Isospin breaking in pion and K_{e4} form factors

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Based on:

[S. Descotes-Genon, M. K., Eur. Phys. J. C 72, 1962 (2012)]

[V. Bernard, S. Descotes-Genon, M. K., Eur. Phys. J. C 72, 1962 (2012)]

OUTLINE

- Introduction - Motivation
- Two-loop representation of pion form factors with IB
- IB in the phases of the two-loop K_{e4} form factors
- Extraction of $\pi\pi$ scattering lengths a_0^0 and a_0^2
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Introduction - Motivation

Several processes carry information on $\pi\pi$ scattering lengths:
$K \to \pi\pi\pi$, pionic atoms, $K \to \pi\pi\ell\nu_\ell$ ($K_{\ell 4}$ decays),...

- Geneva-Saclay: $\sim 30\,000$ events

[L. Rosselet et al., Phys. Rev. D 15, 574 (1977)]

- BNL-E865: $\sim 400\,000$ events

[S. Pislak et al. (BNL-E865 Collaboration), Phys. Rev. Lett. 87, 221801 (2001)]

[Erratum-ibid. 105, 019901 (2010)]

[S. Pislak et al. (BNL-E865 Collaboration), Phys. Rev. D 67, 072004 (2003)]

[Erratum-ibid. D 81, 119903 (2010)]

- NA48/2: $\sim 1\,100\,000$ events

[J.R. Batley et al. (NA48/2 Collaboration), Eur. Phys. J. C 54, 411 (2008)]

[J.R. Batley et al. (NA48/2 Collaboration), Eur. Phys. J. C 70, 635 (2010)]

Standard angular analysis of the form factors provides information on low-energy $\pi\pi$ scattering (Watson's theorem) through the phase difference

$$[\delta_S(s) - \delta_P(s)]_{\text{exp}}$$

[N. Cabibbo, A. Maksymowicz, Phys. Rev. B 137, 438 (1965); Erratum-ibid 168, 1926 (1968)]

[F.A. Berends, A. Donnachie, G.C. Oades, Phys. Rev. 171, 1457(1968)]

Comparison with solutions of the Roy equations

$$[\delta_S(s) - \delta_P(s)]_{\text{exp}} = f_{\text{Roy}}(s; a_0^0, a_0^2)$$

allows to extract the values of the $\pi\pi$ S-wave scattering lengths in the isospin channels $I = 0, 2$

$f_{\text{Roy}}(s; a_0^2, a_0^2)$ follows from:

- dispersion relations (analyticity, unitarity, crossing, Froissard bound)
- $\pi\pi$ data at energies $\sqrt{s} \geq 1$ GeV
- isospin symmetry

[S.M. Roy, Phys. Lett. B 36, 353 (1971)]

Solutions can be constructed for $(a_0^0, a_0^2) \in$ Universal Band

[B. Ananthanarayan, G. Colangelo, J. Gasser, H. Leutwyler, Phys. Rep. 353, 207 (2001)]

Important to take isospin-breaking corrections ($M_\pi \neq M_{\pi^0}$) into account before analysing data

[J. Gasser, PoS KAON, 033 (2008), arXiv:0710.3048]

Evaluation of IB corrections in ChPT

[G. Colangelo, J. Gasser, A. Rusetsky, Eur. Phys. J. C 59, 777 (2009)]

$$\longrightarrow a_0^0 = 0.2220(128)_{\text{stat}}(50)_{\text{syst}}\,(37)_{\text{th}} \qquad a_0^2 = -0.0432(86)_{\text{stat}}(34)_{\text{syst}}(28)_{\text{th}}$$

Corrections evaluated at fixed values of scattering lengths

$$[\delta_S(s) - \delta_P(s)]_{\text{exp}} = f_{\text{Roy}}(s; a_0^0, a_0^2) + \delta f_{\text{IB}}(s; (a_0^2)_{\text{ChPT}}, (a_0^2)_{\text{ChPT}})$$

Drawback shared by other studies devoted to isospin breaking in ChPT

[V. Cuplov, PhD thesis (2004); V. Cuplov, A. Nehme, hep-ph/0311274]

[A. Nehme, Nucl. Phys. B 682, 289 (2004)]

[P. Stoffer, Eur. Phys. J. C 74, 2749 (2004)]

Is it possible to obtain

$$[\delta_S(s) - \delta_P(s)]_{\text{exp}} = f_{\text{Roy}}(s; a_0^2, a_0^2) + \delta f_{\text{IB}}(s; a_0^2, a_0^2)$$

Simple illustration of the problem

$$F^{+-}(s,t,u) = \frac{M_K}{\sqrt{2}F_\pi}\left[1 + \cdots + \frac{2s - M_\pi}{2F_\pi^2} J^r_{\pi\pi}(s) + \cdots\right]$$

[J. Bijnens, Nucl. Phys. B 337, 635 (1990)]
J. Bijnens, G. Colangelo, J. Gasser, Nucl. Phys. B 427, 427 (1994)]

Actually comes from

$$F^{+-}(s,t,u) = \frac{M_K}{\sqrt{2}F_\pi}\left[1 + \cdots + \frac{2s - 2\widehat{m}B_0}{2F_0^2} J^r_{\pi\pi}(s) + \cdots\right], \qquad [\widehat{m} = (m_u + m_d)/2]$$

together with the lowest order expressions $F_\pi = F_0$ and $M_\pi^2 = 2\widehat{m}B_0$

But there are other possibilities:

the $\pi\pi$ scattering lengths a_0^0 and a_0^2 are both proportional to $2\widehat{m}B_0$ at lowest order
there are infinitely many combinations of M_π^2, a_0^0 that sum up to $2\widehat{m}B_0$

What choice should one make in order to analyze data in terms of a_0^0 and a_0^2 ?

