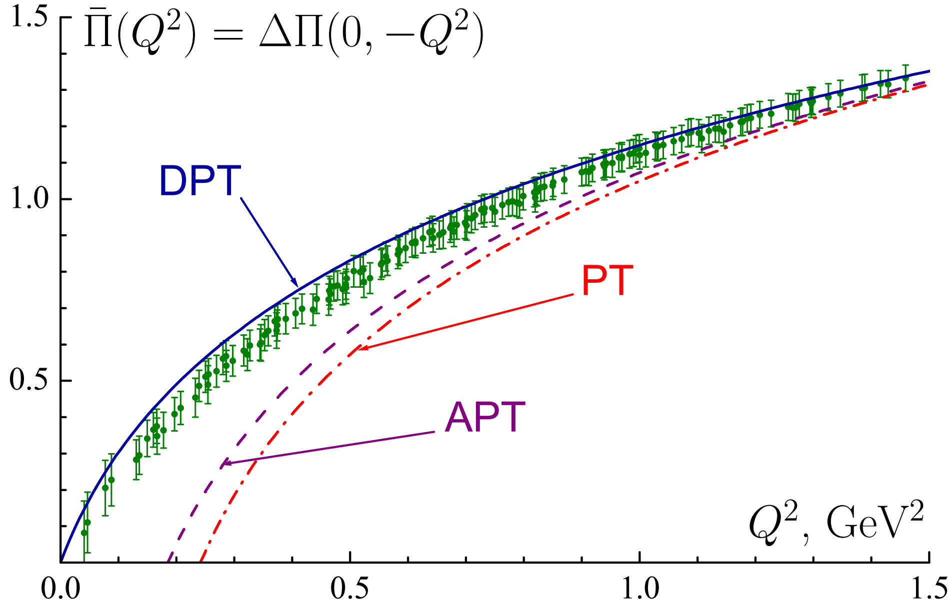
For $\rho(\sigma) = \rho_{\text{pert}}(\sigma)$ two highlighted massless equations become identical to those of the APT ■ Shirkov, Solovtsov, Milton (1997–2007).

But it is essential to keep the threshold m^2 nonvanishing:

- **massless limit loses some of nonperturbative constraints**
- **effects due to $m \neq 0$ become substantial at low energies**

HADRONIC VACUUM POLARIZATION FUNCTION

Comparison of obtained results with lattice simulation data:

