2D wall impedance theory

Nicolas Mounet & Elias Métral

Acknowledgments: N. Biancacci, G. Rumolo, B. Salvant, C. Zannini, B. Zotter.

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- Motivation
- Outline of the theory
- Examples of impedance results
- Computation of wake functions
- Possible extensions
- Conclusions

Motivations for 2D analytical impedance models

- 3D simulation tools can nowadays solve EM problems taking into account material electromagnetic properties, BUT:
  - difficult to have in single simulation both low and high frequency content,
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  ⇒ develop here axisym. and flat 2D theories based on Zotter's formalism (details in EPFL PhD thesis 5305, 2012).

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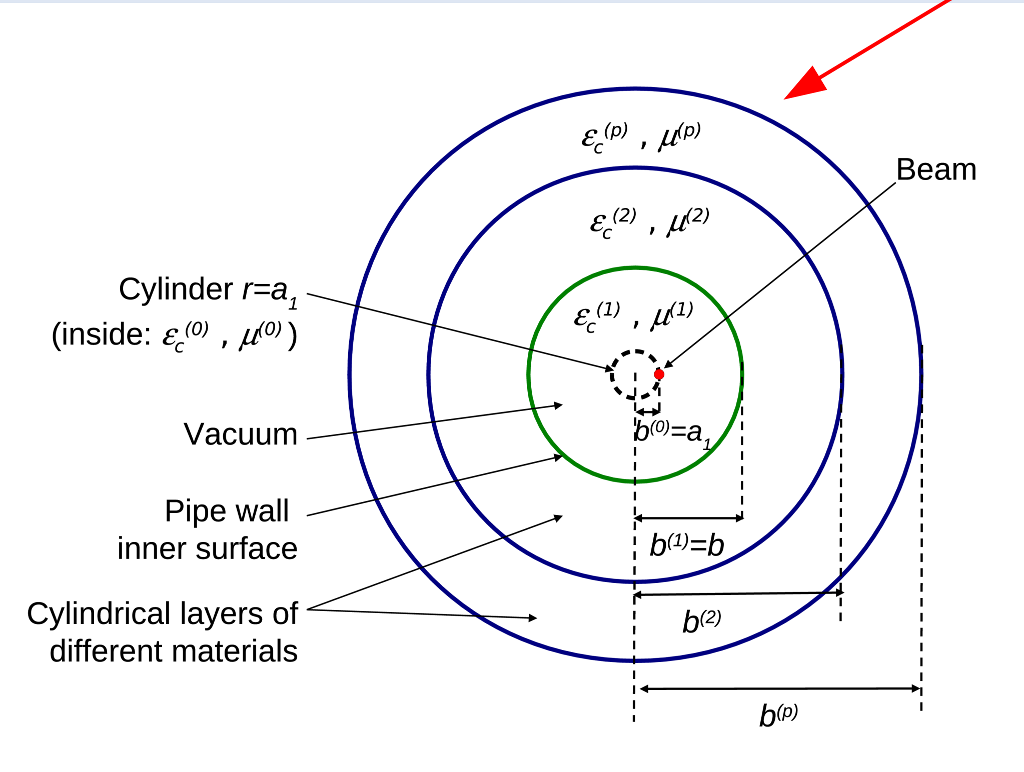