The missing link is provided by unitarity

$$F^{+-}(s,t,u) = \frac{M_K}{\sqrt{2}F_\pi}\left[1 + \cdots + \left(\frac{s - 4M_\pi^2}{F_0^2} + 16\pi a_0^0\right) J^r_{\pi\pi}(s) + \cdots\right]$$

Goal: obtain a representation for K_{e4} form factors that is

a) valid at two loops in the low-energy expansion

b) where the $\pi\pi$ scattering lengths occur as free parameters

c) with IB effects included

adapt the approach described in

[J. Stern, H. Sazdjian, N. H. Fuchs, Phys. Rev. D 47, 3814 (1993), arXiv:hep-ph/9301244]

[M. Knecht, B. Moussallam, J. Stern, N.H. Fuchs, Nucl. Phys. B 457, 513 (1995), arXiv:hep-ph/9507319]

(“reconstruction theorem”)

Two-loop representation
of pion form factors with IB

Description of the general method using the neutral and charged scalar
[$\widehat{m} \equiv (m_u + m_d)/2$]

$$\langle \pi^0(p_1)\pi^0(p_2)|\widehat{m}(\overline{u}u + \overline{d}d)(0)|\Omega\rangle = +F_S^{\pi^0}(s)$$

$$\langle \pi^+(p_+)\pi^-(p_-)|\widehat{m}(\overline{u}u + \overline{d}d)(0)|\Omega\rangle = -F_S^{\pi}(s),$$

and vector

$$\frac{1}{2}\langle \pi^+\pi^-|(\overline{u}\gamma_\mu u - \overline{d}\gamma_\mu d)(0)|\Omega\rangle = (p_- - p_+)_\mu F_V^{\pi}(s),$$

form factors

First ingredient: Dispersive representation of form factors

$$
\begin{aligned}
F_S^{\pi^0}(s) &= F_S^{\pi^0}(0)\left[1+\frac{1}{6}\langle r^2\rangle_S^{\pi^0} s + c_S^{\pi^0} s^2 + U_S^{\pi^0}(s)\right] \\
F_S^{\pi}(s) &= F_S^{\pi}(0)\left[1+\frac{1}{6}\langle r^2\rangle_S^{\pi} s + c_S^{\pi} s^2 + U_S^{\pi}(s)\right] \\
F_V^{\pi}(s) &= 1+\frac{1}{6}\langle r^2\rangle_V^{\pi} s + c_V^{\pi} s^2 + U_V^{\pi}(s).
\end{aligned}
$$

$$
\begin{aligned}
U_S^{\pi^0}(s) &= \frac{s^3}{\pi}\int \frac{dx}{x^3}\,\frac{\mathrm{Im}F_S^{\pi^0}(x)/F_S^{\pi^0}(0)}{x-s-i0} \\
U_S^{\pi}(s) &= \frac{s^3}{\pi}\int \frac{dx}{x^3}\,\frac{\mathrm{Im}F_S^{\pi}(x)/F_S^{\pi}(0)}{x-s-i0} \\
U_V^{\pi}(s) &= \frac{s^3}{\pi}\int \frac{dx}{x^3}\,\frac{\mathrm{Im}F_V^{\pi}(x)}{x-s-i0}\,.
\end{aligned}
$$

[J. Gasser, U.G. Meißner, Nucl. Phys. B 357, 90 (1991)]

Normalization given by Feynman – Hellmann theorem

$$
F_S^{\pi^0}(0) = \widehat{m}\,\frac{\partial M_{\pi^0}^2}{\partial \widehat{m}},\; F_S^{\pi}(0) = \widehat{m}\,\frac{\partial M_{\pi}^2}{\partial \widehat{m}}, \qquad \frac{F_S^{\pi}(0)}{F_S^{\pi^0}(0)} = 1 + \ldots .
$$

For the scattering amplitudes:

Fixed-t dispersion relations with three subtractions

$$A(s,t) = P(t|s,u) + \frac{s^3}{\pi}\int \frac{dx}{x^3}\frac{1}{x-s-i0}\,\mathrm{Im}_s A(x,t) + \frac{u^3}{\pi}\int \frac{dx}{x^3}\frac{1}{x-u-i0}\,\mathrm{Im}_u A(x,t)$$

Second ingredient: Partial wave expansion of $\pi\pi$ amplitudes

[and form factors in the $K_{\ell 4}$ case]

$$A(s,t) \;=\; 16\pi \sum_{\ell \geq 0} (2\ell+1) P_\ell(\cos\theta) f_\ell(s)$$

$$f_\ell(s) \;=\; \frac{1}{32\pi}\int_{-1}^{+1} dz A(s,t) P_\ell(z),$$

00	$\pi^0\pi^0 \to \pi^0\pi^0$
$++$	$\pi^+\pi^+ \to \pi^+\pi^+$
$+-$	$\pi^+\pi^- \to \pi^+\pi^-$
$+0$	$\pi^+\pi^0 \to \pi^+\pi^0$
x	$\pi^+\pi^- \to \pi^0\pi^0$

Crossing properties of $\pi\pi$ amplitudes [and form factors in the $K_{\ell 4}$ case]

Third ingredient: Chiral counting [E denotes a pion momentum or a pion mass]

$$\text{Re}F_S^{\pi(\pi^0)}(s) \sim \mathcal{O}(E^2), \qquad \text{Im}F_S^{\pi(\pi^0)}(s) \sim \mathcal{O}(E^4),$$
$$\text{Re}F_V^{\pi}(s) \sim \mathcal{O}(E^0), \qquad \text{Im}F_V^{\pi}(s) \sim \mathcal{O}(E^2)$$

$$\text{Re}f_\ell(s) \sim \mathcal{O}(E^2),\ \text{Im}f_\ell(s) \sim \mathcal{O}(E^4),\ \ell = 0, 1,$$
$$\text{Re}f_\ell(s) \sim \mathcal{O}(E^4),\ \text{Im}f_\ell(s) \sim \mathcal{O}(E^8),\ \ell \geq 2$$

$$\text{Re}f_\ell(s) = \underbrace{\varphi_\ell(s)}_{\sim\mathcal{O}(E^2)} + \underbrace{\psi_\ell(s)}_{\sim\mathcal{O}(E^4)} + \mathcal{O}(E^6)$$

$$|f_\ell(s)|^2 = [\text{Re}f_\ell(s)]^2 + \mathcal{O}(E^8) = [\varphi_\ell(s)]^2 + 2\varphi_\ell(s)\psi_\ell(s) + \mathcal{O}(E^8),\ \ell = 0, 1$$

Fourth ingredient: Analyticity and unitarity

[cut singularities and their discontinuities]

