Gauge field topology and the hadron spectrum

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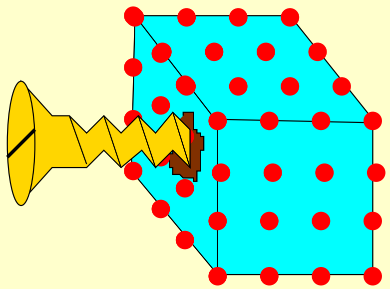


Outline

- what does "topology" mean for gauge fields
- classical versus quantum theories
- implications for fermions
- the pseudoscalar spectrum
- lattice issues

Boundary conditions in R^4 Euclidean space-time

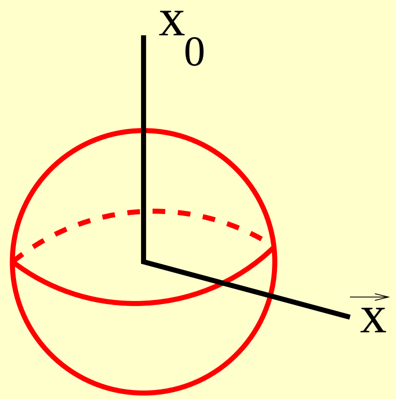
- assume $F_{\mu\nu} \to 0$ at infinity
  - does not require potential $A_\mu \to 0$
- gauge field goes to pure gauge at infinity
- $A_\mu \to -ih^\dagger \partial_\mu h$
  - $h(x) \in$ gauge group

Infinity is a sphere S_3 and the gauge group can contain spheres

- $SU(2) \sim S_3$ $\qquad h = a_0 + \vec{a} \cdot \vec{\sigma}$ with $a_\mu^2 = 1$
- $SU(3) \sim S_3 \otimes S_5$

At infinity $h(x)$ can wrap non-trivially around the gauge group

- still having $F_{\mu\nu} \to 0$



Space of smooth gauge fields divides into topological "sectors"

Instanton solution is an explicit example

- classical solution to the Yang-Mills theory $D_\mu F_{\mu\nu} = 0$
  - $A_\mu = \frac{-ix^2}{g(x^2+\rho^2)} h^\dagger \partial_\mu h$
  - $h(x_\mu) = \frac{t+i\vec{\tau}\cdot\vec{x}}{\sqrt{\vec{x}^2+t^2}} \in SU(2)$
  - ρ is the instanton size
  - size is arbitrary due to conformal invariance of the classical theory
- classical action $S_I = \frac{8\pi^2}{g^2}$

Necessarily non-perturbative

Winding number robust under smooth field deformations

$$\nu = \frac{g^2}{16\pi^2} \int d^4x \ \mathrm{Tr} F\tilde{F}$$

$F\tilde{F}$ is odd under time reversal

- for $x \neq 0$ we have $\langle F\tilde{F}(x) \ F\tilde{F}(0)\rangle < 0$ Seiler 2002

Topological susceptibility

$$\xi = \langle \frac{\nu^2}{V} \rangle = \left(\frac{g^2}{16\pi^2}\right)^2 \int d^4x \langle F\tilde{F}(x) \ F\tilde{F}(0)\rangle$$

- negative contribution from $x \neq 0$
- positive contribution from singularity at $x = 0$, "contact term"

Long and short distances are intimately entwined!

Quantum theory and asymptotic freedom

- effective coupling g goes to zero at short distances
- conformal invariance lost through "dimensional transmutation"

Continuum limit takes $g(a)$ logarithmically to zero

- lattice spacing a represents a cutoff
- $g^2(a) \sim \frac{1}{2\beta_0 \log(1/\Lambda a)}$
- Λ is an integration constant
- sets the scale for particle masses in the quantum theory
  - $\Lambda_{qcd} = \frac{1}{a} e^{1/2\beta_0 g^2} g^{-\beta_1/\beta_0^2} (1 + O(g^2)) \quad \underset{a \to 0}{\longrightarrow} \quad \text{constant}$
  - $\beta_0 = \frac{1}{16\pi^2}(11 - 2N_f/3), \qquad \beta_1 = \left(\frac{1}{16\pi^2}\right)^2 (102 - 22N_f/3)$

Naively combining

- classical instantons suppressed in the continuum limit

$$e^{-S_I} \sim e^{-8\pi^2/g(a)^2} \sim e^{-16\pi^2\beta_0 \log(a\Lambda)} \sim a^{11-2N_f/3}$$

- Power law supression in lattice spacing
  - 3 flavors give a^9 for unit lattice volume
  - a^5 per unit physical volume

Can we ignore instantons at small lattice spacing?