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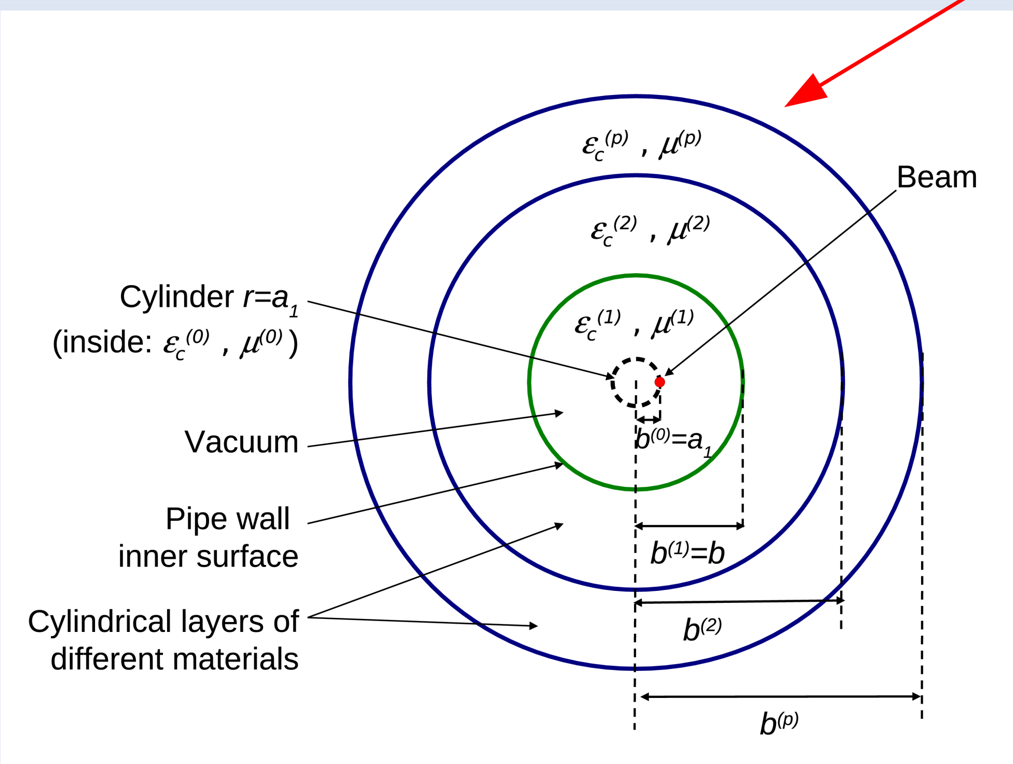

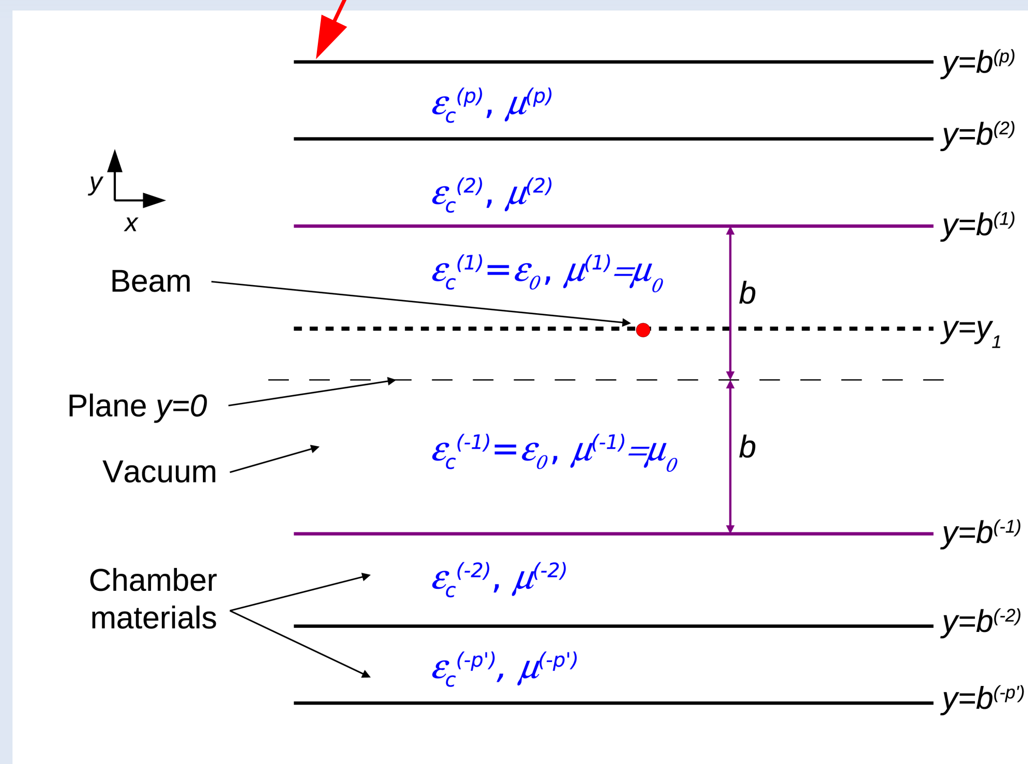

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Continuous Fourier transforms (the $\int \mathrm{d}k'$ and $\int \mathrm{d}k_x$ integrals)

Fourier series decomposition (the sum over m)

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- Finally, put back the Fourier transforms and/or series. In flat case, additional algebra to get a simpler form.

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Outline of the theory

- Electric field longitudinal component in the vacuum:

axisym.