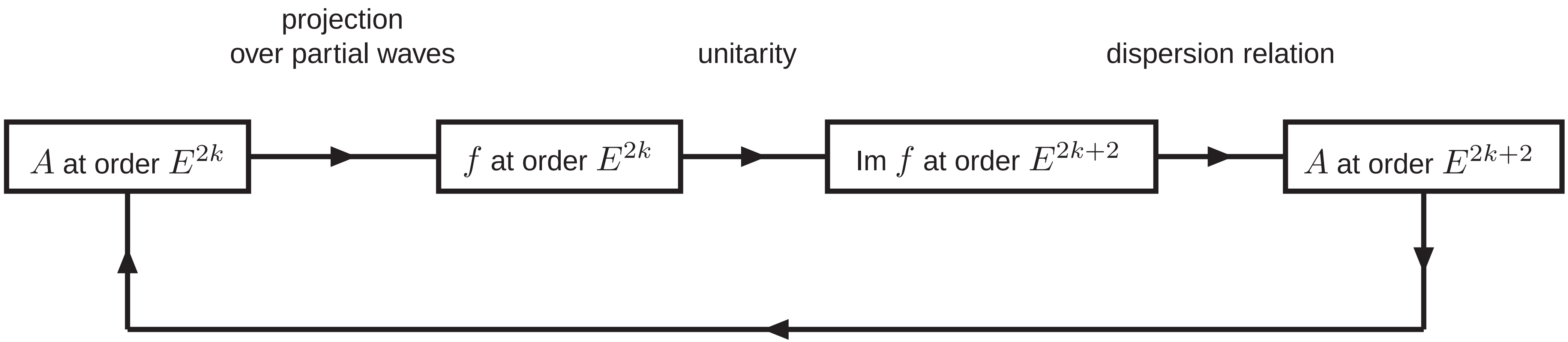
Absorptive parts are given by unitarity

In the low-energy region, only two-pion intermediate states occur up to two loops

$$\begin{aligned}
\text{Im}F_S^{\pi^0}(s) &= \text{Re}\left\{\frac{1}{2}\,\sigma_0(s)f_0^{00}(s)F_S^{\pi^0 *}(s)\theta(s-4M_{\pi^0}^2)-\sigma(s)f_0^{x}(s)F_S^{\pi *}(s)\theta(s-4M_{\pi}^2)\right\}+\mathcal{O}(E^8)\\
\text{Im}F_S^{\pi}(s) &= \text{Re}\left\{\sigma(s)f_0^{+-}(s)F_S^{\pi *}(s)\theta(s-4M_{\pi}^2)-\frac{1}{2}\,\sigma_0(s)f_0^{x}(s)F_S^{\pi^0 *}(s)\theta(s-4M_{\pi^0}^2)\right\}+\mathcal{O}(E^8)\\
\text{Im}F_V^{\pi}(s) &= \text{Re}\left\{\sigma(s)f_1^{+-}(s)F_V^{\pi *}(s)\theta(s-4M_{\pi}^2)\right\}+\mathcal{O}(E^6),\\
\text{Im}A(s,t) &= 16\pi\left[\text{Im}f_0(s)+3z\,\text{Im}f_1(s)\right]+\underbrace{\Phi_{\ell\geq 2}(s,t)}_{\sim\mathcal{O}(E^8)}
\end{aligned}$$

$$\sigma_0(s)=\sqrt{1-\frac{4M_{\pi^0}^2}{s}}\,,\;\sigma(s)=\sqrt{1-\frac{4M_{\pi}^2}{s}}.$$

$\longrightarrow$ Iterative two-step construction of two-loop representation for $\pi\pi$ amplitudes and form factors



Pion form factors at one loop

$$\mathrm{Im} F_S^{\pi^0}(s) = \frac{1}{2}\sigma_0(s)\varphi_0^{00}(s)F_S^{\pi^0}(0)\theta(s-4M_{\pi^0}^2) - \sigma(s)\varphi_0^x(s)F_S^{\pi}(0)\theta(s-4M_\pi^2) + \mathcal{O}(E^6)$$

$$\mathrm{Im} F_S^{\pi}(s) = \sigma(s)\varphi_0^{+-}(s)F_S^{\pi}(0)\theta(s-4M_\pi^2) - \frac{1}{2}\sigma_0(s)\varphi_0^x(s)F_S^{\pi^0}(0)\theta(s-4M_{\pi^0}^2) + \mathcal{O}(E^6)$$

$$\mathrm{Im} F_V^{\pi}(s) = \sigma(s)\varphi_1^{+-}(s)\theta(s-4M_\pi^2) + \mathcal{O}(E^4)$$

Requires $\pi\pi$ amplitudes at tree level:

$$A^{00}(s,t) = 16\pi a_{00}, \quad A^x(s,t) = 16\pi\left[a_x + b_x\frac{s-4M_\pi^2}{F_\pi^2}\right], \quad A^{+-}(s,t) = 16\pi\left[a_{+-} + b_{+-}\frac{s-4M_\pi^2}{F_\pi^2} + c_{+-}\frac{t-u}{F_\pi^2}\right]$$

and corresponding partial waves

$$\varphi_0^{00}(s) = a_{00}, \quad \varphi_0^x(s) = a_x + b_x\frac{s-4M_\pi^2}{F_\pi^2}, \quad \varphi_0^{+-}(s) = a_{+-} + b_{+-}\frac{s-4M_\pi^2}{F_\pi^2}, \quad \varphi_1^{+-}(s) = c_{+-}\frac{s-4M_\pi^2}{F_\pi^2}$$

$$F_S^{\pi^0}(s) = F_S^{\pi^0}(0)\left[1 + a_S^{\pi^0} s + 16\pi \frac{\varphi_0^{00}(s)}{2} \bar{J}_0(s)\right] - 16\pi F_S^{\pi}(0)\, \varphi_0^x(s)\, \bar{J}(s)$$

$$F_S^{\pi}(s) = F_S^{\pi}(0)\left[1 + a_S^{\pi}\, s + 16\pi \varphi_0^{+-}(s) \bar{J}(s)\right] - 16\pi F_S^{\pi^0}(0)\, \frac{1}{2}\, \varphi_0^x(s)\, \bar{J}_0(s)$$

$$F_V^{\pi}(s) = 1 + a_V^{\pi}\, s + 16\pi \varphi_1^{+-}(s) \bar{J}(s)$$

$$\bar{J}_0(s) = \frac{s}{16\pi^2} \int_{4M_{\pi^0}^2}^{\infty} \frac{dx}{x} \frac{1}{x - s - i0}\, \sigma_0(x) = \frac{-1}{16\pi^2} \int_0^1 dx \ln\left[1 - x(1-x)\frac{s}{M_{\pi^0}^2}\right]$$