N0: they affect the hadron spectrum at order a^0

  - semiclassical arguments ignore quantum fluctuations
  - quantum paths are far from smooth; non-differentiable

Fermions and the index theorem

Dirac action $\overline{\psi}(D+m)\psi$

- $D^\dagger = -D = \gamma_5 D \gamma_5$

Non-trivial gauge winding gives ν exact zero modes

- $D\psi(x) = 0$
- zero modes modes are chiral, $\gamma_5|\psi\rangle = \pm|\psi\rangle$

On a fixed gauge configuration $\mathrm{Tr}\gamma_5 = \nu$

But doesn't $\mathrm{Tr}\ \gamma_5 = 0$???

Fujikawa: Not in the regulated theory!!!

Use eigenstates of D to define $\mathrm{Tr}\gamma_5$

- $D|\psi_i\rangle = \lambda_i|\psi_i\rangle$
- $\mathrm{Tr}\gamma_5 = \sum_i \langle\psi_i|\gamma_5|\psi_i\rangle$

Non-zero eigenstates in chiral pairs

- $D|\psi\rangle = \lambda|\psi\rangle$
- $D\gamma_5|\psi\rangle = -\lambda\gamma_5|\psi\rangle = \lambda^*\gamma_5|\psi\rangle$

Space spanned by $|\psi\rangle$ and $|\gamma_5\psi\rangle$ gives no contribution to $\mathrm{Tr}\gamma_5$

- $\langle\psi|\gamma_5|\psi\rangle = 0$ when $\lambda \neq 0$
- only the zero modes count!

$$\mathrm{Tr}\gamma_5 = \sum_i \langle\psi_i|\gamma_5|\psi_i\rangle = \nu$$

Where did the opposite chirality states go?

- continuum: lost at "infinity" "above the cutoff"
- Wilson: real eigenvalues in doubler region
- overlap: modes on opposite side of unitarity circle

This phenomenon involves both short and long distances

- zero modes compensated by modes lost at the cutoff

Cannot uniquely separate perturbative and non-perturbative effects

- small instantons can "fall through the lattice"
- scheme and scale dependent

Zero modes break chiral symmetry

- $\psi \to e^{i\gamma_5\theta}\psi$ changes measure
- $(d\psi d\overline{\psi}) \to e^{i\theta \mathrm{Tr}\gamma_5}(d\psi d\overline{\psi}) = e^{i\nu\theta}(d\psi d\overline{\psi})$
- inserts $e^{i\nu\theta}$ into path integral weight
- "Theta vacuum" is an inequivalent theory; violates CP symmetry

Can we work at fixed topology?

Tunneling between winding numbers is hard

- long correlation times as quarks become light
- near zero modes suppress tunneling

Fixed topology

$$Z_\nu = \int d\Theta \; e^{i\nu\Theta} \; Z(m_q, \Theta)$$

- combines physically different theories

Path integral and volume dependence: JLQCD

$$Z = \int (dA)(d\overline{\psi})(d\psi)\ e^{-S(m_q,\Theta)}\ =\ e^{-VF(m_q,\Theta)}$$

- every value of (m_q, Θ) is a physically different field theory

As $V \to \infty$ with fixed ν

$$Z_\nu = \int d\Theta\ e^{i\nu\Theta}\ Z(m_q, \Theta) = \int d\Theta\ e^{i\nu\Theta}\ e^{-VF(m_q,\Theta)}$$

- dominated by saddle point at $\Theta = 0$
- extra instantons can hide "behind the moon"

Despite long correlation time in winding

- local physical observables likely to be better behaved

Importance for the particle spectrum

Consider two flavor QCD with light but non-degenerate masses

- pseudoscalar operators

$$\overline{u}\gamma_5 u$$
$$\overline{d}\gamma_5 d$$
$$\overline{u}\gamma_5 d \sim \pi_+$$
$$\overline{d}\gamma_5 u \sim \pi_-$$

