# The effects of geometry on longitudinal space charge impedance

ICFA Mini-Workshop on "Electromagnetic wake fields and impedances in particle accelerators"

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  *in free space and in rectangular chambers*
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# Motivation

- **LSC in long wavelength limit has been intensively studied, but not in full wavelength**

- **The analysis of LSC is done for simple geometry: both beam and beam pipe**

- **The goal is to calculate LSC for arbitrary geometry (both beam and beam pipe) and in full wavelength**

# How LSC affects beam

$$Z_{||}(k) = -\frac{1}{I}\int_0^{\infty} \boldsymbol{E_z} dz$$

**Density modulation along the bunch**

$$\Delta Ez = I Z_{||}$$

Energy modulation

β<<1

$$\rho(x,y,z,t) = \lambda_z \rho_{\perp}(x,y) e^{-i\omega(t-z/\upsilon)},$$

$$\boldsymbol{J} = \rho\upsilon = \hat{\mathbf{z}}\beta c \lambda_z \rho_{\perp}(x,y) e^{-i\omega(t-z/\upsilon)},$$

$$I = \lambda_z \upsilon e^{-i\omega(t-z/\upsilon)} = \bar{I} e^{-i\omega(t-z/\upsilon)},$$

$$\mathbf{E_z}(x,y,z,t) = \hat{\mathbf{z}} \mathrm{E_z}(x,y) e^{-i\omega(t-z/\upsilon)}$$

$$\nabla^2 \boldsymbol{E} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} = \frac{\nabla\rho}{\varepsilon_0} + \mu_0 \frac{\partial \boldsymbol{J}}{\partial t}$$

# Small Isochronous Ring(SIR) of NSCL/MSU

[1]. E. Pozdeyev et al, PRST-AB 12, 054202, 2009

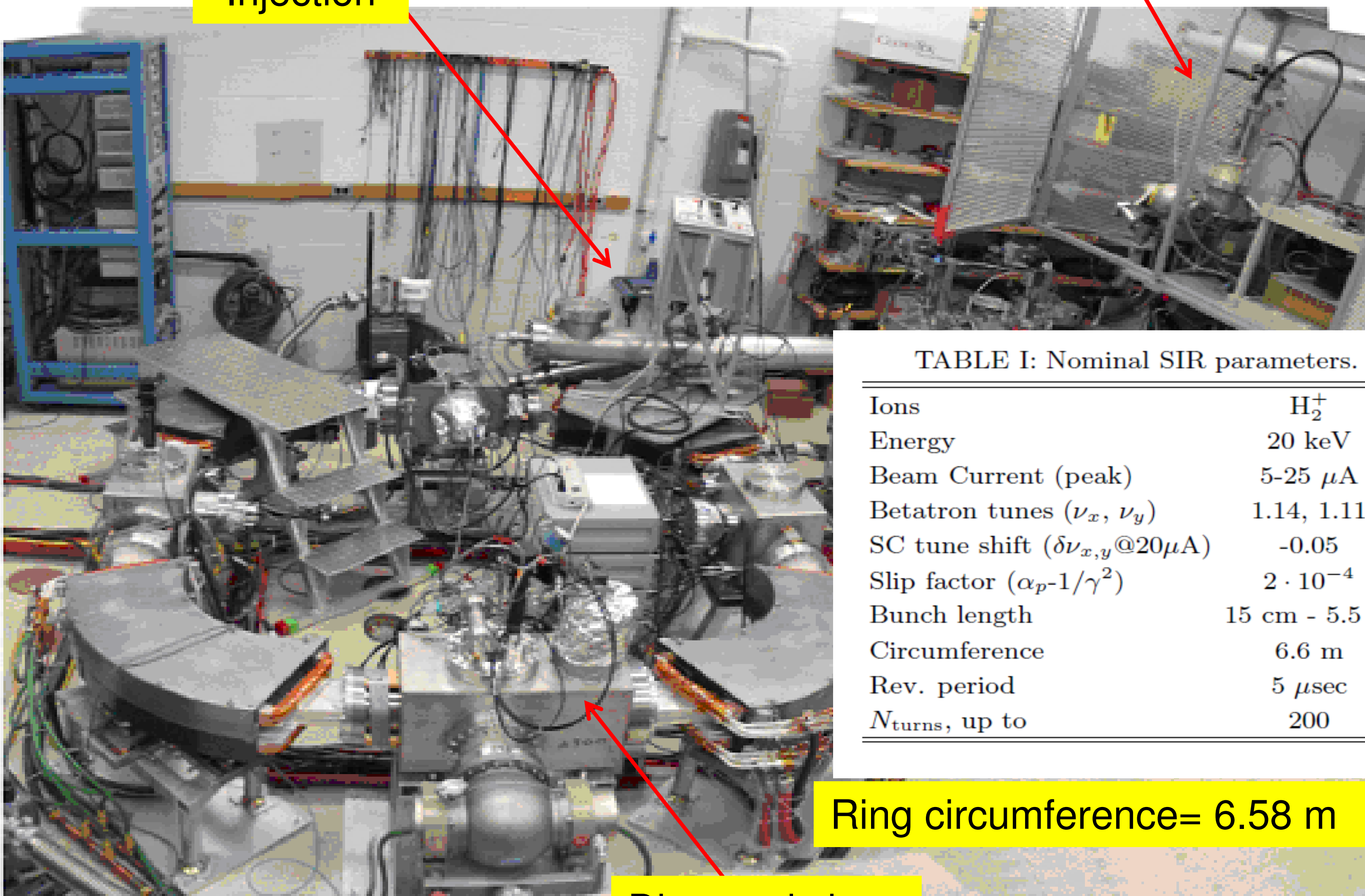
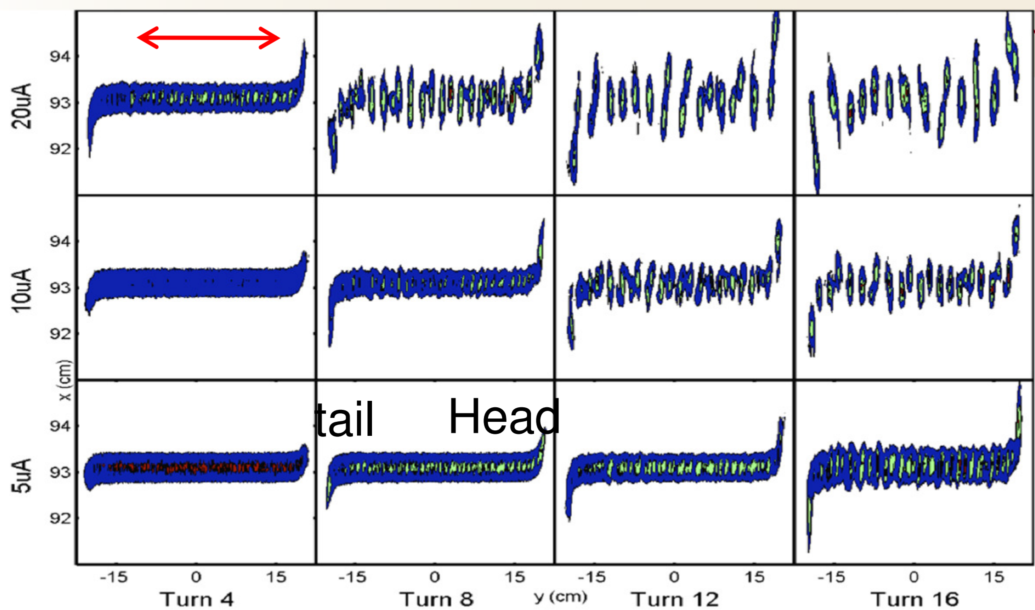



TABLE I: Nominal SIR parameters.

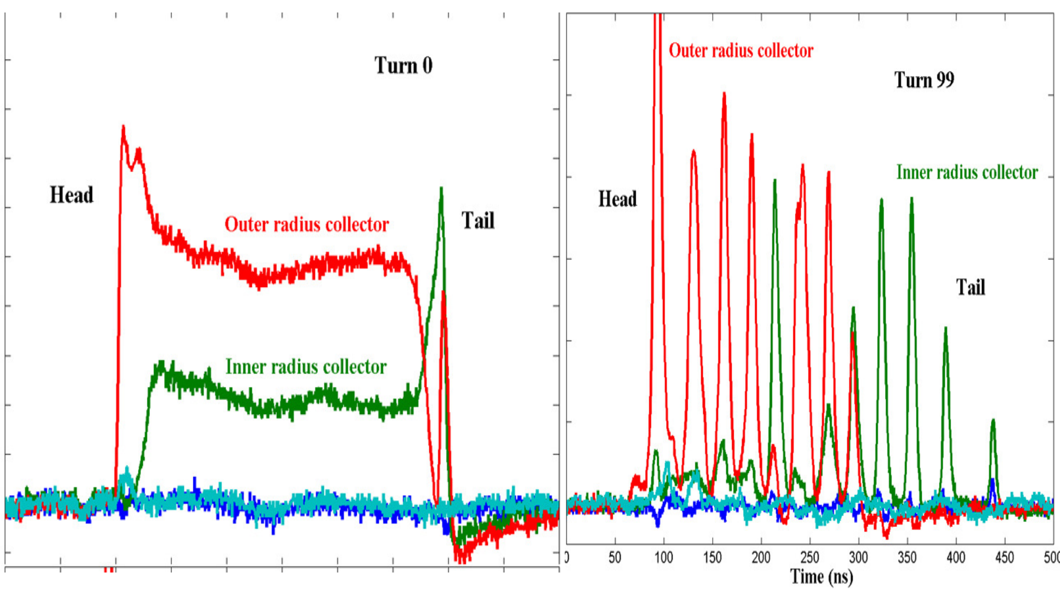
| | |
|---|---|
| Ions | $H_2^+$ |
| Energy | 20 keV |
| Beam Current (peak) | 5-25 $\mu$A |
| Betatron tunes ($\nu_x$, $\nu_y$) | 1.14, 1.11 |
| SC tune shift ($\delta\nu_{x,y}$@20$\mu$A) | -0.05 |
| Slip factor ($\alpha_p$-1/$\gamma^2$) | $2 \cdot 10^{-4}$ |
| Bunch length | 15 cm - 5.5 m |
| Circumference | 6.6 m |
| Rev. period | 5 $\mu$sec |
| $N_{\text{turns}}$, up to | 200 |

# Beam Instability of SIR




Simulations of beam breakup by CYCO [1] shows the onset of the instability and its dependence on the beam current.