$$\bar{J}(s) = \frac{s}{16\pi^2} \int_{4M_{\pi}^2}^{\infty} \frac{dx}{x} \frac{1}{x - s - i0}\, \sigma(x) = \frac{-1}{16\pi^2} \int_0^1 dx \ln\left[1 - x(1-x)\frac{s}{M_{\pi}^2}\right]$$

$$a_{+-} = \frac{2}{3}\, a_0^0 + \frac{1}{3}\, a_0^2 - 2a_0^2\, \frac{\Delta_\pi}{M_\pi^2}\,, \quad b_{+-} = c_{+-} = \frac{1}{24} \frac{F_\pi^2}{M_\pi^2} \left(2a_0^2 - 5a_0^2\right)\,,$$

$$a_x = -\frac{2}{3}\, a_0^0 + \frac{2}{3}\, a_0^2 + a_0^2\, \frac{\Delta_\pi}{M_\pi^2}\,, \quad b_x = -\frac{1}{12} \frac{F_\pi^2}{M_\pi^2} \left(2a_0^2 - 5a_0^2\right)\,,$$

$$a_{00} = \frac{2}{3}\, a_0^0 + \frac{4}{3}\, a_0^2 - \frac{2}{3} \left(a_0^0 + 2a_0^2\right) \frac{\Delta_\pi}{M_\pi^2} \qquad \Delta_\pi \equiv M_\pi^2 - M_{\pi^0}^2$$

$$A^{00}(s,t,u) \;=\; P^{00}(s,t,u) + W_0^{00}(s) + W_0^{00}(t) + W_0^{00}(u) + \mathcal{O}(E^8).$$

$$\mathrm{Im}W_0^{00}(s) = 16\pi\, \mathrm{Im} f_0^{00}(s)\, \theta(s-4M_{\pi^0}^2)$$

$$\frac{1}{16\pi}\,\mathrm{Im}W_0^{00}(s) = \frac{1}{2}\,\sigma_0(s)\,\left|f_0^{00}(s)\right|^2 \theta(s-4M_{\pi^0}^2) + \sigma(s)\,\left|f_0^{x}(s)\right|^2 \theta(s-4M_{\pi}^2) + \mathcal{O}(E^8)$$

Discontinuity at one-loop:

$$\frac{1}{16\pi}\,\mathrm{Im}W_0^{00}(s) = \frac{1}{2}\,\sigma_0(s)\,\left[\varphi_0^{00}(s)\right]^2 \theta(s-4M_{\pi^0}^2) + \sigma(s)\,\left[\varphi_x(s)\right]^2 \theta(s-4M_{\pi}^2) + \mathcal{O}(E^6)$$

one-loop amplitude

$$W_0^{00}(s) = \frac{1}{2}\left[16\pi\varphi_0^{00}(s)\right]^2 \bar{J}_0(s) + \left[16\pi\varphi_0^{x}(s)\right]^2 \bar{J}(s)$$

$P^{00}(s,t,u)$ polynomial of third order in s, t, u, symmetric under any permutation of its variables

$$P^{00}(s,t,u) \;=\; 16\pi a_{00} - w_{00} + \frac{3\lambda_{00}^{(1)}}{F_\pi^4}\left[s(s-4M_{\pi^0}^2) + t(t-4M_{\pi^0}^2) + u(u-4M_{\pi^0}^2)\right]$$

$$w_{00} \;=\; \mathrm{Re}\left[W_0^{00}(4M_{\pi^0}^2) + W_0^{00}(0) + W_0^{00}(0)\right]$$

$$\lambda_{00}^{(1)} = \frac{1}{3}\left(\lambda_1 + 2\lambda_2\right)$$

Two-loop representation of K_{e4} form factors with IB

$K \to \pi^a \pi^b \ell^+ \nu_\ell$

$(K, a, b) \in \{(K^+, +, -), (K^+, 0, 0), (K^0, 0, -)\}$

matrix elements involving the $\Delta S = \Delta Q = +1$ axial current

$$\langle \pi^a(p_a) \pi^b(p_b) | i A_\mu^{4-i5}(0) | K(k) \rangle$$

and the matrix elements related through crossing

$$\langle \pi^a(p_a) \bar{K}(k) | i A_\mu^{4-i5}(0) | \bar{\pi}^b(p_b) \rangle \qquad \langle \bar{K}(k) \pi^b(p_b) | i A_\mu^{4-i5}(0) | \bar{\pi}^a(p_a) \rangle$$

More generally

$$\mathcal{A}_\mu^{ab}(p_a, p_b; p_c) = \langle a(p_a)\, b(p_b) | i A_\mu(0) | \bar{c}(p_c) \rangle$$

$\{a, b, c\} = \{\pi^+, \pi^-, K^-\}, \{\pi^0, \pi^0, K^-\}$ or $\{\pi^0, \pi^-, \bar{K}_0\}$

$$\mathcal{A}_\mu^{ab}(p_a, p_b; p_c) = (p_a + p_b)_\mu F^{ab}(s, t, u) + (p_a - p_b)_\mu G^{ab}(s, t, u) + (p_c - p_a - p_b)_\mu R^{ab}(s, t, u)$$

“mass-shell” condition $s + t + u = M_a^2 + M_b^2 + M_c^2 + s_\ell \equiv \Sigma_\ell$, with $s_\ell \equiv (p_c - p_a - p_b)^2$

Partial-wave projections

$$\mathcal{F}^{ab}(s,t,u) = F^{ab}(s,t,u) + \left[\frac{M_a^2 - M_b^2}{s} + \frac{M_c^2 - s - s_\ell}{s}\,\frac{\lambda_{ab}^{\frac{1}{2}}(s)}{\lambda_{\ell c}^{\frac{1}{2}}(s)}\cos\theta_{ab}\right] G^{ab}(s,t,u),$$

$$\mathcal{G}^{ab}(s,t,u) = G^{ab}(s,t,u),$$

$$\mathcal{R}^{ab}(s,t,u) = R^{ab}(s,t,u) + \frac{M_c^2 - s - s_\ell}{2s_\ell} F^{ab}(s,t,u) + \frac{1}{2ss_\ell}\left[(M_a^2 - M_b^2)(M_c^2 - s - s_\ell) + \lambda_{ab}^{\frac{1}{2}}(s)\lambda_{\ell c}^{\frac{1}{2}}(s)\cos\theta_{ab}\right] G^{ab}(s,t,u)$$

$$\mathcal{F}^{ab}(s,t,u) = \sum_{l\geq 0} f_l^{ab}(s,s_\ell) P_l(\cos\theta_{ab}),$$

$$\mathcal{G}^{ab}(s,t,u) = \sum_{l\geq 1} g_l^{ab}(s,s_\ell) P_l'(\cos\theta_{ab}),$$

$$\mathcal{R}^{ab}(s,t,u) = \sum_{l\geq 0} r_l^{ab}(s,s_\ell) P_l(\cos\theta_{ab})$$