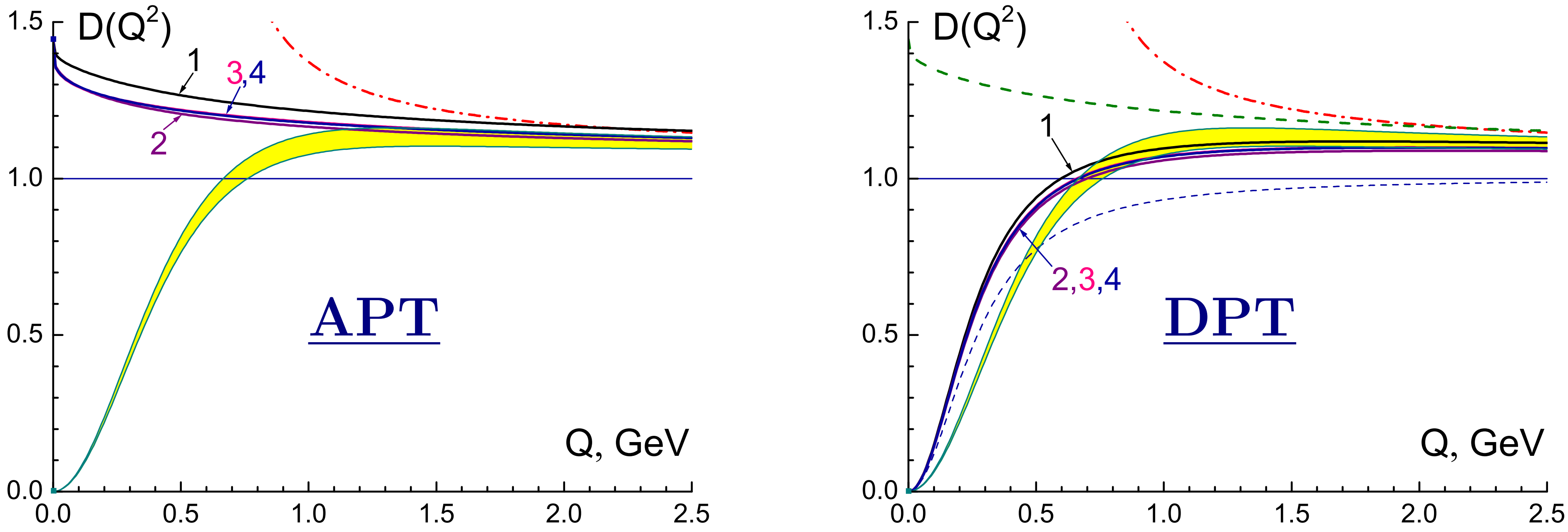


■ **Della Morte, Jager, Juttner, Wittig (2011–2013); Nesterenko (2013, 2014).**

	unphysical singularities	agreement with lattice
PT	contains	disagrees
APT	free	disagrees
DPT	free	agrees

ADLER FUNCTION

massless limit ($m = 0$) **realistic case ($m \neq 0$)**



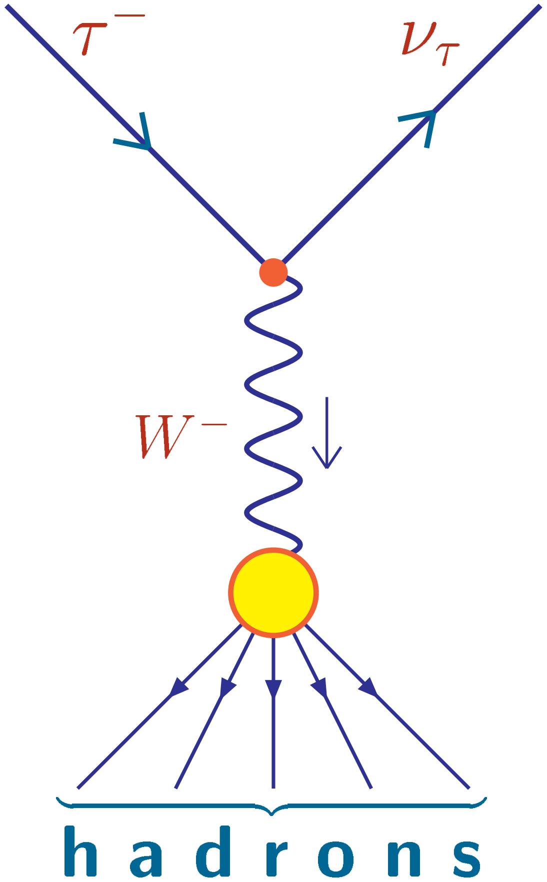
■ Nesterenko, Papavassiliou (2006); Nesterenko (2007–2009).

	unphysical singularities	agreement with data
PT	contains	disagrees
APT	free	disagrees
DPT	free	agrees

INCLUSIVE τ LEPTON HADRONIC DECAY

The interest to this process is due to

- **The only lepton with hadronic decays**
- **Precise experimental data**
- **No need in phenomenological models**
- **Probes infrared hadron dynamics**



The experimentally measurable quantity:

$$R_\tau = \frac{\Gamma(\tau^- \to \text{hadrons}^- \, \nu_\tau)}{\Gamma(\tau^- \to e^- \, \bar{\nu}_e \, \nu_\tau)} = R_{\tau,\text{V}} + R_{\tau,\text{A}} + R_{\tau,\text{S}},$$

$$R_{\tau,\text{V}} = R_{\tau,\text{V}}^{J=0} + R_{\tau,\text{V}}^{J=1} = 1.782 \pm 0.009,$$

$$R_{\tau,\text{A}} = R_{\tau,\text{A}}^{J=0} + R_{\tau,\text{A}}^{J=1} = 1.694 \pm 0.010.$$