Bunch length ~35cm




Experimental results obtained at SIR showing the breakup of the beam[2]

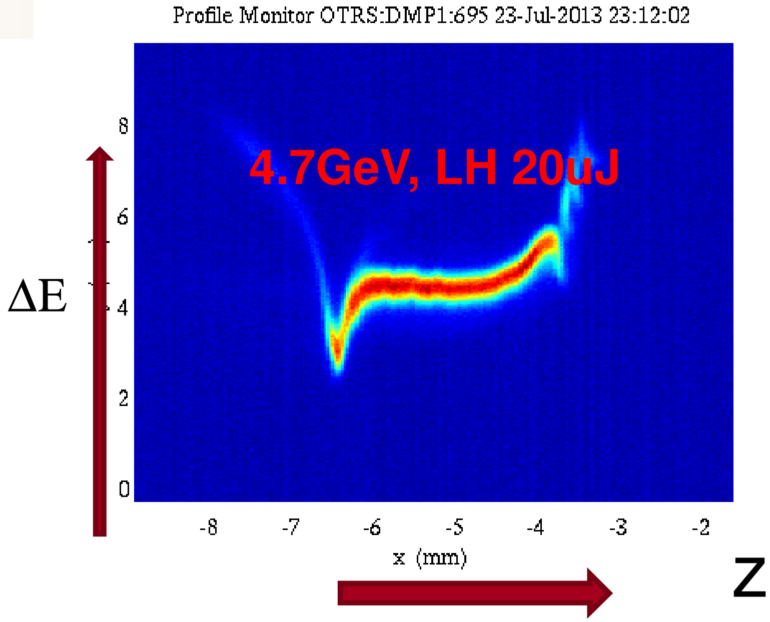

[1]E.Pozdeyev, Ph.D thesis, Michigan State University(2003)

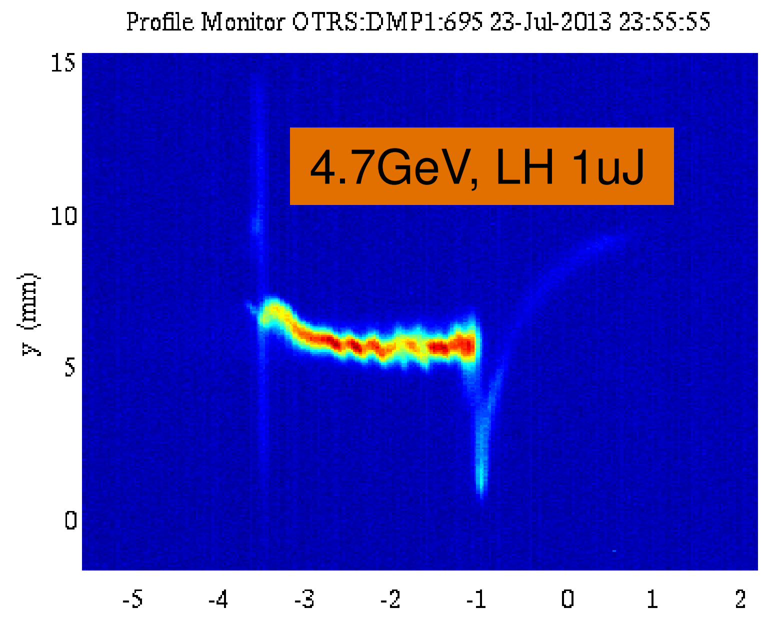
[2] J.A.Rodriguez, Ph.D thesis, Michigan State University(2004)

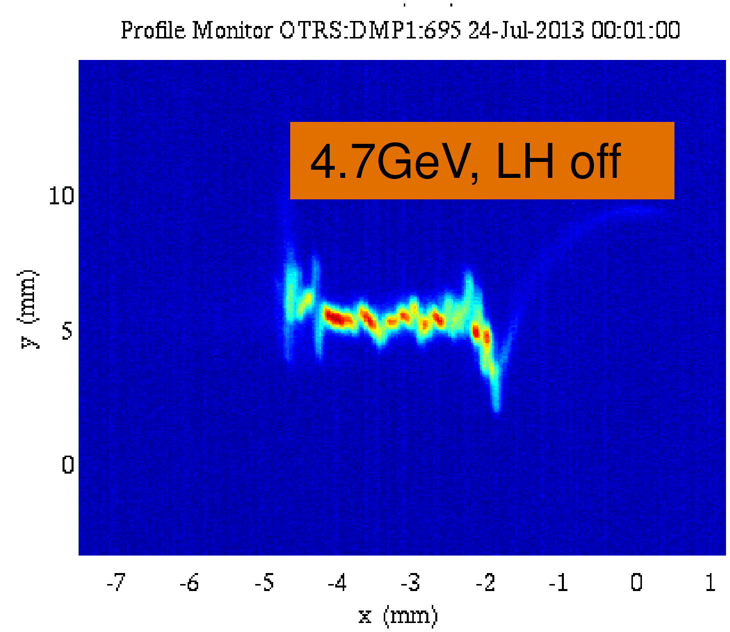
# XTcav measured micro-bunching images in LCLS

150pC, peak current ~1.2kA-2kA, ~25um bunch length

(Yuntao Ding)

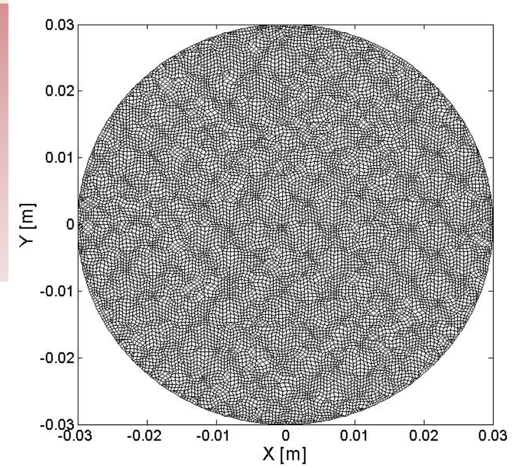


# FEM (Finite Element Method) for LSC

- Arbitrary geometry of beam pipes
- Any shape of beam
- Parallel computation




Large beam

FEM equation

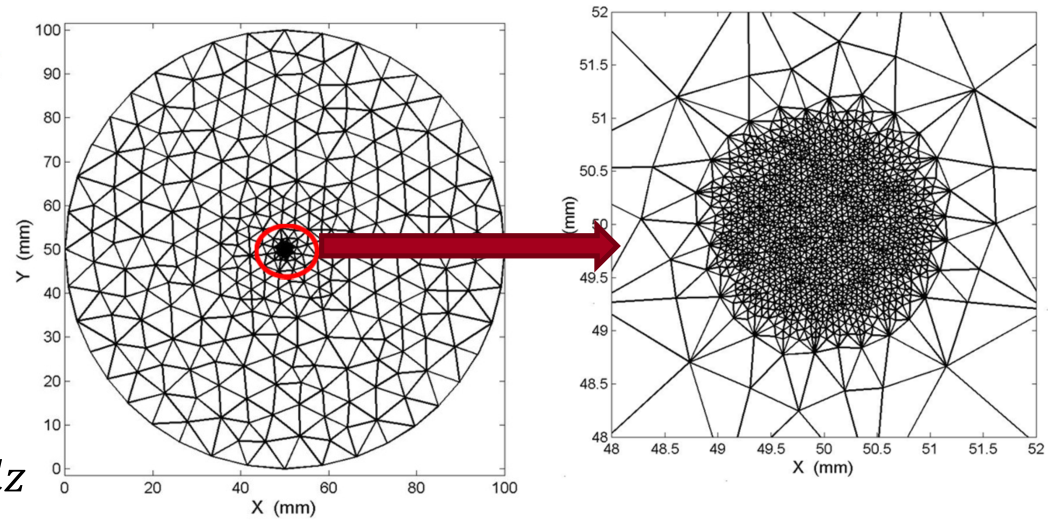
$$\left(\mathbf{M} + \frac{k^2}{\gamma^2}\mathbf{B}\right)\mathbf{E} = \mathbf{Q}$$

$$M_{ij}^e = \iint_S \left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right) dxdy$$

$$B_{ij}^e = \iint_S N_i N_j \, dxdy$$

$$Q_i^e = -i\frac{kq_i}{\varepsilon_0\gamma^2}$$

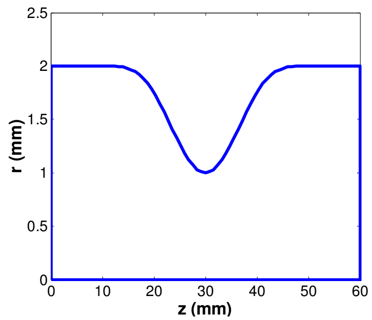
$$Z_{||}(k) = -\frac{1}{\mathrm{I}}\int_0^\infty \boldsymbol{E_z} dz = -\frac{1}{\lambda_z \beta c}\int_0^\infty E_z dz$$



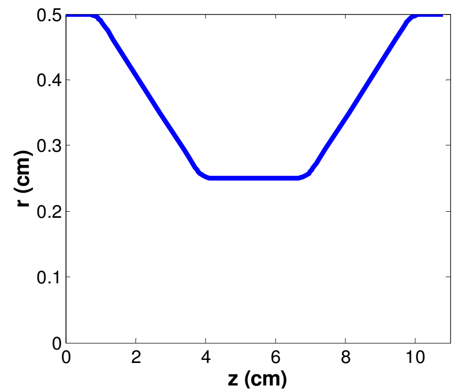

Small beam

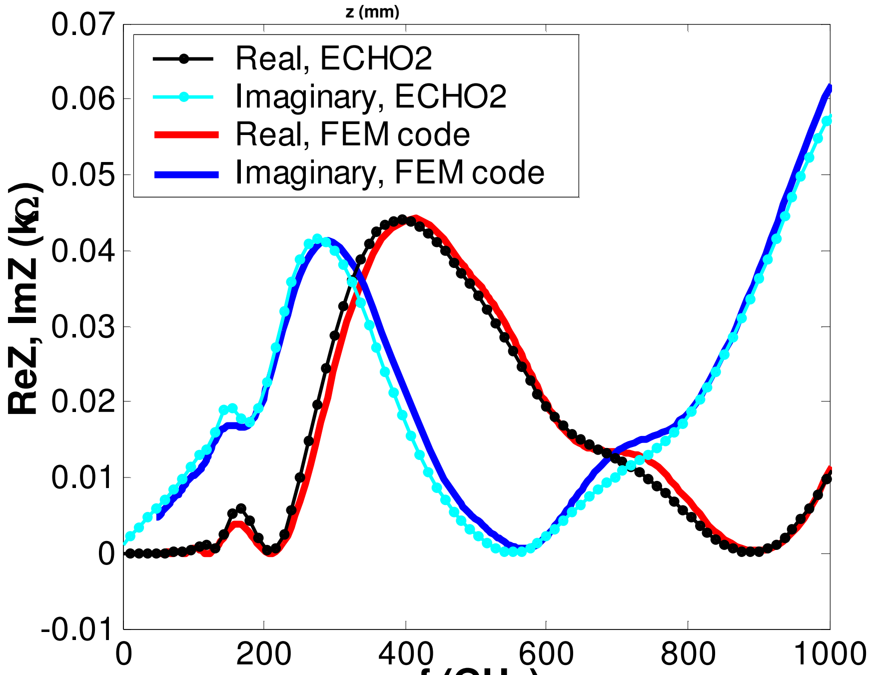
# 2D parabolic solver for calculating impedance