Helicity conservation naively suggests mixing of

  - $\overline{u}\gamma_5 u = \overline{u}_L\gamma_5 u_R + \overline{u}_R\gamma_5 u_L$
- with
  - $\overline{d}\gamma_5 d = \overline{d}_L\gamma_5 d_R + \overline{d}_R\gamma_5 d_L$
- suppressed by $m_u m_d$

Wrong: the anomaly couples u and d through $F\tilde{F}$

- strongly mixes $\overline{u}\gamma_5 u$ and $\overline{d}\gamma_5 d$
  - topology induces the effective "t'Hooft vertex"
- physical $\eta' \sim \overline{u}\gamma_5 u + \overline{d}\gamma_5 d$ not a pseudo-Goldstone boson
  - $M_{\eta'} \sim \Lambda_{qcd} + O(m_u, m_d)$

Leaves the orthogonal combination $\pi_0 \sim \overline{u}\gamma_5 u - \overline{d}\gamma_5 d$

  - $M_{\pi_0}^2 \sim \frac{m_u + m_d}{2}$
- isospin breaking suppressed to higher order
  - $M_{\pi_0}^2 = M_{\pi_\pm}^2 - O((m_u - m_d)^2)$

The η' meson

- $\psi \to e^{i\gamma_5\theta}\psi$ is not a symmetry of the quantum theory
- $\eta' \sim i\overline{\psi}\gamma_5\psi$ is not a Goldstone boson
  - $m_{\eta'} \propto \Lambda_{qcd} + O(m_q)$
- does not vanish as the quark masses go to zero
  - $m_{\eta'}^2 \sim \frac{1}{a^2} e^{1/\beta_0 g^2} g^{-\beta_1/\beta_0^2}(1 + O(g^2)) \sim a^0$