[F.A. Berends, A. Donnachie, G.C. Oades, Phys. Rev. 171, 1457 (1968)]

Chiral counting

$$\text{Re} f_0^{ab}(s,s_\ell),\ \text{Re} f_1^{ab}(s,s_\ell),\ \text{Re}\, g_1^{ab}(s,s_\ell) \sim \mathcal{O}(E^0) \qquad \text{Im} f_0^{ab}(s,s_\ell),\ \text{Im} f_1^{ab}(s,s_\ell),\ \text{Im}\, g_1^{ab}(s,s_\ell) \sim \mathcal{O}(E^2)$$

$$\text{Re} f_l^{ab}(s,s_\ell),\ \text{Re}\, g_l^{ab}(s,s_\ell) \sim \mathcal{O}(E^2),\ l \geq 2 \qquad \text{Im} f_l^{ab}(s,s_\ell),\ \text{Im}\, g_l^{ab}(s,s_\ell) \sim \mathcal{O}(E^6),\ l \geq 2$$

G. Colangelo, M. Knecht, J. Stern, Phys. Lett. B 336, 543 (1994), arXiv:hep-ph/9406211]

$$\begin{aligned} F^{ab}(s,t,u) &= F_S^{ab}(s,s_\ell) + F_P^{ab}(s,s_\ell)\cos\theta_{ab} + F_>^{ab}(s,\cos\theta_{ab},s_\ell) \\ G^{ab}(s,t,u) &= G_P^{ab}(s,s_\ell) + G_>^{ab}(s,\cos\theta_{ab},s_\ell) \end{aligned}$$

$$\text{Re}F_>^{ab}(s,\cos\theta_{ab},s_\ell),\ \text{Re}G_>^{ab}(s,\cos\theta_{ab},s_\ell) \sim \mathcal{O}(E^2)$$

$$\text{Im}F_>^{ab}(s,\cos\theta_{ab},s_\ell),\ \text{Im}G_>^{ab}(s,\cos\theta_{ab},s_\ell) \sim \mathcal{O}(E^6)$$

$$\begin{aligned} F_S^{ab}(s,s_\ell) &= f_0^{ab}(s,s_\ell) - \frac{M_a^2 - M_b^2}{s}\, g_1^{ab}(s,s_\ell), \\ F_P^{ab}(s,s_\ell) &= f_1^{ab}(s,s_\ell) - \frac{M_c^2 - s - s_\ell}{s}\, \frac{\lambda_{ab}^{\frac{1}{2}}(s)}{\lambda_{\ell c}^{\frac{1}{2}}(s)}\, g_1^{ab}(s,s_\ell)\,, \\ G_P^{ab}(s,s_\ell) &= g_1^{ab}(s,s_\ell) \end{aligned}$$

Analyticity, unitarity

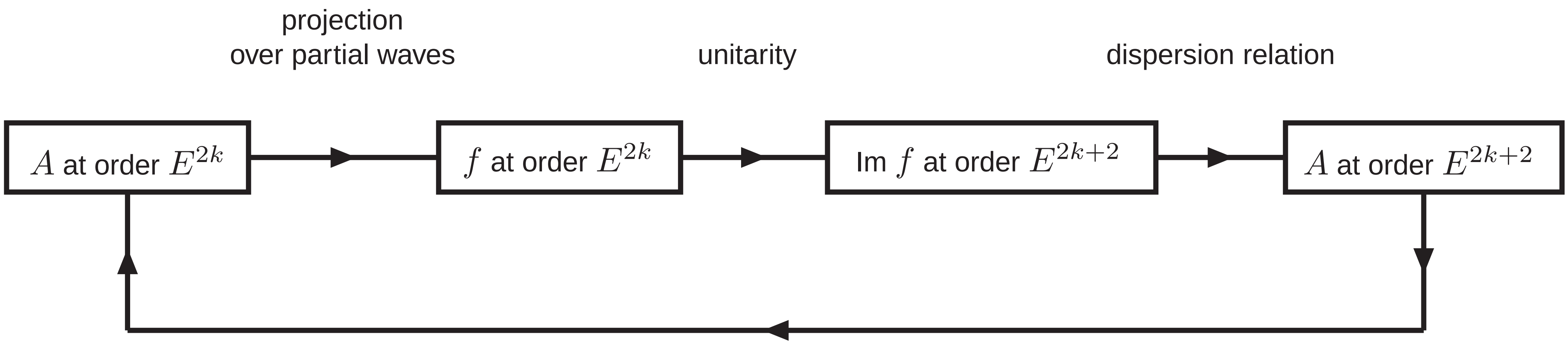
$$\text{Im}\, f_l^{ab}(s, s_\ell) = \sum_{\{a',b'\}} \frac{1}{\mathcal{S}_{a'b'}} \frac{\lambda_{a'b'}^{\frac{1}{2}}(s)}{s}\, \text{Re}\left\{ t_l^{a'b';ab}(s) \left[f_l^{a'b'}(s, s_\ell) \right]^\star \right\} \theta(s - s_{a'b'}) + \mathcal{O}(E^8),$$

$$\text{Im}\, g_l^{ab}(s, s_\ell) = \sum_{\{a',b'\}} \frac{1}{\mathcal{S}_{a'b'}} \frac{\lambda_{a'b'}^{\frac{1}{2}}(s)}{s} \frac{\lambda_{a'b'}^{\frac{1}{2}}(s)}{\lambda_{ab}^{\frac{1}{2}}(s)}\, \text{Re}\left\{ t_l^{a'b';ab}(s) \left[g_l^{a'b'}(s, s_\ell) \right]^\star \right\} \theta(s - s_{a'b'}) + \mathcal{O}(E^8)$$

mesonic scattering amplitudes $A^{a'b';ab}(s, \hat{t})$, $\hat{t} = (p_a - p_{a'})^2$

$$A^{a'b';ab}(s, \hat{t}) = 16\pi \sum_l (2l+1) t_l^{a'b';ab}(s) P_l(\cos\hat{\theta})$$