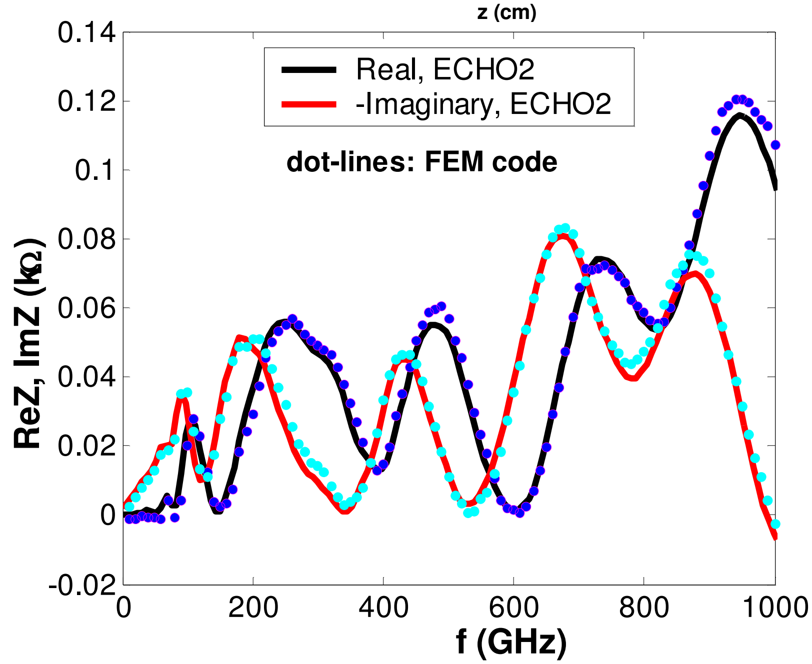
- **L. Wang, L. Lee, G. Stupakov,** *fast 2D solver* **(IPAC10)**




Echo2, sigl=0.1mm



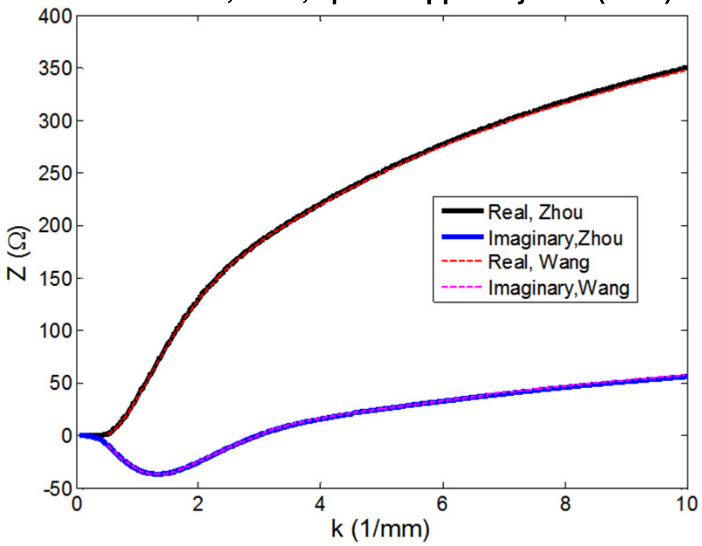



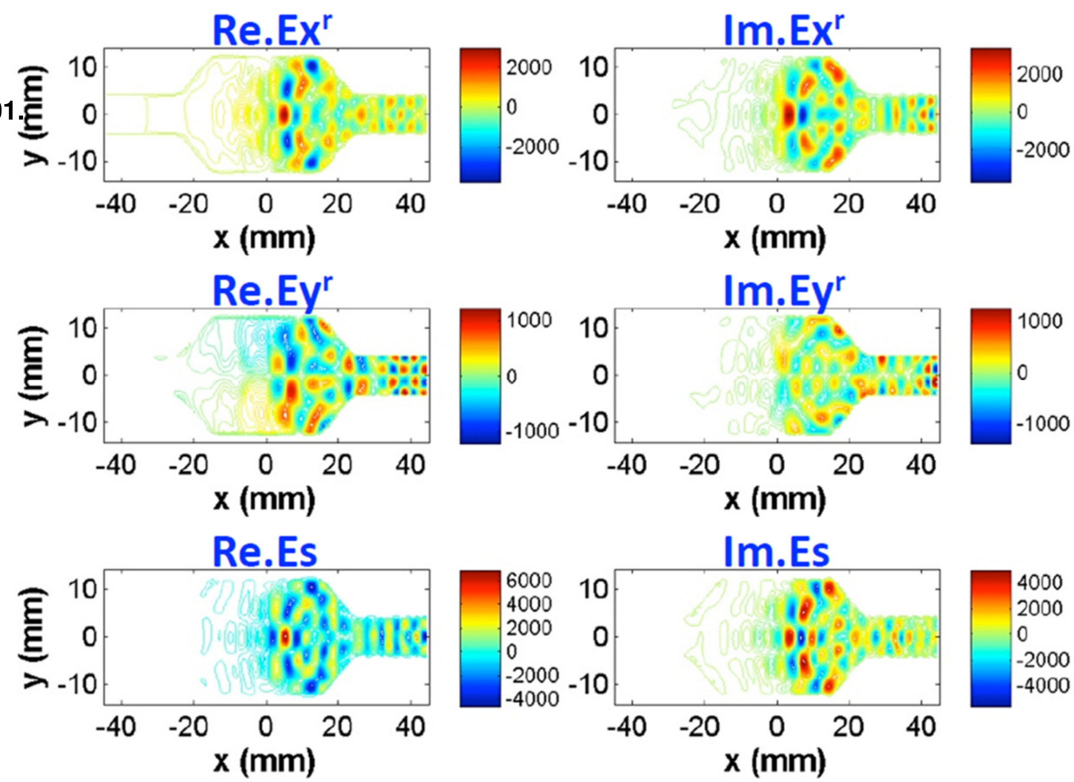
# New FEM CSR code with an arbitrary cross-section of the pipe

$$\left(\left[\nabla_\perp^2 + \frac{2k^2 x}{R}\right] + 2ki\frac{\partial}{\partial s}\right)\tilde{\mathbf{E}}_\perp = -\frac{e}{\varepsilon_0}\nabla_\perp n_0$$

Agoh, Yokoya, PRSTAB 054403,2004
G. Stupakov, PRSTAB 104401, 2009
K. Oide, PAC09
D. Zhou, JJAP (2012) 016401.

CSRZ: **D. Zhou, et al., Jpn. J. Appl. Phys. 51 (2012) 016401.**



# LSC of round uniform beams in free space

(J.L. Laclare, CERN Accelerator School, CERN 85-19, p. 381)

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Round uniform beam

$$\frac{Z_{||}^{rd,free}(k)}{L} = -\frac{E_z}{\bar{I}} = -\frac{E_z}{\lambda_z \beta c} = i\frac{1}{k\pi a^2 \varepsilon_0 \beta c}\left[1 - \frac{ka}{\gamma}K_1\left(\frac{ka}{\gamma}\right)\right]$$

Long wavelength limit : $\gamma$/(ka)>4

$$\frac{Z_{||}^{rd,free,small\,k}(k)}{L} \approx -i\frac{k}{4\pi\varepsilon_0 \beta c\gamma^2}\left[2\ln\left(\frac{ka}{2\gamma}\right) + 2C - 1\right]$$

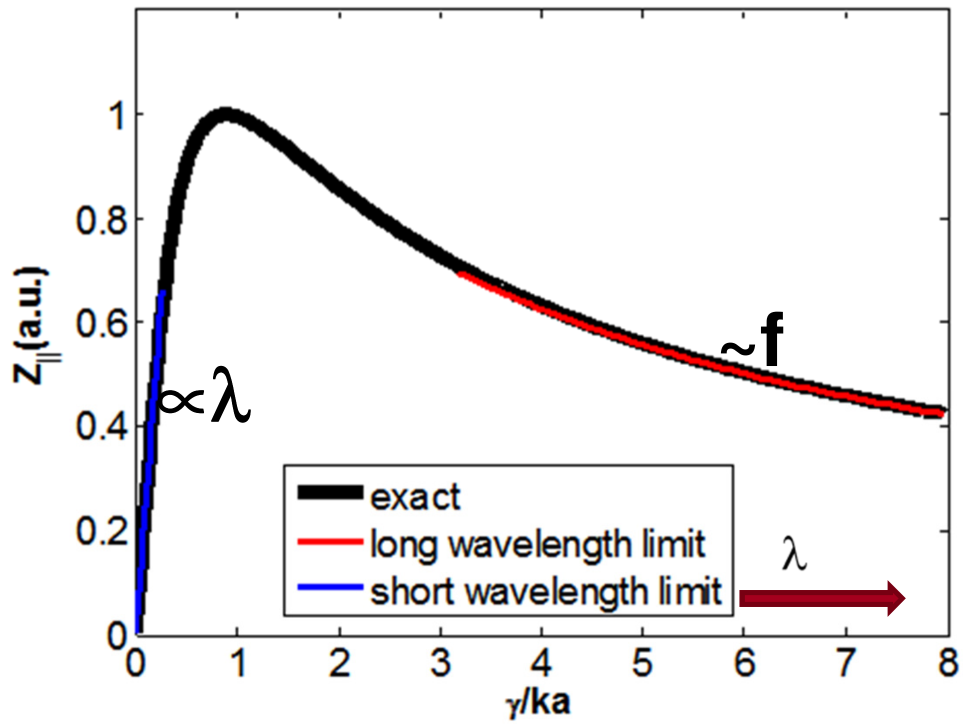
Short wavelength limit $\gamma$/ka<0.2

$$\frac{Z_{||}^{rd,free,high\,k}(k)}{L} \approx i\frac{1}{k\varepsilon_0 \beta c}\frac{1}{\pi a^2}$$

$$C = 0.577216$$

Simple rule for u-instability wavelength

**ka/ γ =1**



# LSC of Gaussian beams in free space

Gaussian beam

$$\frac{Z_{||}^{gau,free}(k)}{L} = i\frac{1}{4\pi\varepsilon_0}\frac{k}{\gamma^2\beta c}e^{-\frac{k^2\sigma_r^2}{2\gamma^2}}Ei(-\frac{k^2\sigma_r^2}{2\gamma^2})$$

Long wavelength limit $\gamma/(ka)>6$

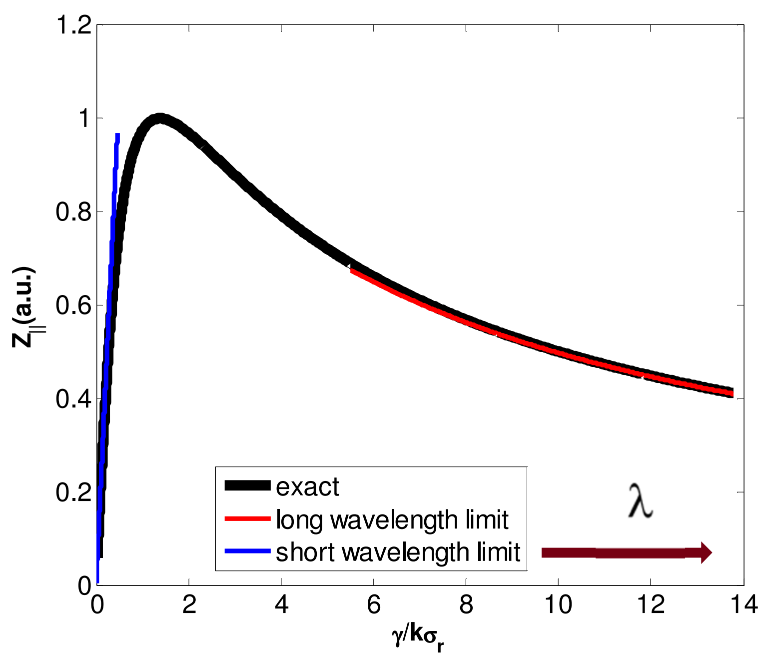
$$\frac{Z_{||}^{gau,free,small\,k}(k)}{L} \approx -i\frac{k}{4\pi\varepsilon_0\beta c\gamma^2}[C + 2\ln\left(\frac{k\sigma_r}{\sqrt{2}\gamma}\right)]$$