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Direct space-charge term | Constants | Wall term

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Direct space-charge term

Constants

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Direct space-charge term　Constants　Wall term

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Direct space-charge term Constants Wall term

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Direct space-charge term — Constants — Wall term

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Direct space-charge term | Constants | Wall term

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Final wall impedances

- From EM force in vacuum, upon integration over a finite length L and normalization (by the test and sources charges) we obtain the **beam-coupling impedances** as simple functions of the first few α_{TM}^{m} (axisymmetric) or α_{mn} (flat) (**freq. dependent**).

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Axisymmetric case

$$Z_{\parallel}^{Wall} \approx \frac{j\omega\mu_0 L}{2\pi\beta^2\gamma^2}\alpha^0_{\text{TM}}(\omega),$$

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Quadrupolar term also in axisymmetric !

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Constant term in vertical when no top-bottom symmetry

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Flat case: **quadrupolar terms not exactly opposite to one another** (≠ A. Burov –V. Danilov, PRL 1999, ultrarelativistic case)

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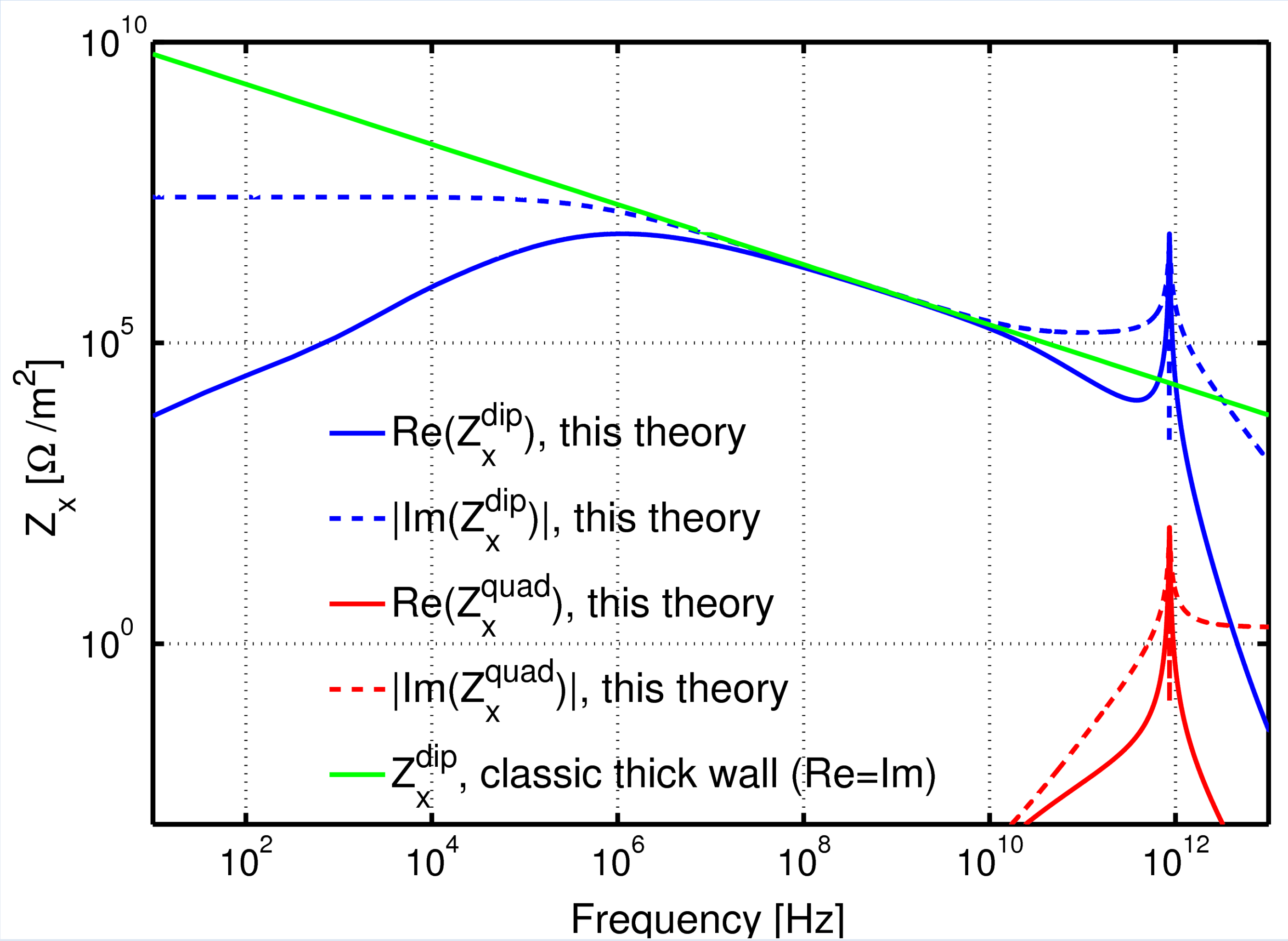
Note: it is easy to go to higher order than linear !