■ **ALEPH Collaboration (1997–2005); Davier, Hocker, Malaescu, Yuan, Zhang (2014).**

The theoretical prediction for the quantities on hand reads

$$R^{J=1}_{\tau,\mathrm{V/A}} = \frac{N_{\mathrm{c}}}{2}\,|V_{\mathrm{ud}}|^2\,S_{\mathrm{EW}}\Bigl(\Delta^{\mathrm{V/A}}_{\mathrm{QCD}} + \delta'_{\mathrm{EW}}\Bigr),$$

$N_{\mathrm{c}}=3$, $|V_{\mathrm{ud}}|=0.9738\pm0.0005$, $S_{\mathrm{EW}}=1.0194\pm0.0050$, $\delta'_{\mathrm{EW}}=0.0010$,

$$\Delta^{\mathrm{V/A}}_{\mathrm{QCD}} = 2\int_{m^2_{\mathrm{V/A}}}^{M_\tau^2} f\Bigl(\frac{s}{M_\tau^2}\Bigr) R^{\mathrm{V/A}}(s)\,\frac{ds}{M_\tau^2}\,,$$

where $M_\tau = 1.777\,\mathrm{GeV}$, $f(x) = (1-x)^2\,(1+2x)$,

$$R^{\mathrm{V/A}}(s)=\frac{1}{2\pi i}\lim_{\varepsilon\to 0_+}\Bigl[\Pi^{\mathrm{V/A}}(s+i\varepsilon)-\Pi^{\mathrm{V/A}}(s-i\varepsilon)\Bigr]=\frac{1}{\pi}\,\mathrm{Im}\lim_{\varepsilon\to 0_+}\Pi^{\mathrm{V/A}}(s+i\varepsilon)$$

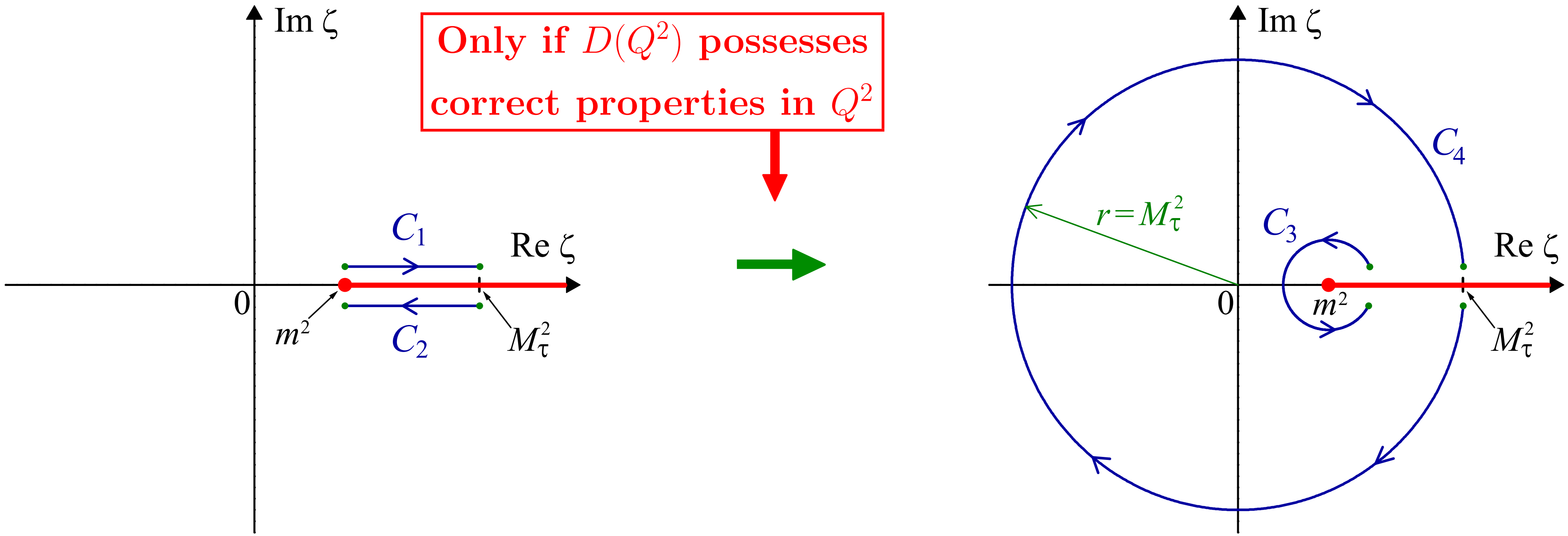
- **Braaten, Narison, Pich (1992); Pivovarov (1992).**