Short wave length limit $\gamma/(ka)<0.2$

$$\frac{Z_{||}^{gau,free,high\,k}(k)}{L} \approx i\frac{1}{k\varepsilon_0\beta c}\frac{1}{2\pi\sigma_r^2}$$

Where $E_i$ is the exponential integral function

**example: LCLS: laser heater**
**λ=758nm γ=264,a=64μm, γ/(ka)=0.3**



# Comparison of round uniform and Gaussian beams in free space

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Round uniform beam

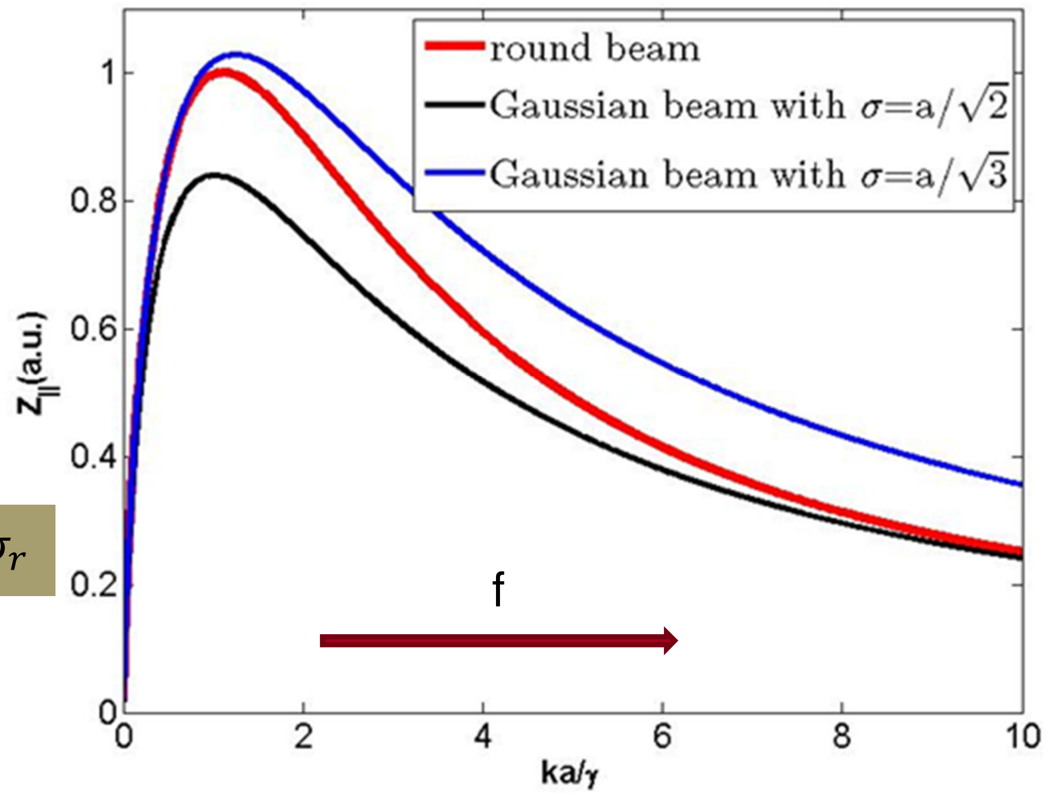
$$\frac{Z_{||}^{rd,free}(k)}{L} = -\frac{E_z}{\bar{I}} = -\frac{E_z}{\lambda_z \beta c} = i\frac{1}{k\pi a^2 \varepsilon_0 \beta c}\left[1 - \frac{ka}{\gamma} K_1\left(\frac{ka}{\gamma}\right)\right]$$

Gaussian beam

$$\frac{Z_{||}^{gau,free}(k)}{L} = i\frac{1}{4\pi\varepsilon_0}\frac{k}{\gamma^2 \beta c} e^{-\frac{k^2\sigma_r^2}{2\gamma^2}} Ei\left(-\frac{k^2\sigma_r^2}{2\gamma^2}\right)$$

In a low frequency regime with $ka/\gamma < 1$ we have $a_{eff} = \sqrt{3}\sigma_r$ while $a_{eff} = \sqrt{2}\sigma_r$ in the high frequency regime with $ka > 10$

$exactly$: $a_{eff} = 1.747\sigma_r$




For instance:

# LSC of a rectangular beam in a rectangular chamber

(NIMA747_30_2014, Yingjie Li and Lanfa Wang)

chamber full height=2h
full width=2w
Beam full height=2b
full width=2a

**Variable separation method**

$$E_h = X(x)Y(y)\cos(kz).$$

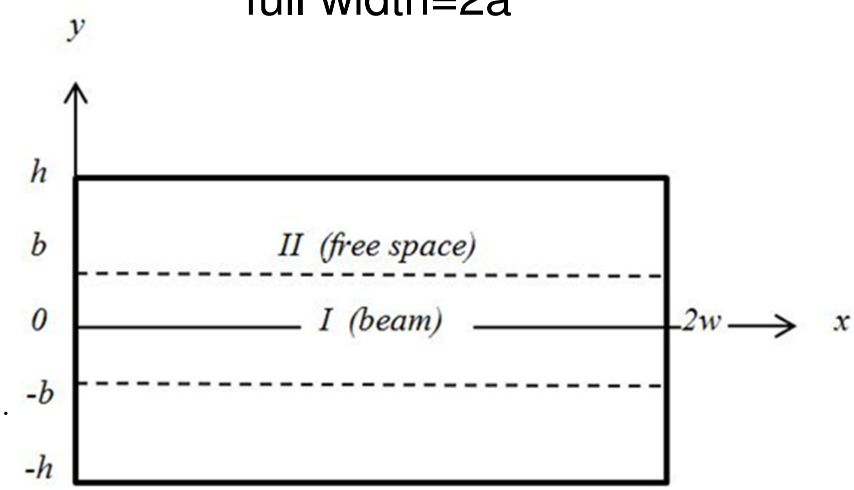
$$X(x) \sim \cos(\eta_n x), \sin(\eta_n y)$$

$$Y(y) \sim \cosh(\nu_n y), \sinh(\nu_n y)$$

$$\eta_n = \frac{n\pi}{2w}. \qquad \rho(x,y,z) = \begin{cases} \frac{\Lambda_k \cos(kz)}{2b} G(x), \\ 0, \end{cases}$$

$$g_n = \frac{2}{\eta_n a}\sin(\eta_n w)\sin(\eta_n a), \quad g_n = 2\int_0^{2w} G(x)\sin(\eta_n x)dx.$$

$$\nu_n^2 = \eta_n^2 + \frac{k^2}{\gamma^2}.$$



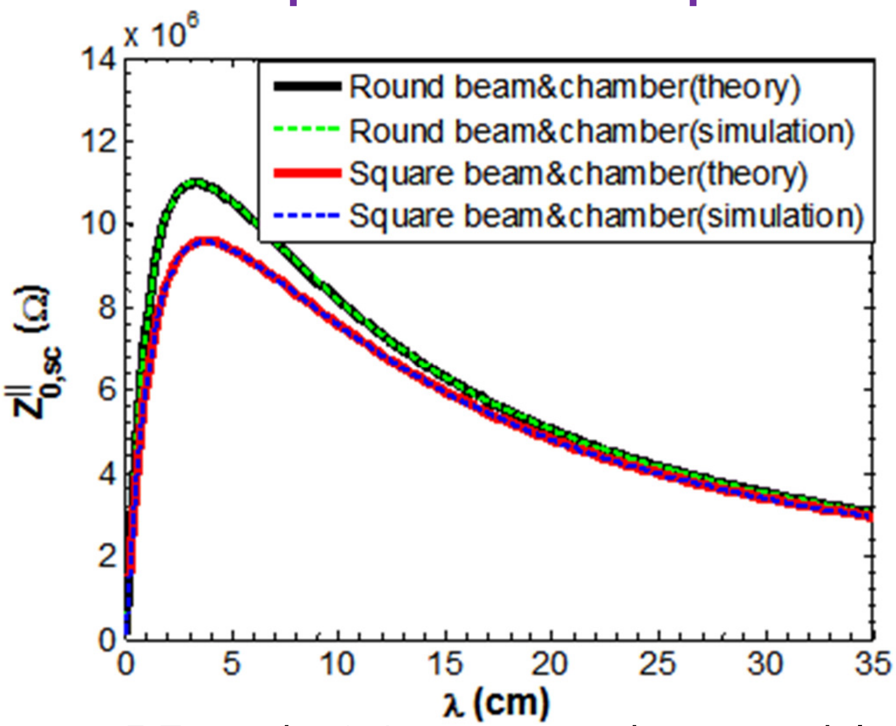

$$E_{z,I}(x,y,z,t) = -\frac{\frac{\partial \Lambda(z,t)}{\partial z}}{4\varepsilon_0 bw\gamma^2}\sum_{n=1}^{\infty}\frac{g_n}{\nu_n^2}\sin(\eta_n x)\{1 - \frac{\cosh[\nu_n(h-b)]}{\cosh(\nu_n h)}\cosh(\nu_n y)\}.$$

LSC on axis: $\frac{Z_{0,sc}^{\parallel}(k)}{C_0} = i\frac{Z_0 k}{4\beta bw\gamma^2}\chi(k), \quad \chi_{axis}(k) = \sum_{n=1}^{\infty}\frac{g_n}{\nu_n^2}\sin(\frac{n\pi}{2})\{1 - \frac{\cosh[\nu_n(h-b)]}{\cosh(\nu_n h)}\}.$

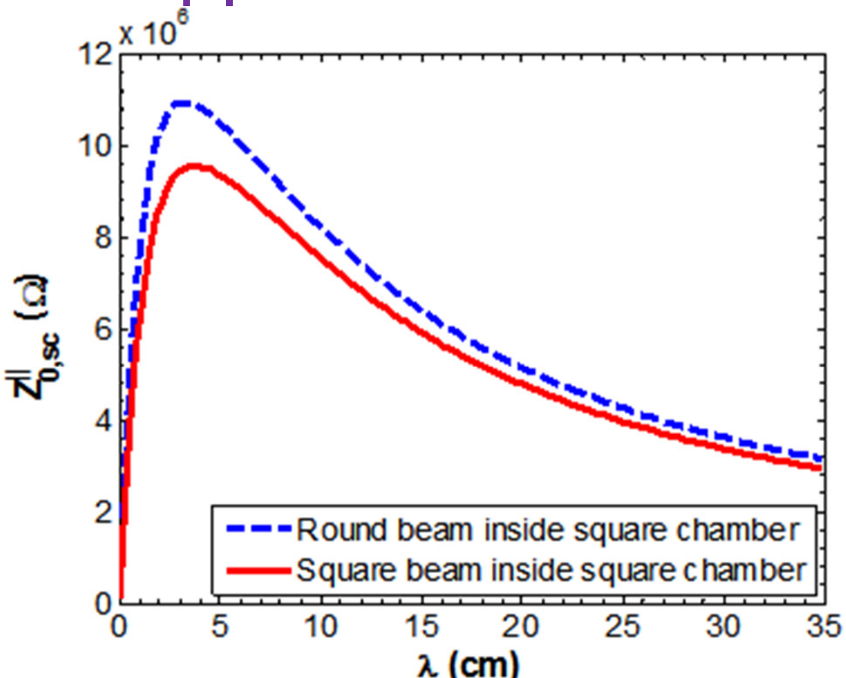
# Squared beam has less LSC impedance

(NIMA747_30_2014, Yingjie Li and Lanfa Wang)