Quantum fluctuations allow more topology than semi-classical estimate

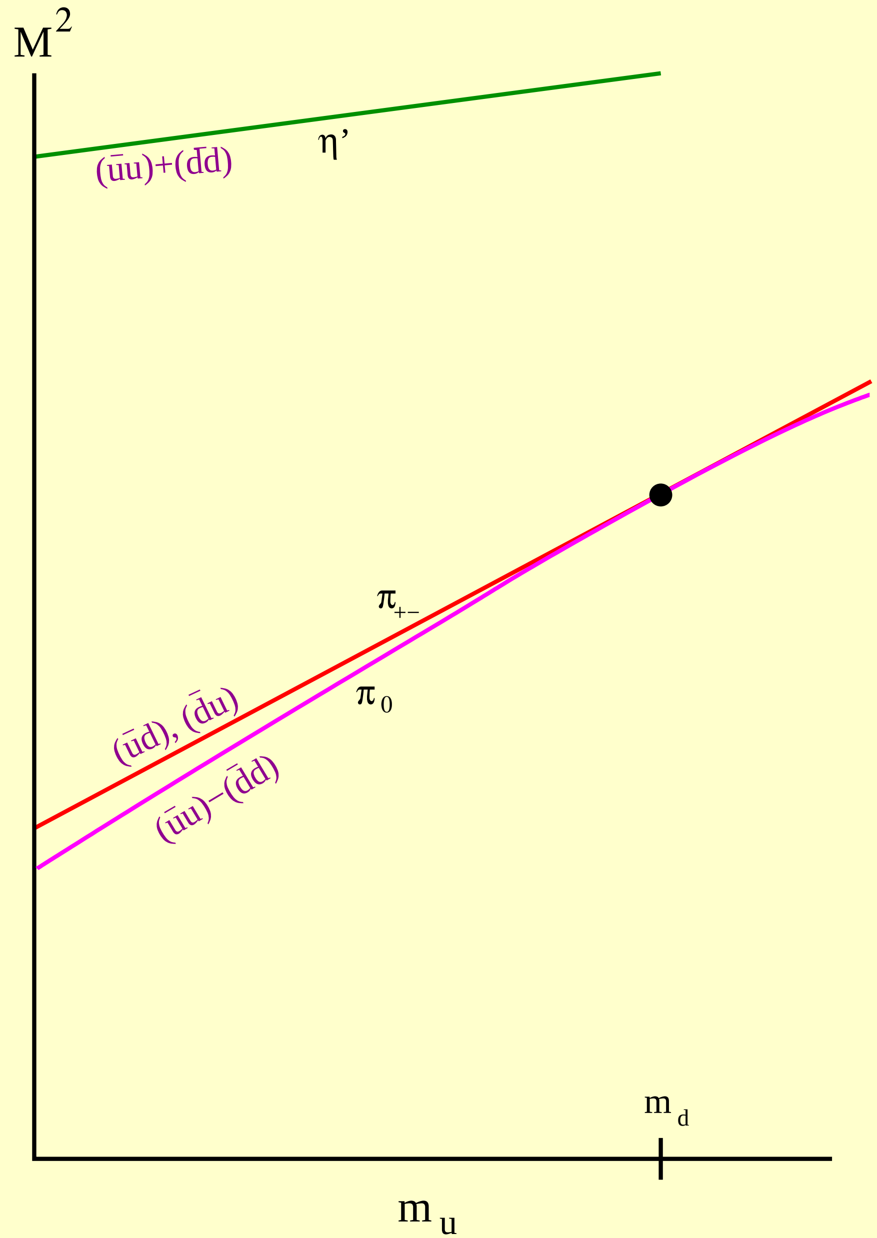
Fix m_d, vary m_u

- $M_\pi^2 \propto \frac{m_u+m_d}{2} + O(m_q^2)$
- $M_{\eta'} \sim \Lambda_{qcd}$

With isospin broken

- $M_{\pi_\pm}^2 - M_{\pi_0}^2 \propto (m_d - m_u)^2$
- η', π_0, glueballs all mix

Mass gap survives at $m_u = 0$

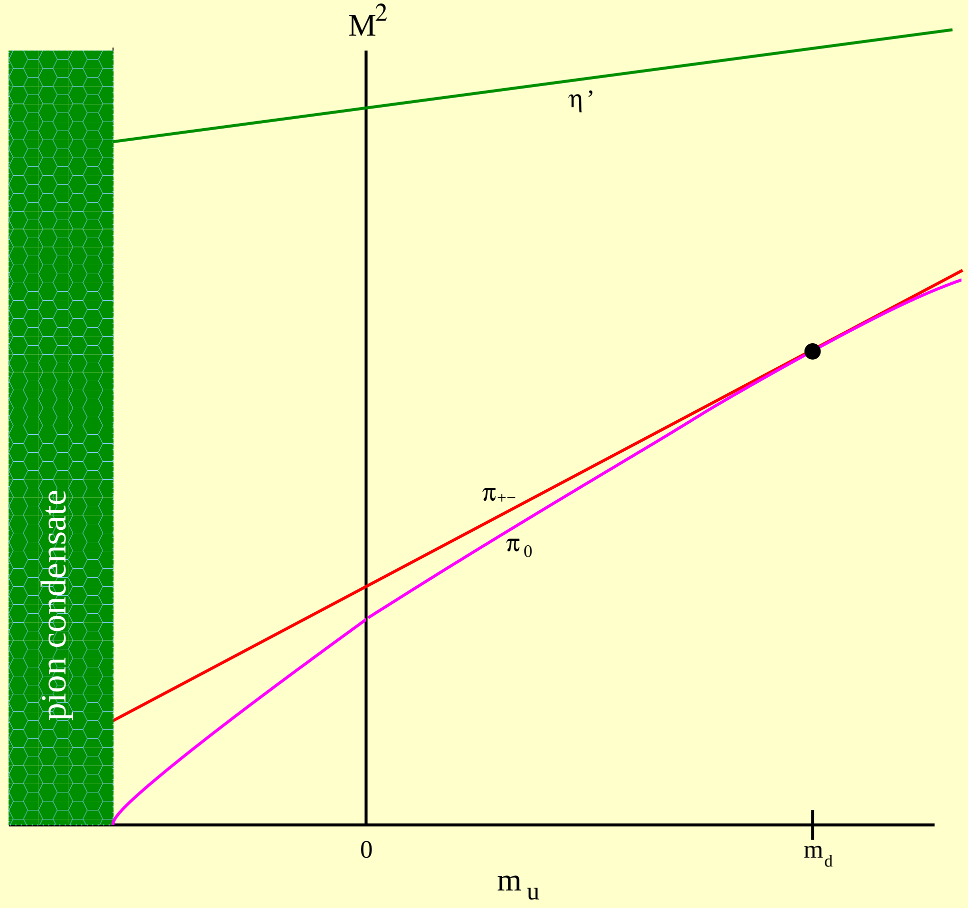


The Dashen phenomenon

No singularity at $m_u = 0$

- extrapolate to negative m_u
- $M^2_{\pi_0}$ can go negative
- pion condensate forms
  - $\langle \pi_0 \rangle \neq 0$
  - CP broken
- formally at $\Theta = \pi$
  - $\prod_q m_q < 0$