Integration by parts leads to

$$\Delta_{\mathrm{QCD}} = g(1)R(M_\tau^2) - g(\chi)R\bigl(m^2\bigr) + \frac{1}{2\pi i}\int\limits_{C_1+C_2} g\left(\frac{\zeta}{M_\tau^2}\right) D(-\zeta)\frac{d\zeta}{\zeta},$$

where $\chi = m^2/M_\tau^2$ **and** $g(x) = x(2-2x^2+x^3)$.

Continuous deformation of the integration contour:



$$\Delta_{\mathrm{QCD}} = g(1)R(M_\tau^2) - g(\chi)R\left(m^2\right) + \frac{1}{2\pi i}\int\limits_{C_3+C_4} g\left(\frac{\zeta}{M_\tau^2}\right) D(-\zeta)\frac{d\zeta}{\zeta}.$$

In the massless limit ($m=0$) this equation acquires the form

$$\Delta_{\mathrm{QCD}} = \frac{1}{2\pi}\lim_{\varepsilon\to 0_+}\int_{-\pi+\varepsilon}^{\pi-\varepsilon}\left[1 - g\left(-e^{i\theta}\right)\right] D\left(M_\tau^2 e^{i\theta}\right) d\theta.$$

Inclusive τ decay within perturbative approach:

Commonly, perturbative $D(Q^2)$ is directly employed here

$$D(Q^2) \simeq D_{\text{pert}}^{(\ell)}(Q^2) = 1 + \sum_{j=1}^{\ell} d_j \left[\alpha_{\text{pert}}^{(\ell)}(Q^2)\right]^j, \quad Q^2 \to \infty$$

with $\alpha_{\text{pert}}^{(1)}(Q^2) = 4\pi/[\beta_0 \ln(Q^2/\Lambda^2)]$**,** $\beta_0 = 11 - 2n_{\text{f}}/3$**, and** $d_1 = 1/\pi$**.**

In what follows the one–loop level ($\ell = 1$) with $n_{\text{f}} = 3$ active flavors will be assumed.

The one–loop perturbative expression for $\Delta_{\text{QCD}}^{\text{V/A}}$ reads

$$\Delta_{\text{pert}}^{\text{V/A}} = 1 + \frac{4}{\beta_0} \int_0^{\pi} \frac{\lambda A_1(\theta) + \theta A_2(\theta)}{\pi(\lambda^2 + \theta^2)} \, d\theta,$$

where $\lambda = \ln\left(M_\tau^2/\Lambda^2\right)$**,** $A_1(\theta) = 1 + 2\cos(\theta) - 2\cos(3\theta) - \cos(4\theta)$**,** $A_2(\theta) = 2\sin(\theta) - 2\sin(3\theta) - \sin(4\theta)$**.**