- We used the small isochronous ring (SIR) at Michigan State University in this example. The beam was a 20 keV $H_2^+$ beam ($\beta \approx 0.0046$, $\gamma \approx 1.0$) with a rectangular SIR chamber. SIR circumference was 6.58 *m*.
- Square beam & chamber gives smaller LSC impedance
- **Beam profile is more important than beam pipe**

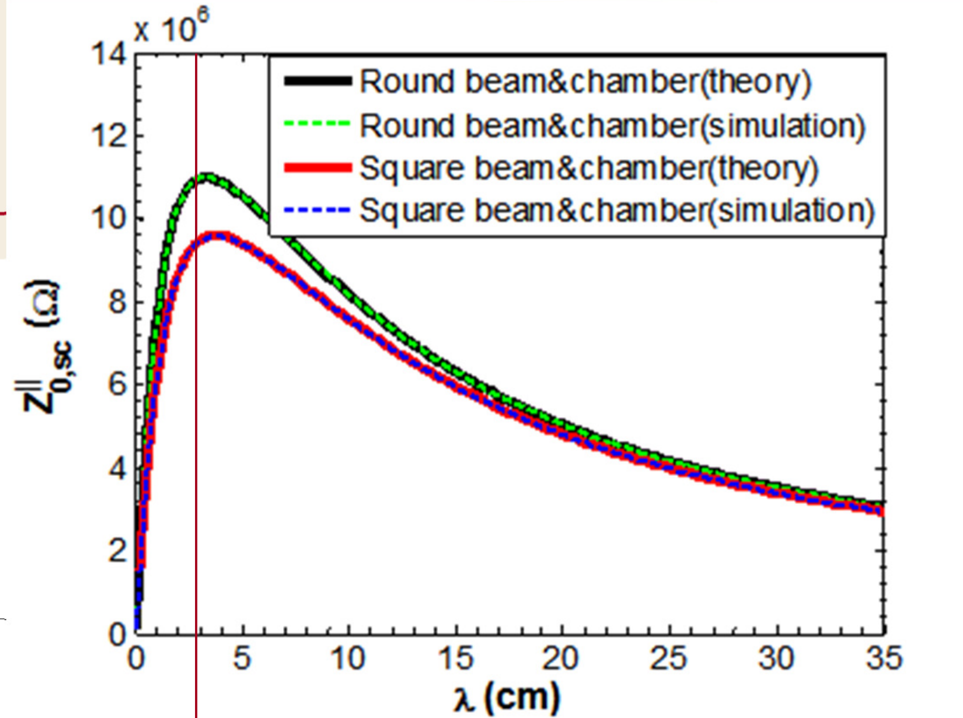


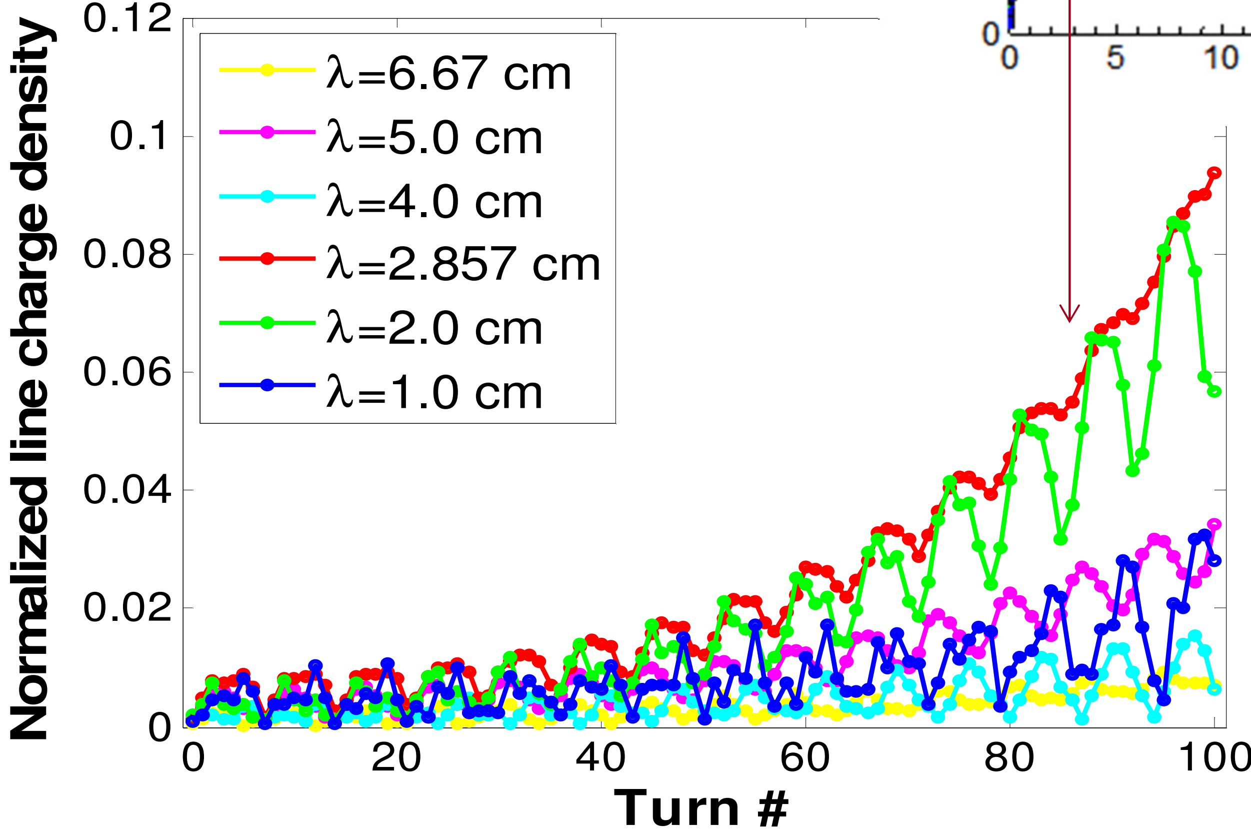

$w$ = 5.7 cm, $h$ =2.4 cm, square beam model with $a = b = r_0 = 0.5$ cm




Simulated LSC impedances of square and round beam models in square chambers ($w = h$ =3.0 cm, $a = b = r_0$ = 0.5 cm)

# Simulated instability
# (Y. LI, MSU)

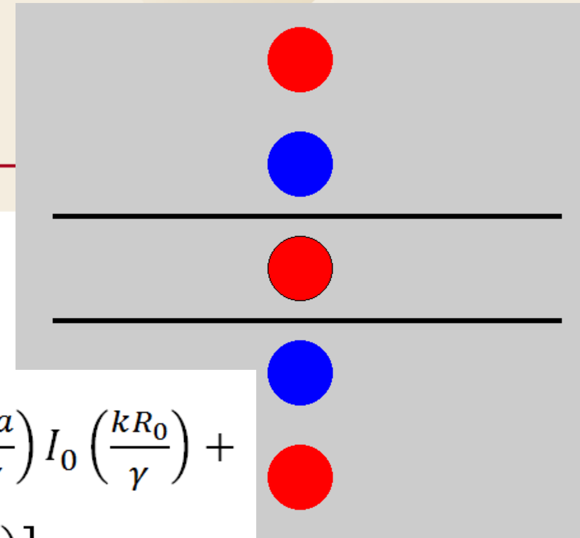


# LSC of a round beam in a parallel plate

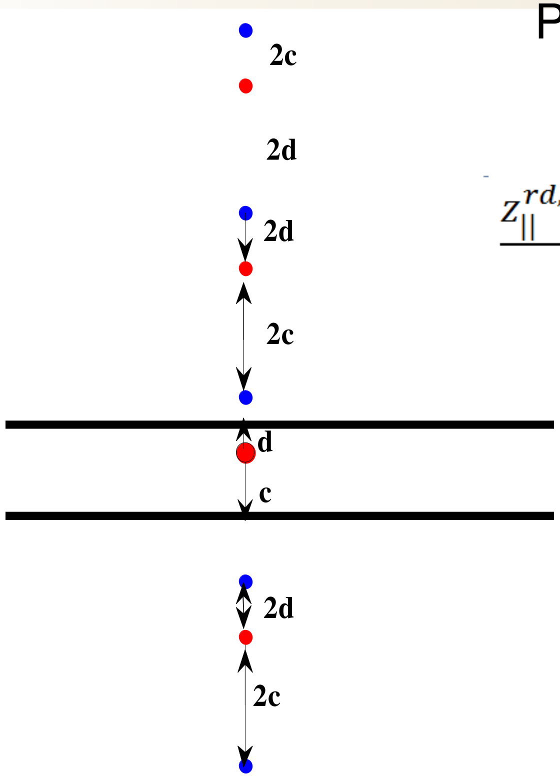
L. Wang and Y. Li, http://arxiv.org/abs/1404.5637

Point charge $\Longrightarrow$ symmetric bunched charge



$$\frac{Z_{||}^{rd,pp}(k,x,y)}{L} = i\frac{1}{k\pi a^2\varepsilon_0\beta c}\left[1-\frac{ka}{\gamma}K_1\left(\frac{ka}{\gamma}\right)I_0\left(\frac{kR_0}{\gamma}\right)+\frac{ka}{\gamma}I_1\left(\frac{ka}{\gamma}\right)\sum_{|n|=1}^{\infty}(-1)^{|n|}K_0\left(\frac{kR_n}{\gamma}\right)\right] \quad (18)$$