$\longrightarrow$ Iterative two-step construction of two-loop representation for meson scattering amplitudes and K_{e4} form factors



Extraction of $\pi\pi$ scattering lengths a_0^0 and a_0^2

Phases of the form factors

$$
\begin{aligned}
F(s,t,u) &= \widehat{F}_S(s,s_\ell)e^{i\delta_S(s,s_\ell)} + \widehat{F}_P(s,s_\ell)e^{i\delta_P(s,s_\ell)}\cos\theta + \text{Re}F_>(s,\cos\theta,s_\ell) + \mathcal{O}(E^6),\\
G(s,t,u) &= \widehat{G}_P(s,s_\ell)e^{i\delta_P(s,s_\ell)} + \text{Re}G_>(s,\cos\theta,s_\ell) + \mathcal{O}(E^6)
\end{aligned}
$$

$$
\text{Re}\,F_S(s,s_\ell) = F_{S[0]} + F_{S[2]}(s,s_\ell) + \mathcal{O}(E^4), \qquad \text{Re}\,G_P(s,s_\ell) = G_{P[0]} + G_{P[2]}(s,s_\ell) + \mathcal{O}(E^4)
$$

$$
\text{Re}\,t_l^{a'b';+-}(s) = \varphi_l^{a'b';+-}(s) + \psi_l^{a'b';+-}(s) + \mathcal{O}(E^6)
$$

$$
\delta_S(s,s_\ell) = \sum_{\{a',b'\}} \frac{1}{\mathcal{S}_{a'b'}} \frac{\lambda^{\frac{1}{2}}_{a'b'}(s)}{s} \left[\varphi_0^{a'b';+-}(s)\,\frac{F_{S[0]}^{a'b'} + F_{S[2]}^{a'b'}(s,s_\ell)}{F_{S[0]} + F_{S[2]}(s,s_\ell)} + \psi_0^{a'b';+-}(s)\,\frac{F_{S[0]}^{a'b'}}{F_{S[0]}} \right] \theta(s - s_{a'b'}) + \mathcal{O}(E^6)
$$

$$
\delta_P(s,s_\ell) = \sum_{\{a',b'\}} \frac{\lambda^{\frac{1}{2}}_{a'b'}(s)}{s} \frac{\lambda^{\frac{1}{2}}_{a'b'}(s)}{\lambda^{\frac{1}{2}}_{ab}(s)} \left[\varphi_1^{a'b';+-}(s)\,\frac{G_{P[0]}^{a'b'} + G_{P[2]}^{a'b'}(s,s_\ell)}{G_{P[0]} + G_{P[2]}(s,s_\ell)} + \psi_1^{a'b';+-}(s)\,\frac{G_{P[0]}^{a'b'}}{G_{P[0]}} \right] \theta(s - s_{a'b'}) + \mathcal{O}(E^6)
$$

IB in the phases of the form factors

$$\begin{aligned}
\delta_S(s, s_\ell) - \delta_0(s) &= \sigma(s) \left\{ \left[\varphi_0^{+-}(s) - \overset{o}{\varphi}{}_0^{+-}(s) \right] + \left[\psi_0^{+-}(s) - \overset{o}{\psi}{}_0^{+-}(s) \right] \right\} \\
&+ \frac{1}{2}\sigma_0(s) \left[\varphi_0^x(s) \frac{F_{S[0]}^{00} + F_{S[2]}^{00}(s, s_\ell)}{F_{S[0]}^{+-} + F_{S[2]}^{+-}(s, s_\ell)} + \psi_0^x(s) \frac{F_{S[0]}^{00}}{F_{S[0]}^{+-}} \right] \\
&+ \frac{1}{2}\sigma_0(s) \left[\overset{o}{\varphi}{}_0^x(s) + \overset{o}{\psi}{}_0^x(s) \right] + \mathcal{O}(E^6)
\end{aligned}$$

$$\delta_P(s) - \delta_1(s) = \sigma(s) \left\{ \left[\varphi_1^{+-}(s) - \overset{o}{\varphi}{}_1^{+-}(s) \right] + \left[\psi_1^{+-}(s) - \overset{o}{\psi}{}_1^{+-}(s) \right] \right\} + \mathcal{O}(E^6)$$

Note the dependence on s_ℓ in $\delta_S(s, s_\ell)$, resulting from IB effects

Numerically, it turns out to be negligible $\longrightarrow$ use $\delta_S(s) \equiv \delta_S(s, 0)$

Now we have

$$[\delta_S(s) - \delta_P(s)]_{\text{exp}} = f_{\text{Roy}}(s; a_0^2, a_0^2) + \delta f_{\text{IB}}(s; a_0^2, a_0^2)$$