Dashen 1971

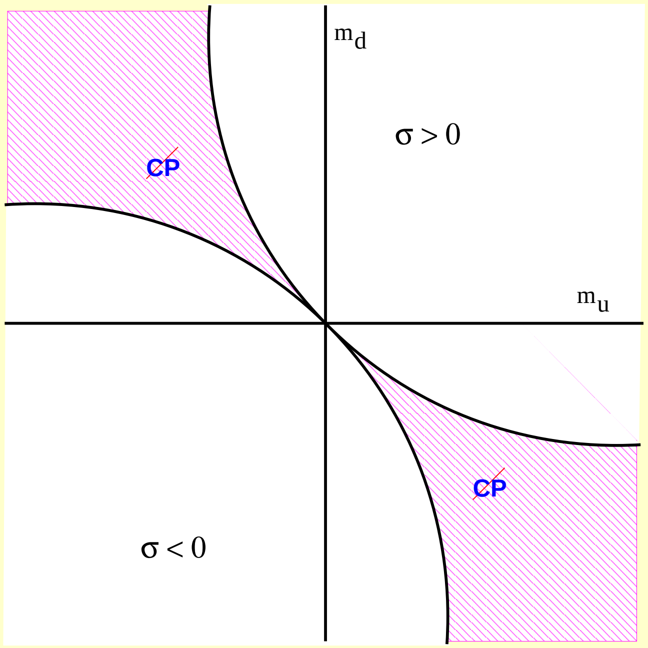


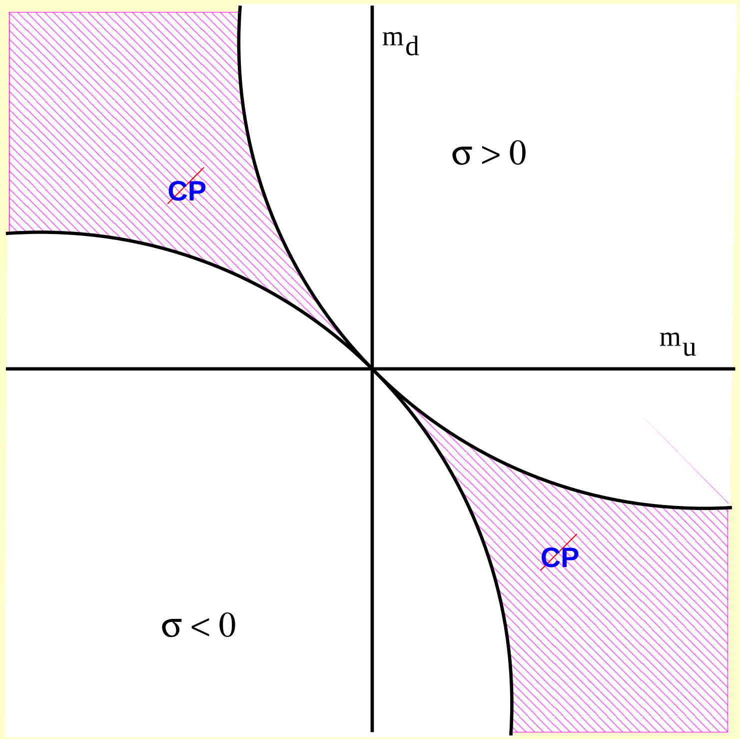
Structure manifest in both "linear" and "nonlinear" sigma models