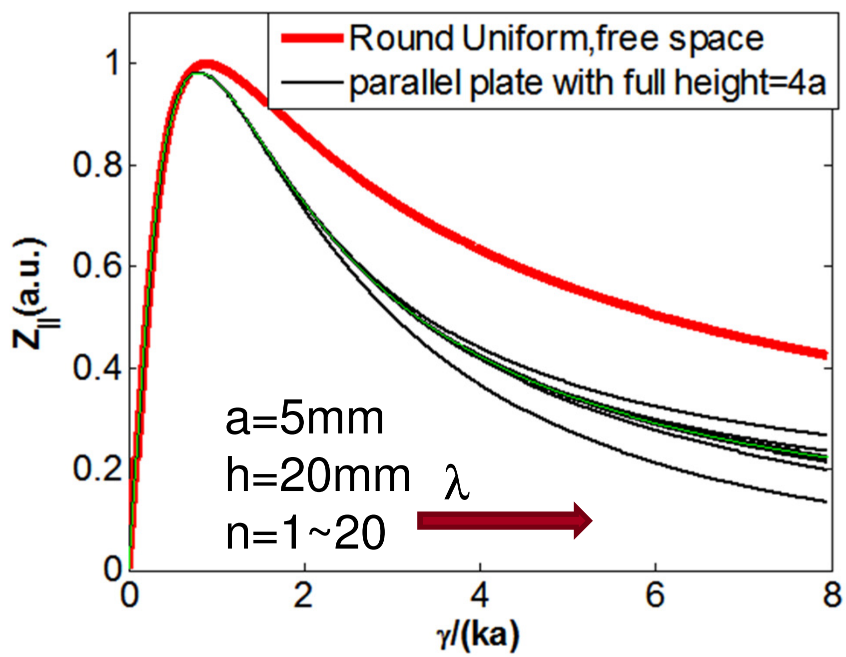
$$R_n = \sqrt{x^2+(y(n)-y)^2}$$




Sequence of images of a point charge in a parallel plate

**position** $y(n) = nh + (-1)^{|n|}y_c$

**Image charge has** $(n) = \lambda_z(-1)^{|n|}$, $n = \pm 1, \pm 2 \cdots$



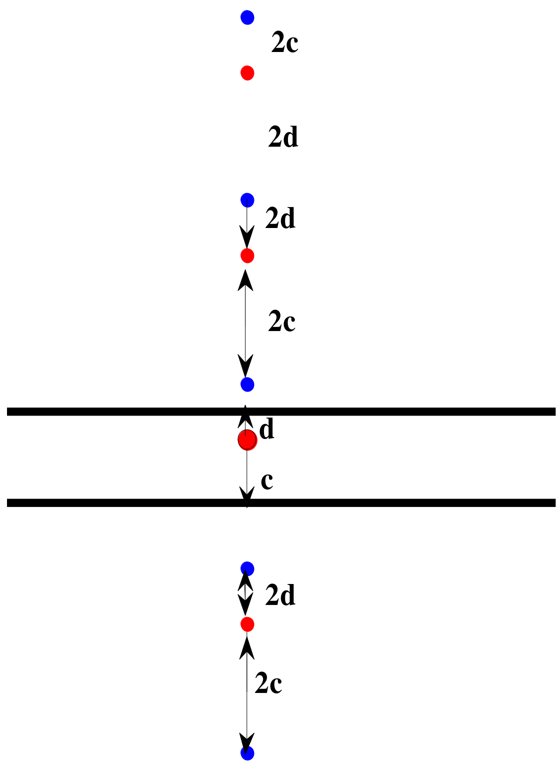
# LSC of a round beam in a p

pos

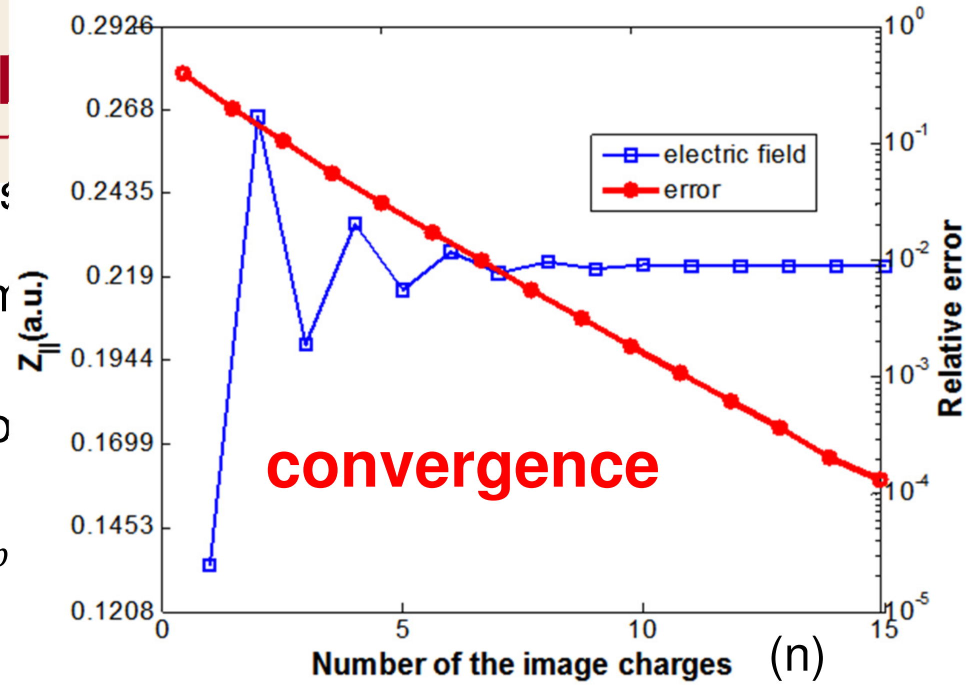
Im

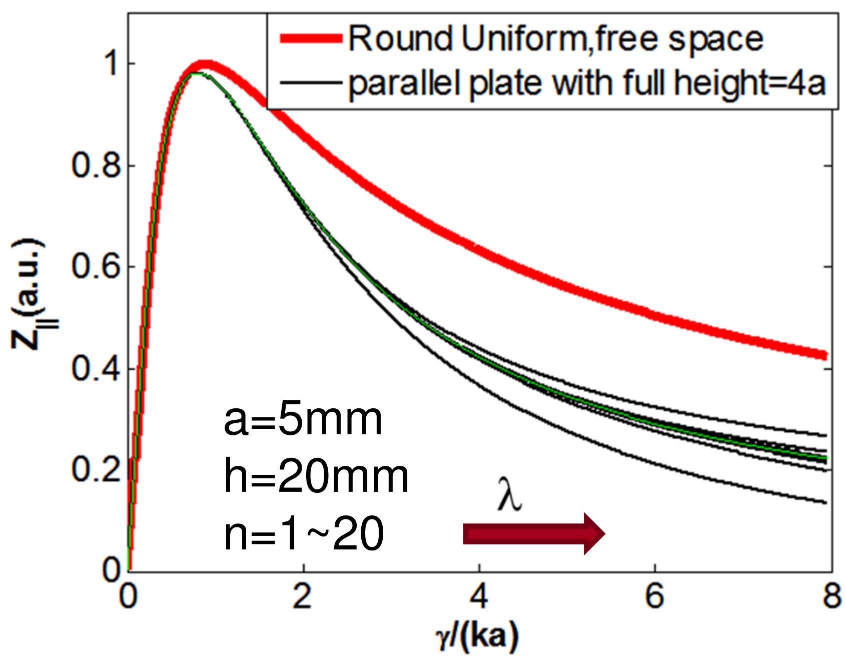
Po

$E_z^{rd,pp}$ ... ι)



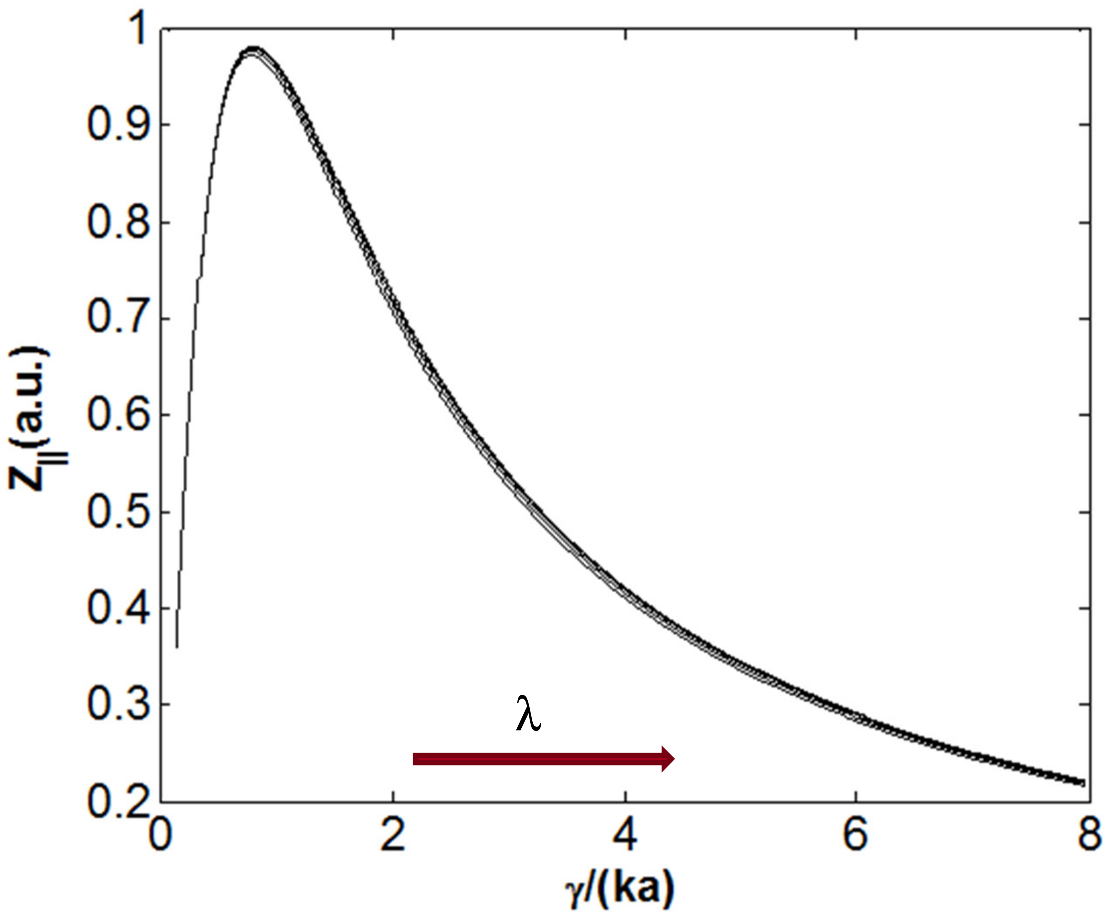

Sequence of images of a point charge in a parallel plate



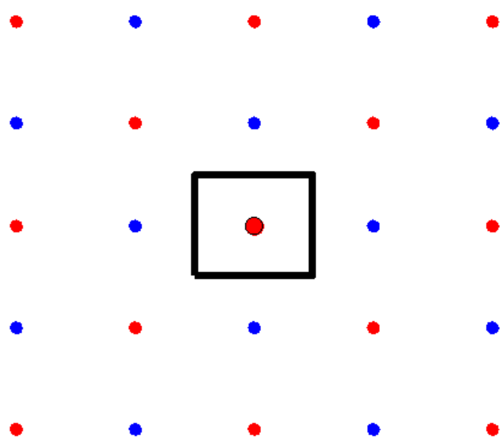
# Effects of Beam offset

- LSC impedance at long wavelength is slightly reduced
- In general case, the effect of beam offset is negligible (asymmetry geometry of beam pipe? To be studied)




Beam offset effect on LSC in a parallel plate chamber with $h$=4$a$. The offset in the plot are for $y_c$=0, 0.2$a$,0.4$a$,0.6$a$ and 0.8$a$