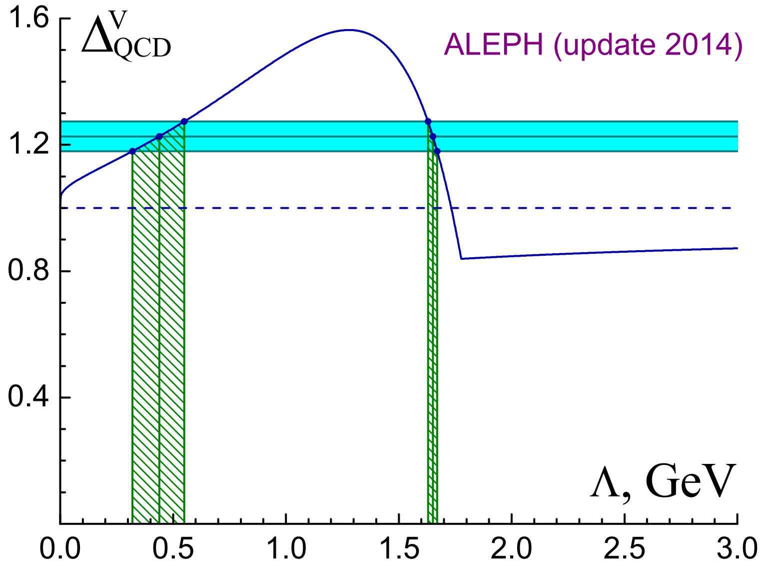
Perturbative approach gives $\Delta^{\rm V}_{\rm pert} \equiv \Delta^{\rm A}_{\rm pert}$ **, however** $\Delta^{\rm V}_{\rm exp} \neq \Delta^{\rm A}_{\rm exp}$ **:**

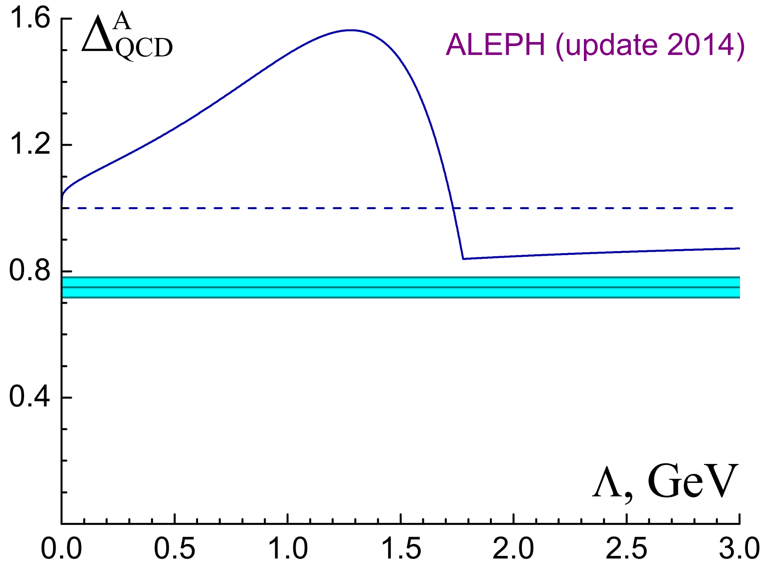
OPAL (update 2012): $\Delta^{\rm V}_{\rm exp} = 1.229 \pm 0.088$, $\Delta^{\rm A}_{\rm exp} = 0.741 \pm 0.058$

- **OPAL Collaboration (1999); Boito, Golterman, Jamin et al. (2012).**

ALEPH (update 2014): $\Delta^{\rm V}_{\rm exp} = 1.227 \pm 0.047$, $\Delta^{\rm A}_{\rm exp} = 0.749 \pm 0.032$

- **ALEPH Collaboration (2005); Davier, Hocker, Malaescu, Yuan, Zhang (2014).**





$\Lambda = \left(439^{+110}_{-119}\right)$ MeV $\Lambda = \left(1651^{+20}_{-22}\right)$ MeV