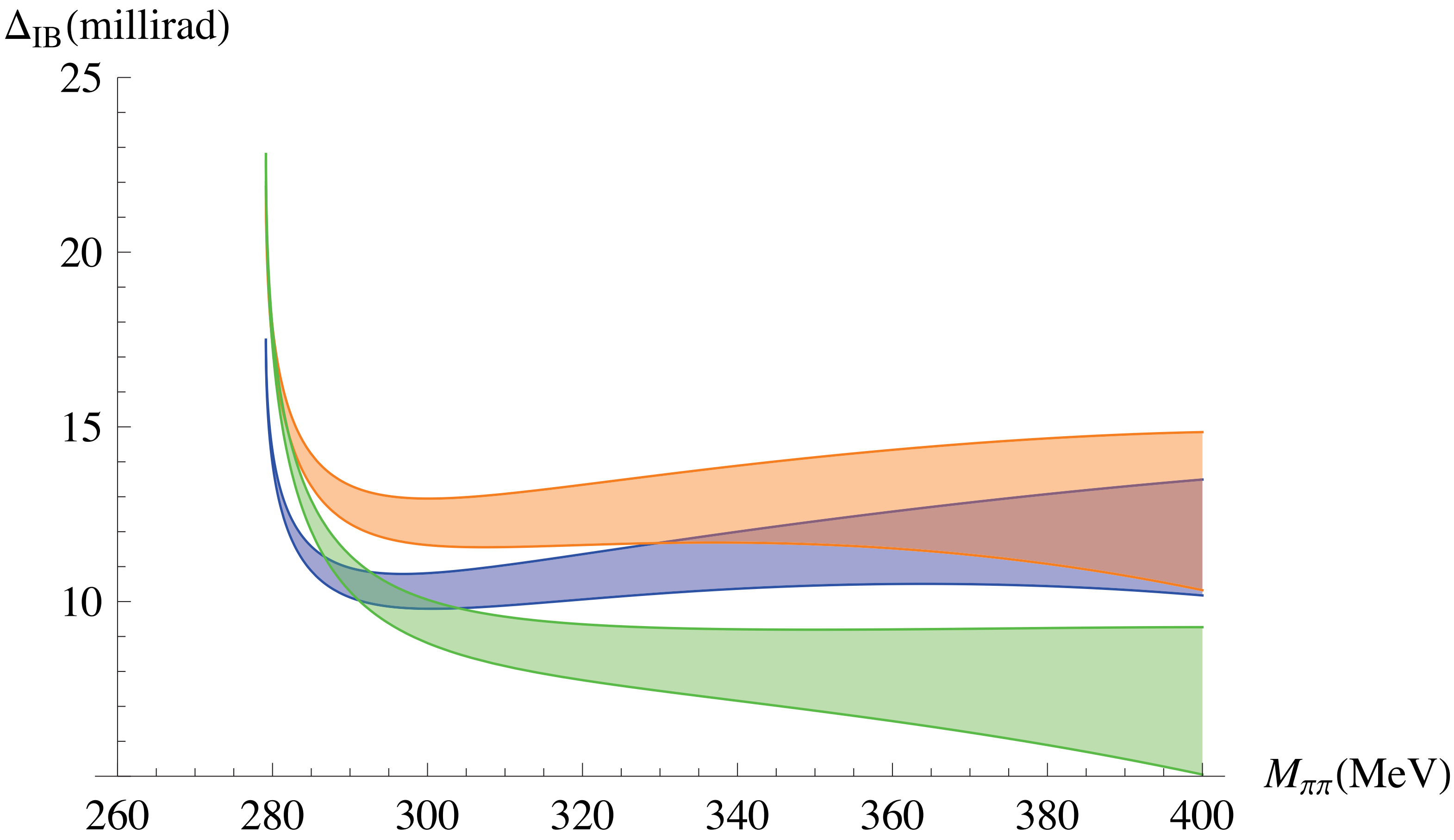


Figure 1: Isospin breaking in the phase of the two-loop form factors, $\Delta_{\rm IB}(s, s_\ell)$ as a function of the dipion invariant mass $M_{\pi\pi} = \sqrt{s}$, for $s_\ell = 0$. The middle (light-blue) band corresponds to the $(a_0^0, a_0^2) = (0.182, -0.052)$, whereas the other two cases shown correspond to $(a_0^0, a_0^2) = (0.205, -0.055)$ (upper orange band) and to $(a_0^0, a_0^2) = (0.24, -0.035)$ (lower green band). The widths of these bands result from the uncertainty on the various inputs needed at two loops.

Re-analysis of NA48/2 data

NA48/2 data alone provide a strong correlation between a_0^0 and a_0^2, but a weaker constraint on each of them separately

$\longrightarrow$ supply additional information, either from

- $I = 2$ data in S-wave (“extended fit”)

[S. Descotes-Genon, N.H. Fuchs, L. Girlanda, J. Stern, Eur. Phys. J. C 24, 469 (2002)]

- $N_f = 2$ ChPT and scalar radius of the pion

$$a_0^2 = -0.0444 + .236(a_0^0 - 0.22) - 0.61(a_0^0 - 0.22)^2 - 9.9(a_0^0 - 0.22)^3 \pm 0.0008$$

[G. Colangelo, J. Gasser, H. Leutwyler, Phys. Lett. B 488, 261 (2000)]

	With isospin-breaking corrections			Without isospin-breaking corrections		
	S-P	Extended	Scalar	S-P	Extended	Scalar
a_0^0	0.221 ± 0.018	0.232 ± 0.009	0.226 ± 0.007	0.247 ± 0.014	0.247 ± 0.008	0.242 ± 0.006
a_0^2	-0.0453 ± 0.0106	-0.0383 ± 0.0040	-0.0431 ± 0.0019	-0.0357 ± 0.0096	-0.0349 ± 0.0038	-0.0396 ± 0.0015
$\rho_{a_0^0, a_0^2}$	0.964	0.881	0.914	0.945	0.842	0.855
θ_0	$(82.3 \pm 3.4)^\circ$	$(82.3 \pm 3.4)^\circ$	82.3°	$(82.3 \pm 3.4)^\circ$	$(82.3 \pm 3.4)^\circ$	82.3°
θ_1	$(108.9 \pm 2)^\circ$	$(108.9 \pm 2)^\circ$	108.9°	$(108.9 \pm 2)^\circ$	$(108.9 \pm 2)^\circ$	108.9°
χ^2/N	7.6/6	16.6/16	7.8/8	7.2/6	15.7/16	7.3/8
α	1.043 ± 0.548	1.340 ± 0.231	1.179 ± 0.123	1.637 ± 0.472	1.672 ± 0.208	1.458 ± 0.098
β	1.124 ± 0.053	1.088 ± 0.020	1.116 ± 0.007	1.103 ± 0.055	1.098 ± 0.021	1.128 ± 0.008
$\rho_{\alpha\beta}$	0.47	0.31	0.02	0.47	0.32	0.00
$\lambda_1 \cdot 10^3$	-3.56 ± 0.68	-3.80 ± 0.58	-3.89 ± 0.10	-3.79 ± 0.68	-3.78 ± 0.57	-3.74 ± 0.11
$\lambda_2 \cdot 10^3$	9.08 ± 0.28	8.94 ± 0.10	9.14 ± 0.04	9.02 ± 0.23	9.02 ± 0.11	9.21 ± 0.42
$\lambda_3 \cdot 10^4$	2.38 ± 0.18	2.30 ± 0.14	2.32 ± 0.04	2.34 ± 0.18	2.34 ± 0.14	2.41 ± 3.67
$\lambda_4 \cdot 10^4$	-1.46 ± 0.10	-1.39 ± 0.04	-1.45 ± 0.02	-1.41 ± 0.10	-1.40 ± 0.04	-1.46 ± 0.02
$\bar{\ell}_3$	3.15 ± 9.9	-10.2 ± 5.7	-2.7 ± 6.6	-39.9 ± 20.3	-43.5 ± 19.1	-19.6 ± 7.8
$\bar{\ell}_4$	5.3 ± 0.8	4.4 ± 0.6	5.1 ± 0.3	5.2 ± 0.8	5.2 ± 0.7	6.0 ± 0.4
$X(2)$	0.88 ± 0.05	0.80 ± 0.06	0.82 ± 0.02	0.72 ± 0.05	0.71 ± 0.05	0.75 ± 0.03
$Z(2)$	0.87 ± 0.03	0.89 ± 0.02	0.86 ± 0.01	0.87 ± 0.02	0.87 ± 0.02	0.85 ± 0.01