# LSC of a round beam in a rectangular chamber



Sequence of images of a point charge in a rectangular chamber

$$x(m,n) = mw + (-1)^{|m|} x_c$$

$$y(m,n) = nh + (-1)^{|n|} y_c$$

$$\lambda(m,n) = \lambda_z (-1)^{|m|+|n|}, m = n = 0, \pm 1,2,3 \cdots$$

$$\frac{Z_{||}^{rd,rect}(k,x,y)}{L} =$$

$$i \frac{1}{k\pi a^2 \varepsilon_0 \beta c}\left[1 - \frac{ka}{\gamma} K_1\left(\frac{ka}{\gamma}\right) I_0\left(\frac{kR_{0,0}}{\gamma}\right) + \right.$$

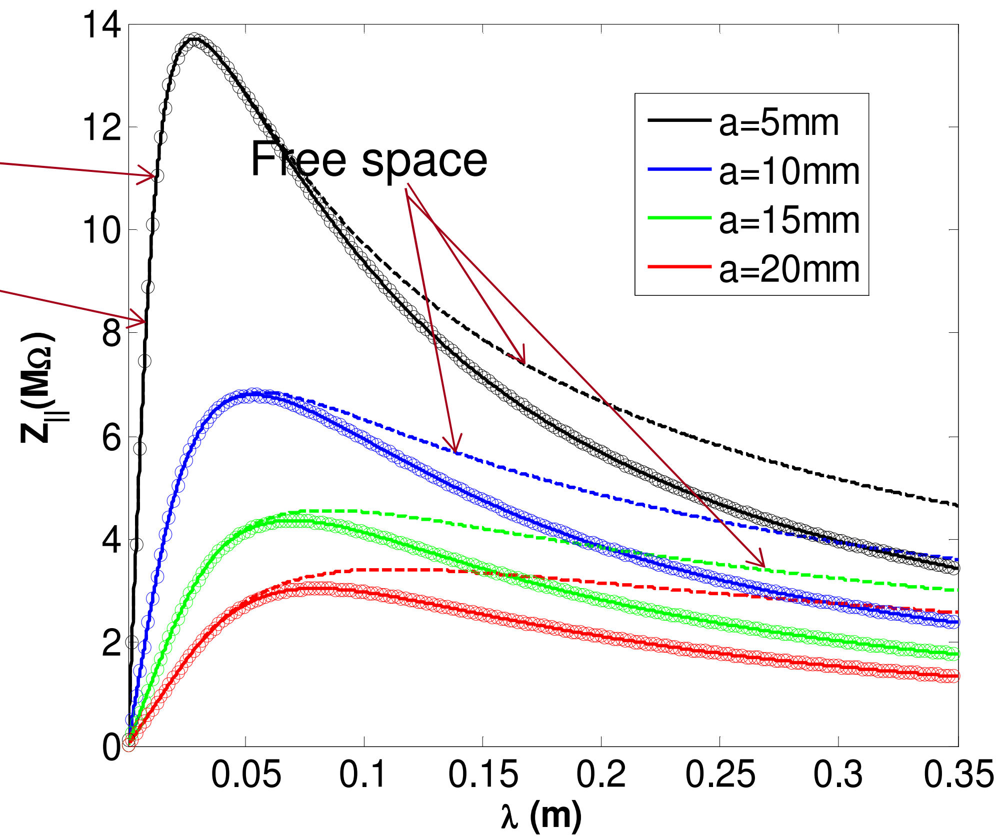
$$\left. \frac{ka}{\gamma} I_1\left(\frac{ka}{\gamma}\right) \sum_{\substack{m=-\infty \\ m\neq 0}}^{m=\infty} \sum_{\substack{n=-\infty \\ n\neq 0}}^{n=\infty} (-1)^{|m|+|n|} K_0\left(\frac{kR_{m,n}}{\gamma}\right)\right]$$

$$R_{m,n} = \sqrt{(x(m,n)-x)^2 + (y(m,n)-y)^2}$$

# Comparisons of analysis and FEM
## (*LSC in a rectangular chamber*)

FEM : dots

Analysis: solid line



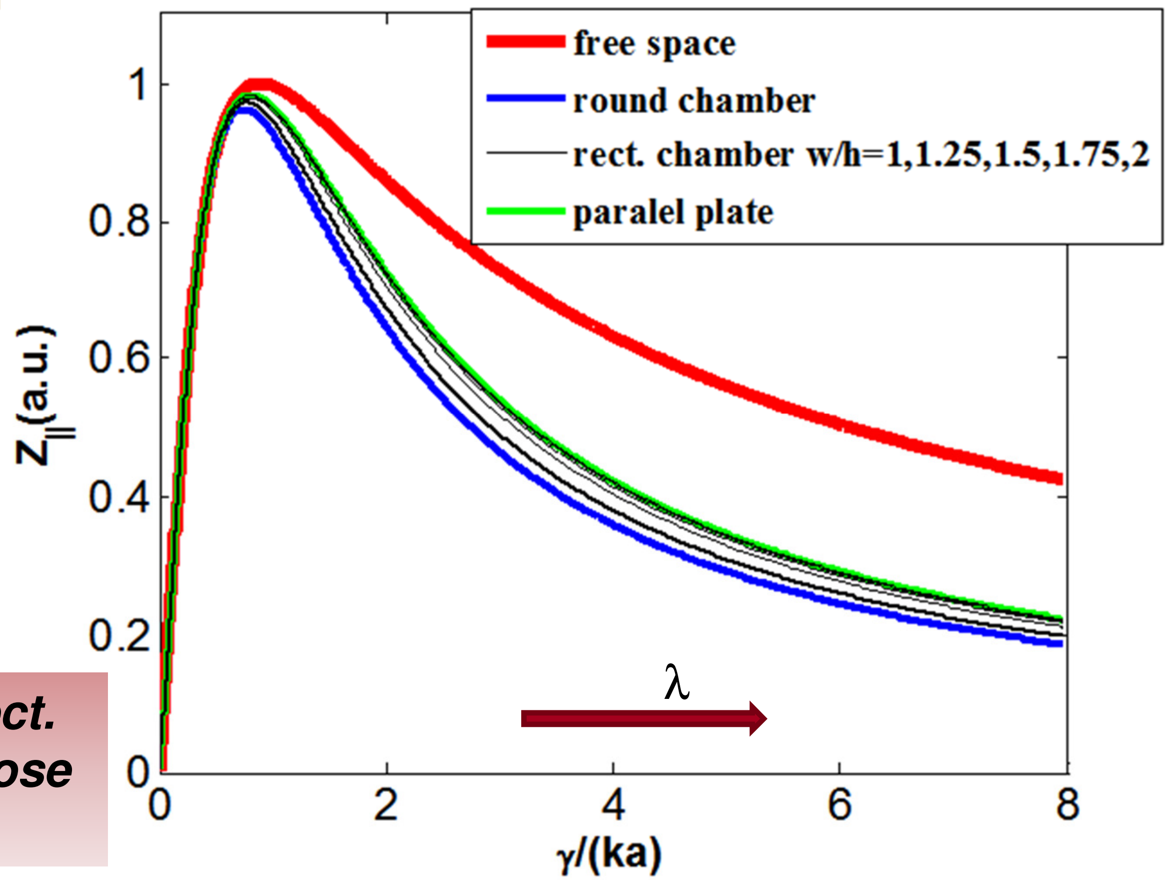

We used the small isochronous ring (SIR) at Michigan State University [1] in this example. The beam was a 20 keV $H_2^+$ beam (β ≈ 0.0046, γ ≈ 1.0) with a rectangular SIR chamber. Calculation parameters were *w* = 11.4 *cm*, *h* = 4.8 *cm*. The beam radius varied from *a* = 0.5, 1.0, 1.5 and 2.0 *cm*. SIR circumference was 6.58 *m*.

Comparisons of the impedance of round uniform beams in a rectangular chamber by analysis (solid lines) and FEM (dots). The free space case is also presented (dashed lines) to show the beam chamber shielding effect.

# Shielding effects of a round vs rectangular chamber

- ***The round chamber has better shielding than a squared chamber. It is about 6% (rw=2a) (1%,rw=10a) less than that corresponding to a square pipe at long wavelength regime with $\gamma/ka > 2$.***

- ***When the aspect ratio of rect. Chamber is >2, it is very close to parallel plate model***




Comparisons of the shielding effects of a round chamber with a rectangular chamber as a function of the *w/h* ratio between the sides of the rectangle. The height of the rectangular chamber is fixed and equal to the diameter of the round chamber. ($r_w$=2a)

## Why shielding at long wavelength?

- ✓ The long wavelength space charge wave can propagate longer distance and therefore interacts with the surroundings (beam pipe in our case)
- ✓ while the short wavelength space charge wave is localized around the source beam and can't cross-talk with the beam pipe which is in general far way from it.

$$K_0\left(\frac{k}{\gamma}r\right)$$

# LSC of an elliptical beam in free space and a rectangular chamber

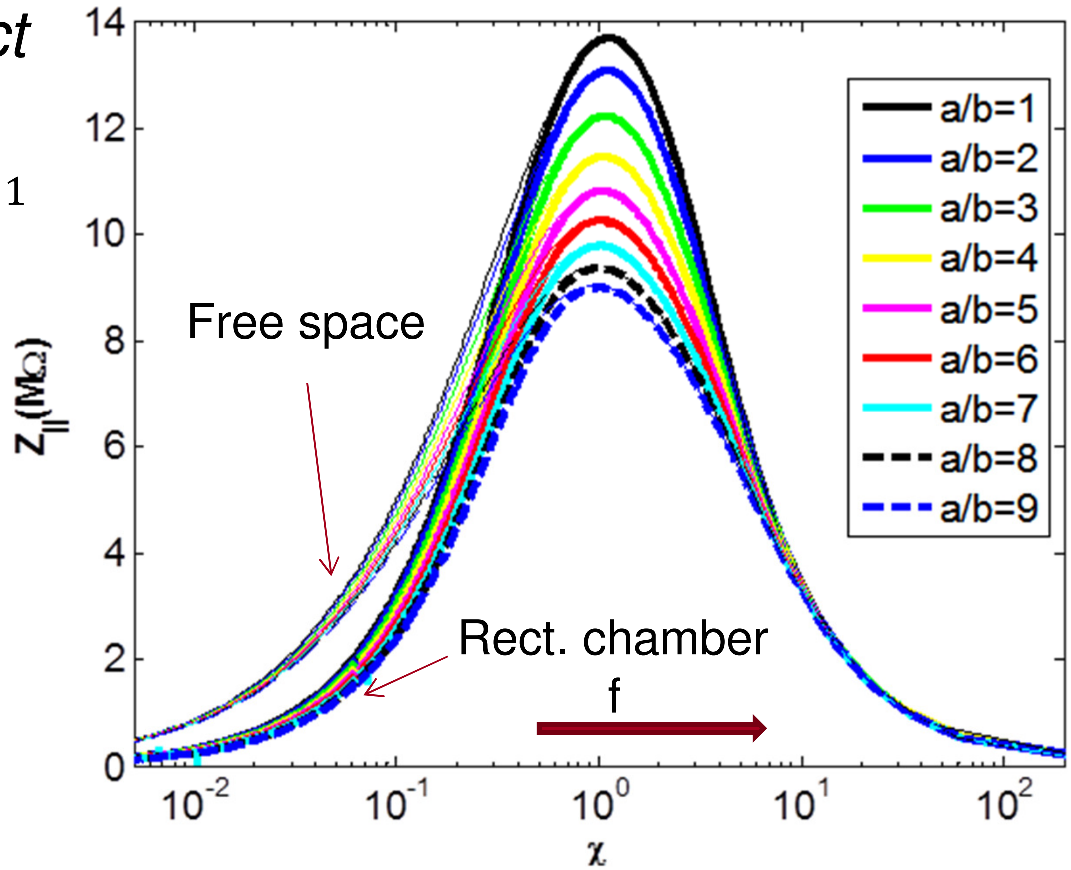
*beam aspect ratio effect*

$$\rho_{\perp}(x,y) = \begin{cases} \frac{1}{\pi ab}, & \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) < 1 \\ 0, & else \end{cases}$$