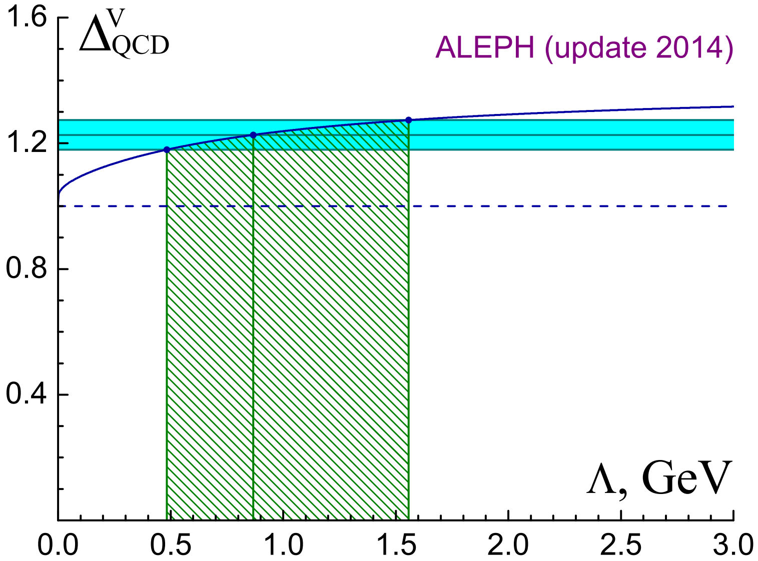
no solution

V–channel: PT gives two equally justified solutions for Λ

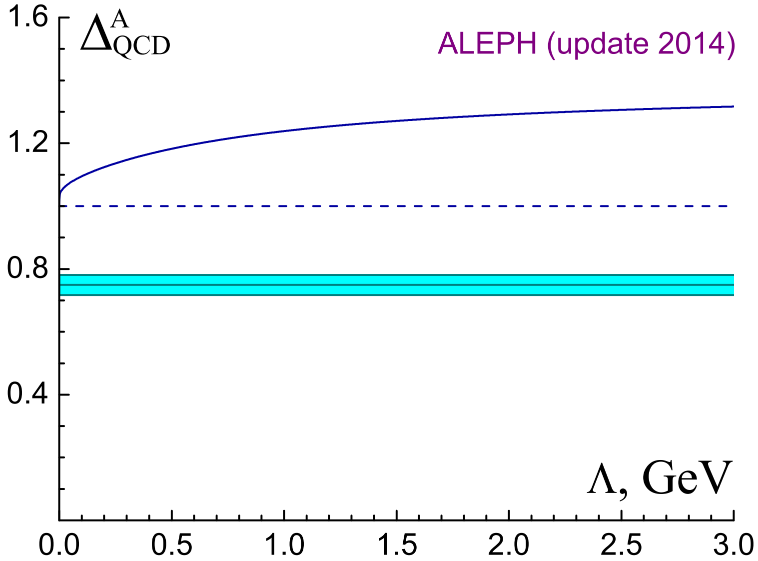
A–channel: PT fails to describe data on τ **lepton decay**

Inclusive τ decay within APT:

Description of the inclusive τ lepton hadronic decay within APT leads to a qualitatively similar result:



$$\Lambda = \left(867^{+690}_{-385}\right)\ \text{MeV}$$



no solution

V–channel: APT gives one solution for Λ

A–channel: APT fails to describe data on τ lepton decay

Both PT and APT leave out the effects due to hadronization.

Inclusive τ decay within dispersive approach:

Description of the inclusive τ lepton hadronic decay within dispersive approach enables one to properly account for

- nonperturbative constraints on the functions on hand
- effects due to hadronic production threshold

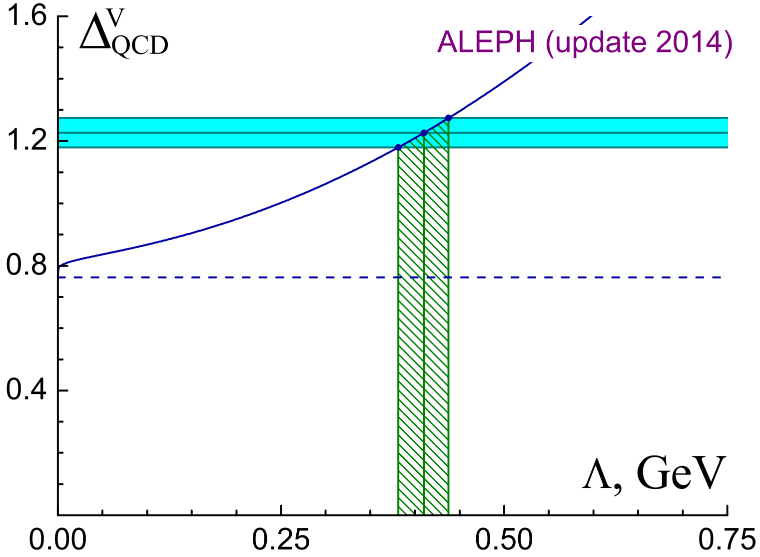
In the framework of dispersive approach the initial expression for $\Delta_{\mathrm{QCD}}^{\mathrm{V/A}}$ eventually acquires the following form