- MC 2004, Aoki & MC 2014, Horkel 2014

Ising-like transition at $m_u < 0$

- order parameter $\langle \pi_0 \rangle \neq 0$
- breaks CP spontaneously



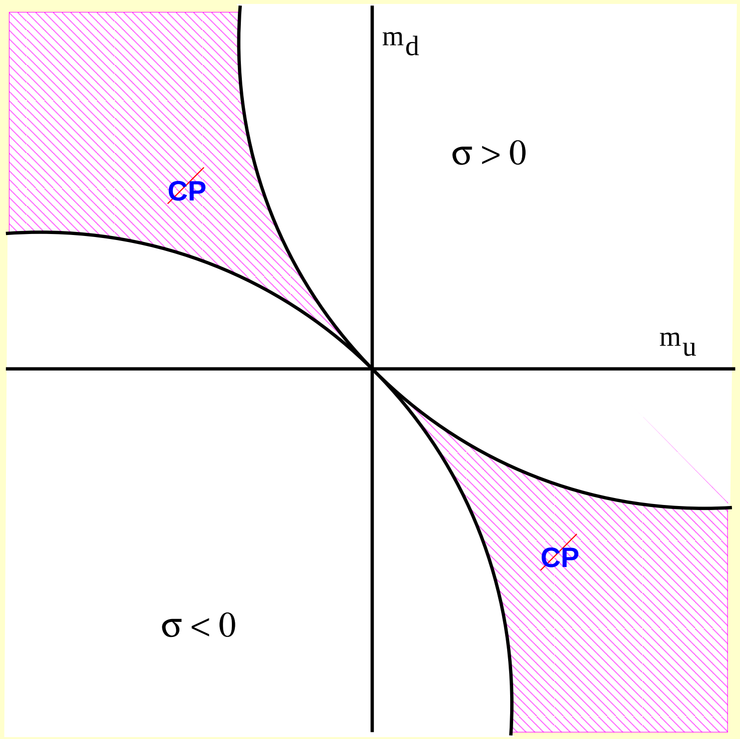


No structure at $m_u = 0$ when $m_d \neq 0$

- no long distance physics despite possible small Dirac eigenvalues

Second order transition at non-vanishing m_u and m_d of opposite sign

- long distance physics without small Dirac eigenvalues

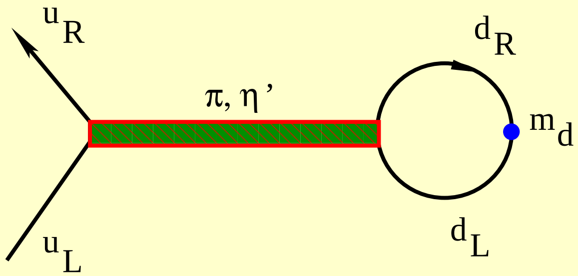


Independent symmetries for isoscalar and isovector mass terms

- $m_u + m_d$ versus $m_u - m_d$ separately multiplicatively renormalized

Non-perturbative mass renormalization not "flavor blind"

Nonperturbative contributions mix masses



- $\delta m_u \sim \frac{(M_{\eta'} - M_\pi)}{\Lambda_{qcd}} m_d$
  - three flavors: $\delta m_u \sim \frac{m_d m_s}{m_d + m_s}$
  - chiral perturbation theory: Kaplan and Manohar ambiguity

A single massless quark is not renormalization group invariant

- Question: Can any experiment tell if $m_u = 0$?
  - can $m_u = 0$ solve the strong CP problem if it is ill-defined?
- $\overline{MS}$ is perturbative and cannot answer this question

Non-perturbative issues require the lattice

- adjust lattice parameters to get the hadron spectrum
- read off the quark masses and see if $m_u = 0$