Condition: same spot size

General conclusions:

- **The same LSC occurs at short wavelength limit**
- **The flat beam has smaller LSC impedance**




LSC in a free space and a rectangular chamber with full width=full height=40 *mm*

$$\chi = \frac{k\,Sqrt(ab)}{\gamma}$$

# Variation of Ez with the transverse position inside the beam

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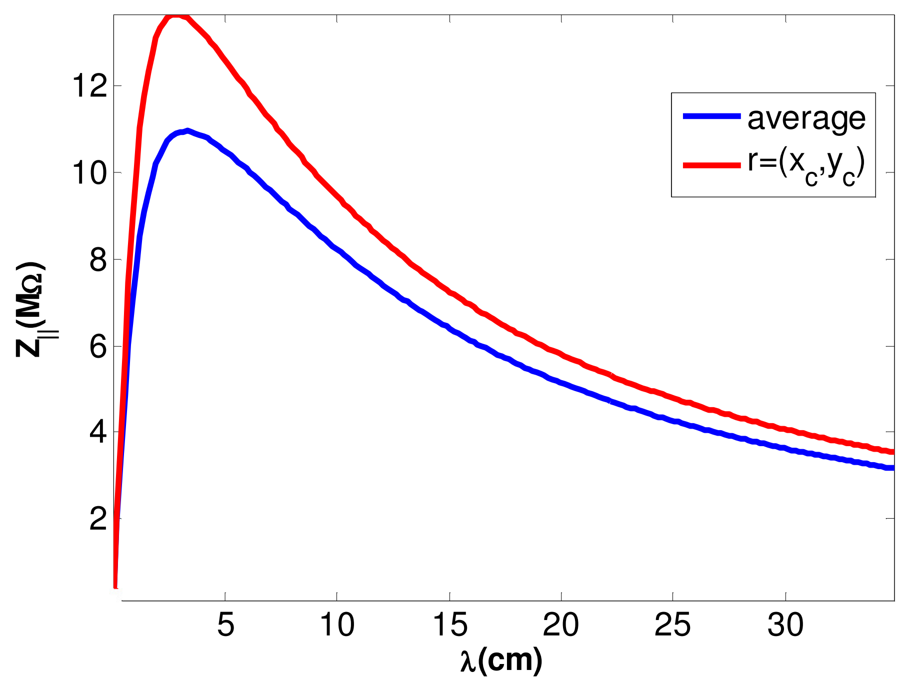
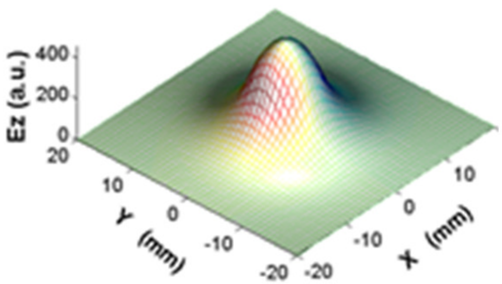
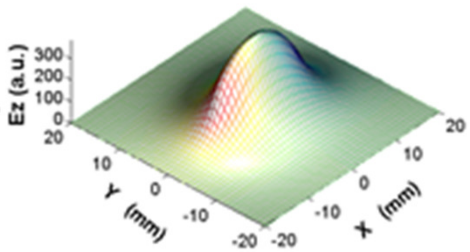
- The Ez field (and thus LSC impedance) varies with the transverse position
- The Ez is maximal at the beam center
- **The position- dependent fields should be used in the instability study (3D LSC), which can mitigate the instability.**




Round beam




Elliptical beam




LSC of Round uniform beam in a squared chamber: ($w = h$ =3.0 cm, $a$= 0.5 cm).

## Summary

- LSC in full wavelength is necessary, especially for FEL
- A FEM code has been developed to calculate LSC impedance for arbitrary geometry of the beam and beam pipe in full wavelength
- Analytical studies for
  - a Gaussian beam in free space
  - a rectangular beam in a rectangular beam pipe
  - a round uniform beam in a rectangular and parallel plate beam pipe

## Conclusions

SLAC

- ✓ **A flat beam** gives less LSC impedance, <u>but the difference disappears at short wavelength limits</u>; We may design the optics to make flat beam in most location where we don't use it for experiments.
- ✓ **Squared beam** has less LSC impedance than round one
- ✓ **a round beam pipe** has slightly smaller LSC impedance than a squared one
- ✓ When a aspect ratio of a rectangular chamber w/h is larger than 2, its shielding effect is close to parallel plate.
- ✓ Shielding is effective for long wavelength $kr_p/\gamma<1$.

# Future work

--- Transverse Space charge Impedance

— Find empirical formulae for:

- different beam aspect ratios
- the average LSC
- shielding effects (of various beam pipe)

---- Apply 3D LSC in micro-bunch study

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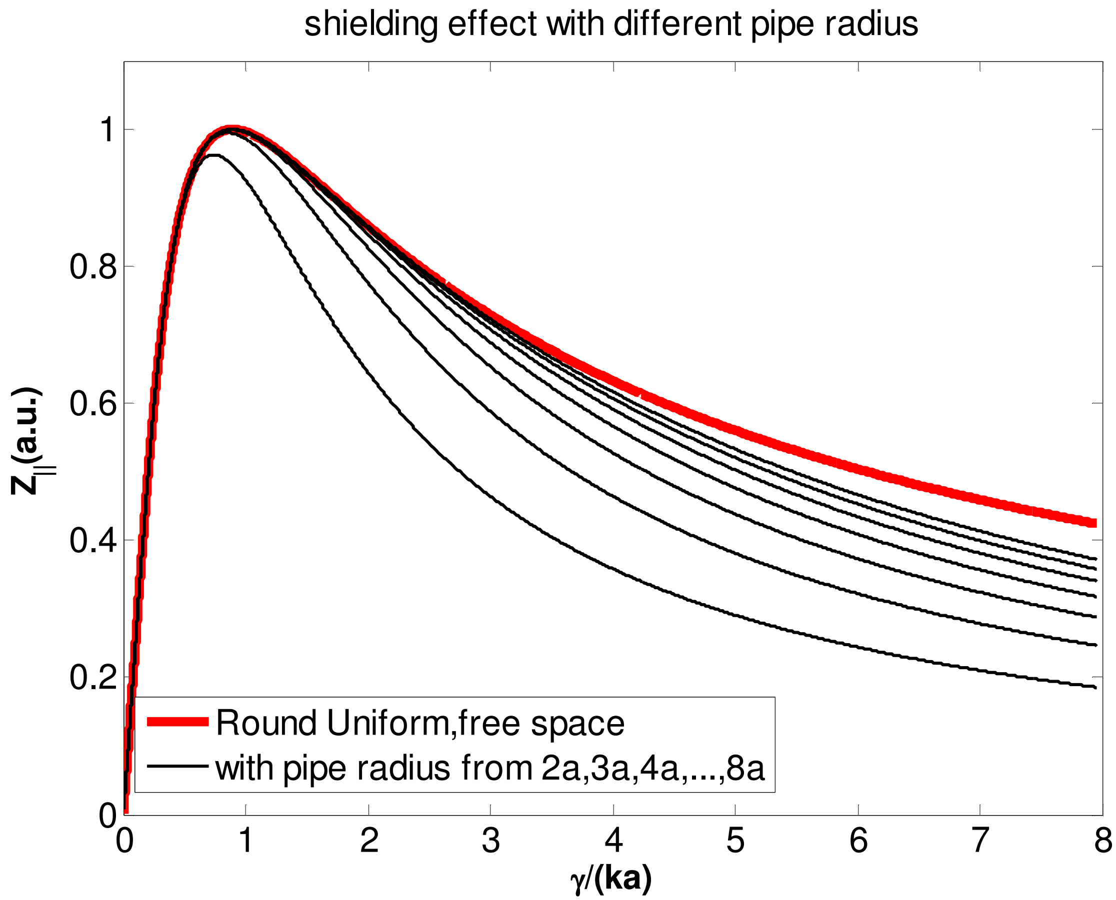
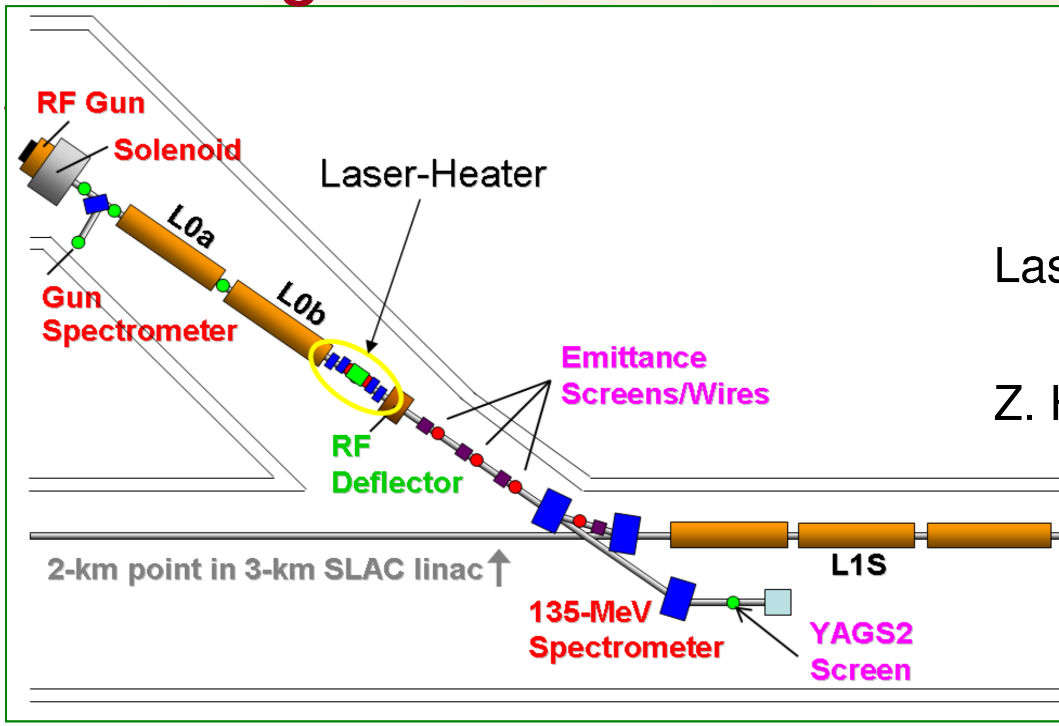
Thank you all!

# Shielding effect of round chamber



# Heating Measurements




Laser heater suppress u-bunch instability

Z. Huang, et.al. PRSTAB **7**, 074401 (2004)