$$\Delta_{\mathrm{QCD}}^{\mathrm{V/A}} = \sqrt{1-\zeta_{\mathrm{V/A}}}\left(1 + 6\zeta_{\mathrm{V/A}} - \frac{5}{8}\zeta_{\mathrm{V/A}}^2 + \frac{3}{16}\zeta_{\mathrm{V/A}}^3\right) + \int_{m_{\mathrm{V/A}}^2}^{\infty} H\!\left(\frac{\sigma}{M_\tau^2}\right)\rho(\sigma)\,\frac{d\sigma}{\sigma}$$
$$- 3\zeta_{\mathrm{V/A}}\left(1 + \frac{1}{8}\zeta_{\mathrm{V/A}}^2 - \frac{1}{32}\zeta_{\mathrm{V/A}}^3\right)\ln\!\left[\frac{2}{\zeta_{\mathrm{V/A}}}\left(1 + \sqrt{1-\zeta_{\mathrm{V/A}}}\right) - 1\right],$$

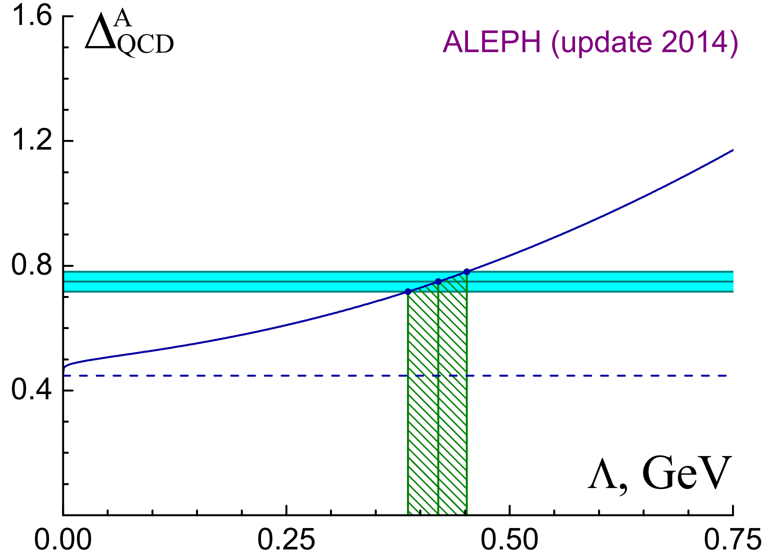
with $\zeta_{\mathrm{V/A}} = m_{\mathrm{V/A}}^2/M_\tau^2$, $H(x) = g(x)\,\theta(1-x) + g(1)\,\theta(x-1) - g\big(\zeta_{\mathrm{V/A}}\big)$

■ Nesterenko (2011–2014)

Comparison of the obtained result with ALEPH–14 data:



$$\Lambda = \bigl(409 \pm 28\bigr)\ \mathrm{MeV}$$



$$\Lambda = \bigl(419 \pm 33\bigr)\ \mathrm{MeV}$$

OPAL–12 data: $\Lambda = \bigl(409 \pm 53\bigr)$ MeV **[V]**, $\Lambda = \bigl(409 \pm 61\bigr)$ MeV **[A]**

■ **Nesterenko (2011–2014)**

	unphysical singularities	description of $R_{\tau,\mathrm{V}}$ and $R_{\tau,\mathrm{A}}$
PT	contains	fails
APT	free	fails
DPT	free	describes

SUMMARY

- **The integral representations for $\Pi(q^2)$, $R(s)$, and $D(Q^2)$ are derived within dispersive approach to QCD**

- **These representations embody the nonperturbative constraints and account for the effects due to hadronization**

- **The obtained results are in a good agreement with relevant lattice data and low–energy experimental predictions**

- **The developed approach proves to be capable of describing experimental data on inclusive τ lepton hadronic decay in vector and axial–vector channels in a self–consistent way**