But there are many lattice formulations

- how do we define the quark mass
- how do we define topology

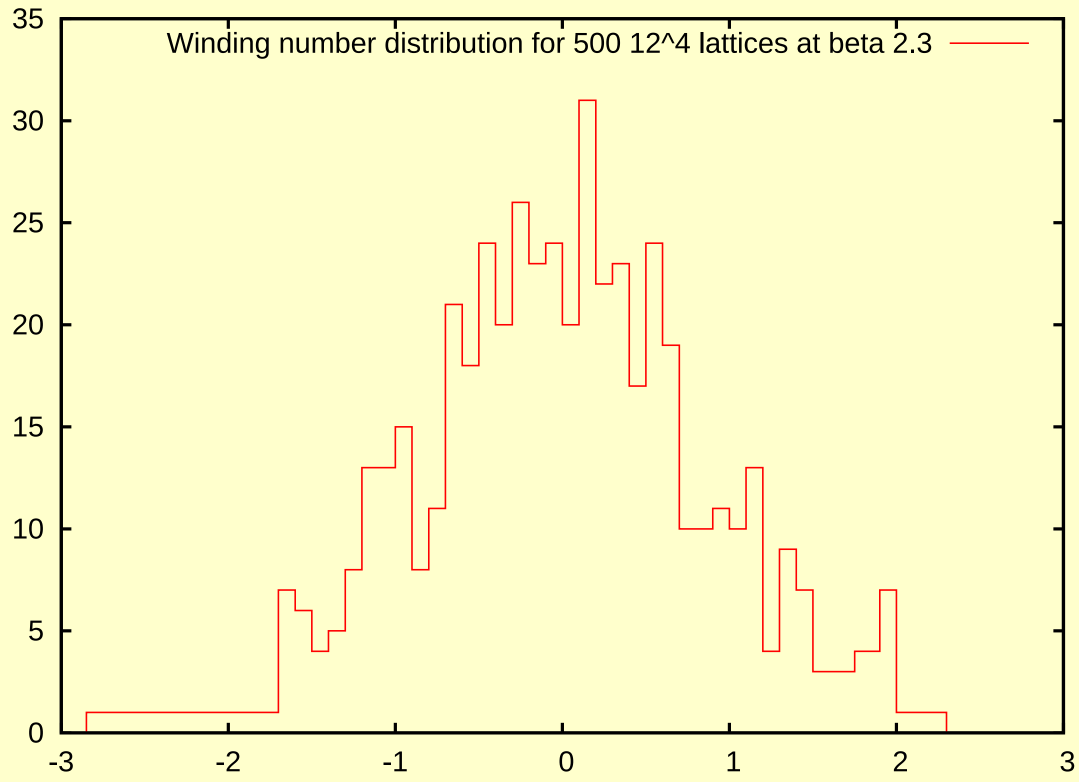
Space of lattice gauge fields is simply connected

Topology is lost at the outset

- small instantons can fall through the lattice

Combine loops around a hypercube to give $F\tilde{F}$ in the naive continuum limit

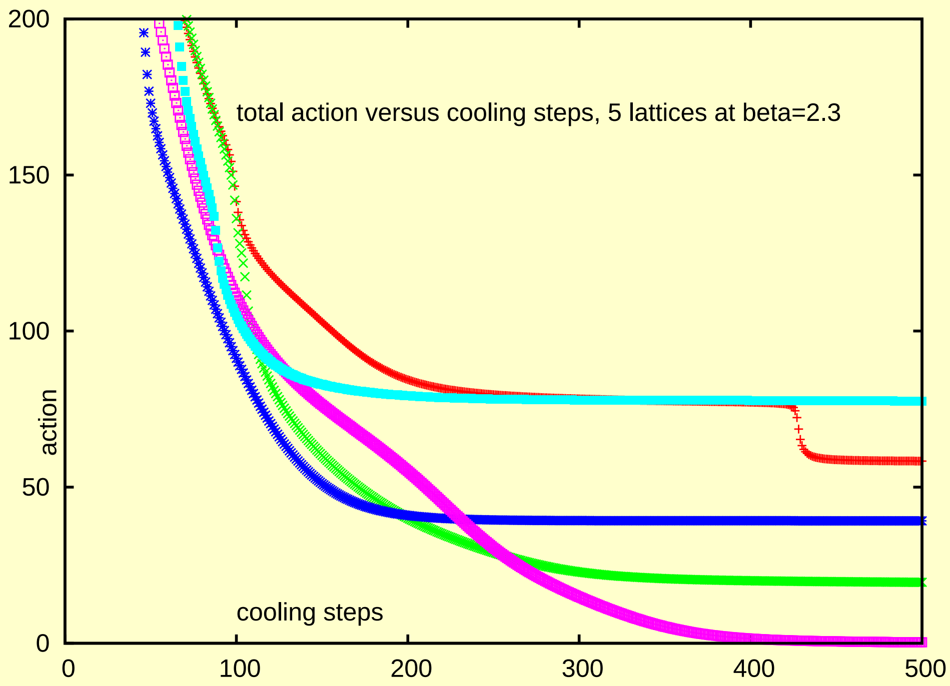
- resulting topological charge is not generally an integer