Table 1: Scattering lengths, subthreshold parameters and chiral low-energy constants for the different fits considered, with and without the isospin-breaking correction Δ_{IB}.

$$\longrightarrow a_0^0 = 0.222(13) \qquad a_0^2 = -0.043(9)$$

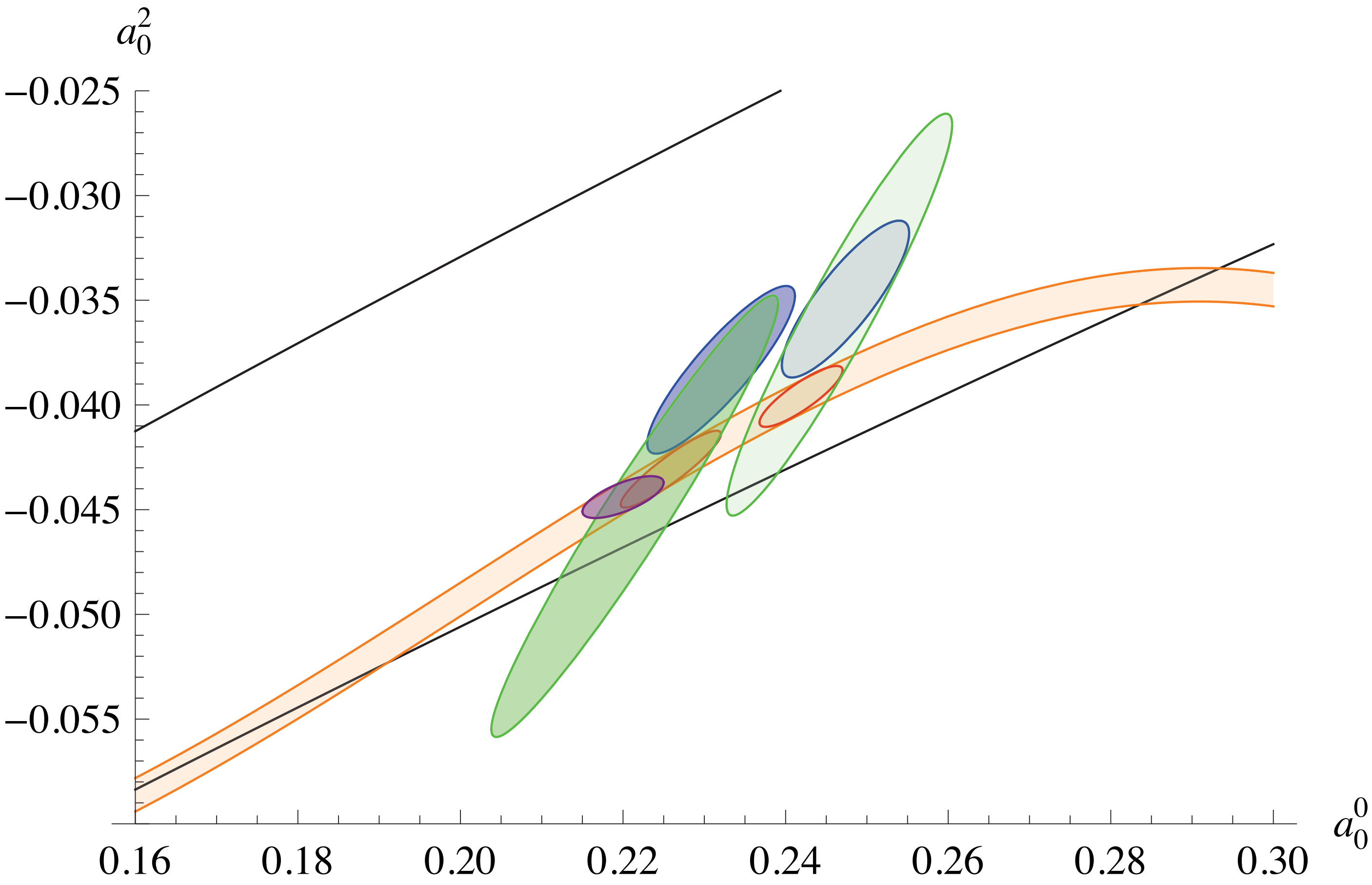


Figure 2: Results of the fits to the NA48/2 data in the (a_0^0, a_0^2) plane. The two black solid lines indicate the universal band where the two S-wave scattering lengths comply with dispersive constraints (Roy equations) and high-energy data on $\pi\pi$ scattering. The orange band is the constraint coming from the scalar radius of the pion. The small dark (purple) ellipse represents the prediction based on $N_f = 2$ chiral perturbation theory. The three other ellipses on the left represent, in order of increasing sizes, the 1-σ ellipses corresponding to the scalar (orange ellipse), S-P (blue ellipse) and extended (green ellipse), respectively, when isospin-breaking corrections are included. The light-shaded ellipses on the right represent the same outputs, but obtained without including isopin-breaking corrections.

Summary - Conclusion

• The high-precision data for $\delta_S(s) - \delta_P(s)$ obtained by the NA48/2 experiment require that isospin-breaking corrections be included

• Since the ultimate goal is to extract a_0^0 and a_0^2, the $\pi\pi$ scattering lengths in the isospin limit, the corrections should not be computed at fixed values of the scattering lengths, but should be parametrized in terms of them

• General properties (analyticity, unitarity, crossing, chiral counting) provide the necessary information to do this in a model independent way

$$[\delta_S(s) - \delta_P(s)]_{\text{exp}} = f_{\text{Roy}}(s; a_0^2, a_0^2) + \delta f_{\text{IB}}(s; a_0^2, a_0^2)$$

with $\delta f_{\text{IB}}(s; a_0^2, a_0^2)$ worked out at NLO

• Fit to NA48/2 data have been redone. Results compatible with those published by NA48/2 within errors

• General set-up can of use and implemented in other cases, e. g. $K_{e4}^{\pm}(\pi^0\pi^0)$

[V. Bernard, S. Descotes-Genon, M. K., work in progress]