MC 2011

Cooling (Wilson flow, ...) can remove short distance fluctuations

Action settles to multiples of the classical instanton

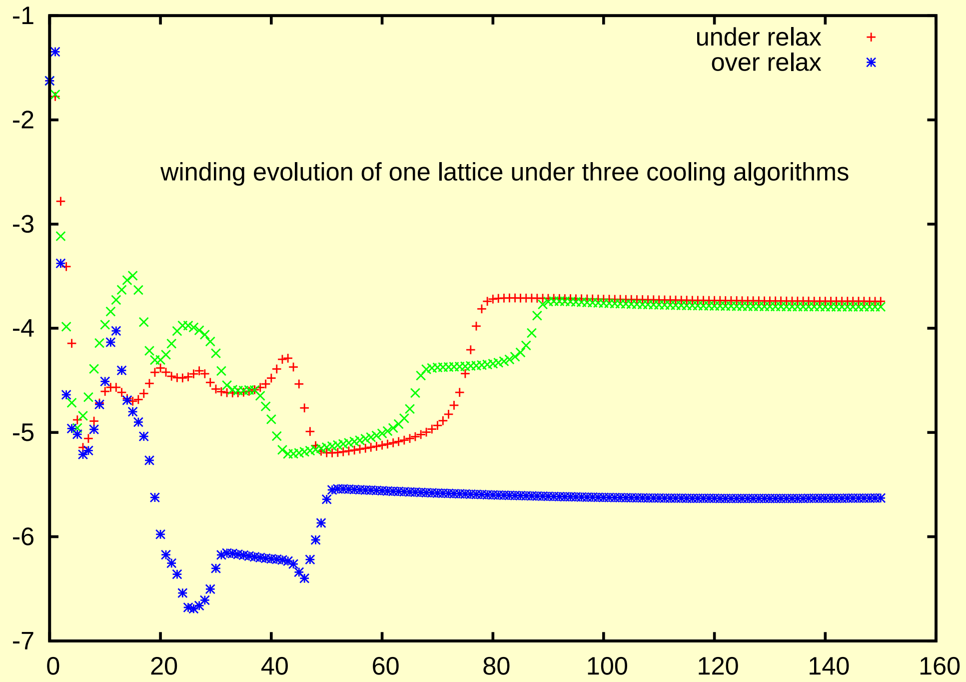


Many studies over the years:

- M. Teper 1985
- de Forcrand, Garcia Perez, Stamatescu 1997
- Del Debbio, Giusti, Pica 2005
- Bruckmann, Gruber, Jansen, Marinkovic, Urbach, Wagner 2010

Often stable but ambiguous cases do appear

Winding can depend on cooling algorithm



- with which action should we cool? How long?
- can small "instantons" fall through the lattice?

Admissibility condition: put a bound on how large a plaquette can be

- allows unique interpolation through hypercubes
- winding number becomes well defined

Luscher: if plaquettes restricted $P <\sim .03$

- unique continuum continuation of gauge fields
- instantons cannot collapse, unique winding number

The admissibility constraint requires a non-Hermitian Hamiltonian

- Hermitan H implies $\langle\psi|e^{-aH}|\psi\rangle > 0$ for every ψ
- requires plaquette weight to be analytic over the gauge group
- inconsistent with the admissibility constraint MC, 2001

Can we use the index theorem?

- count small real eigenvalues of the Wilson operator
  - At finite cutoff not exact zeros
  - How to define “small”?
  - does the eigenvalue density in the first Wilson “circle” go to zero?

Count zero modes of the overlap operator

- operator not unique; depends on chosen “domain wall height”
- relies on density of eigenvalues in the first Wilson “circle”

Should we care if topology is slightly ambiguous?

- topology is not directly measured in laboratory experiments
- concentrate on $M_{\eta'}$ which is physical
  - Witten-Veneziano formula is only a large N_c result
- tied to the $m_u = 0$ issue
  - already ambiguous in chiral perturbation theory

Summary

Topology in gauge theories is a rich topic

- Important consequences for the light hadron spectrum

Perturbation theory misses many issues

- mass mixing effects

Tied to several controversies

  - $m_u = 0$, rooting, topological susceptibility

Interesting phase structure with negative mass quarks

- possible pion condensation
- CP violation: useful in unified models?