Results in axisymmetric

- 25mm-thick graphite of radius 1.5 mm (surrounded by stainless steel), compared to classic formula:



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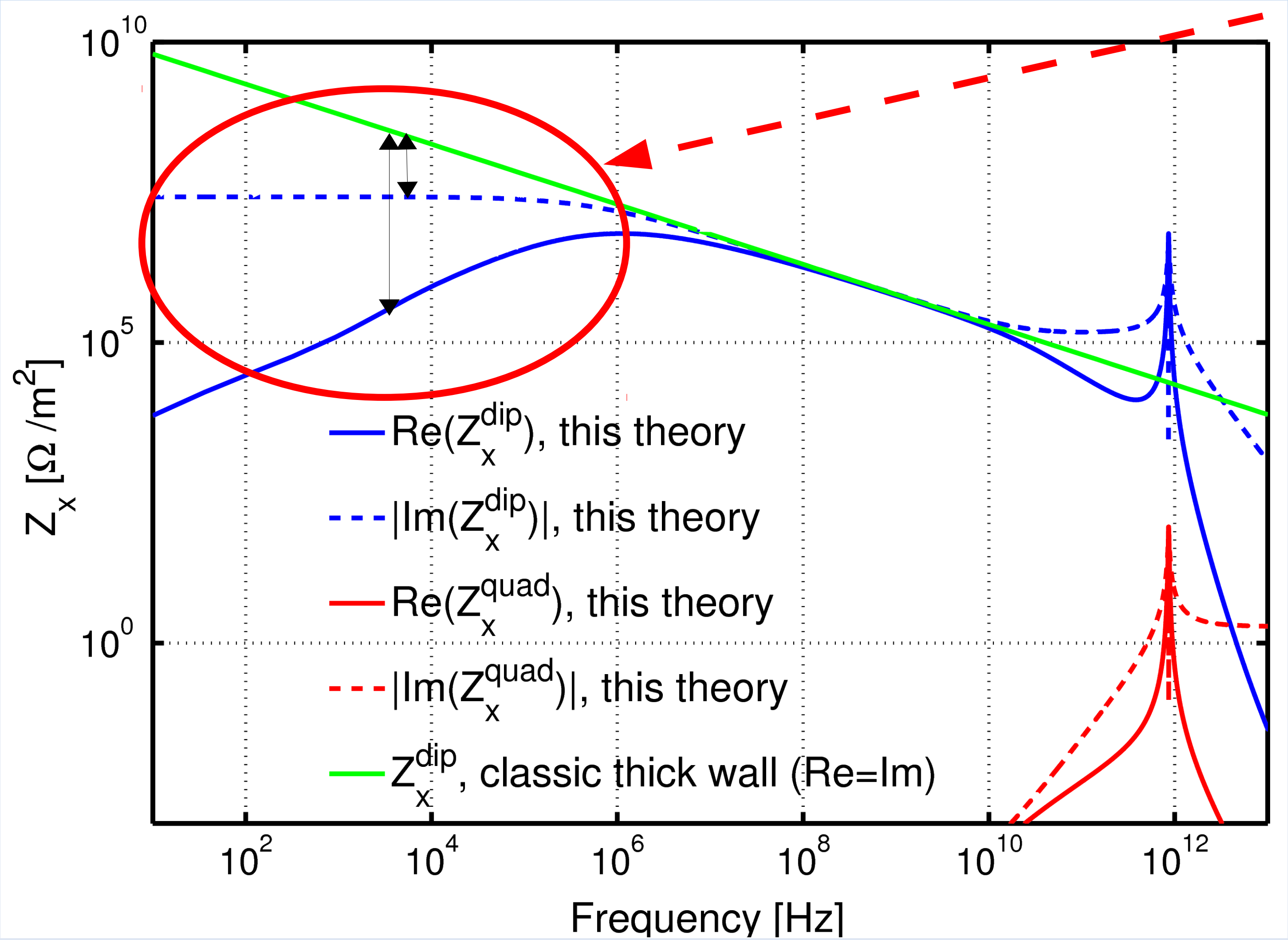

Low frequencies: importance of general theory w.r.t classic formula (factor ~10 for imag. part, >100 for real part)

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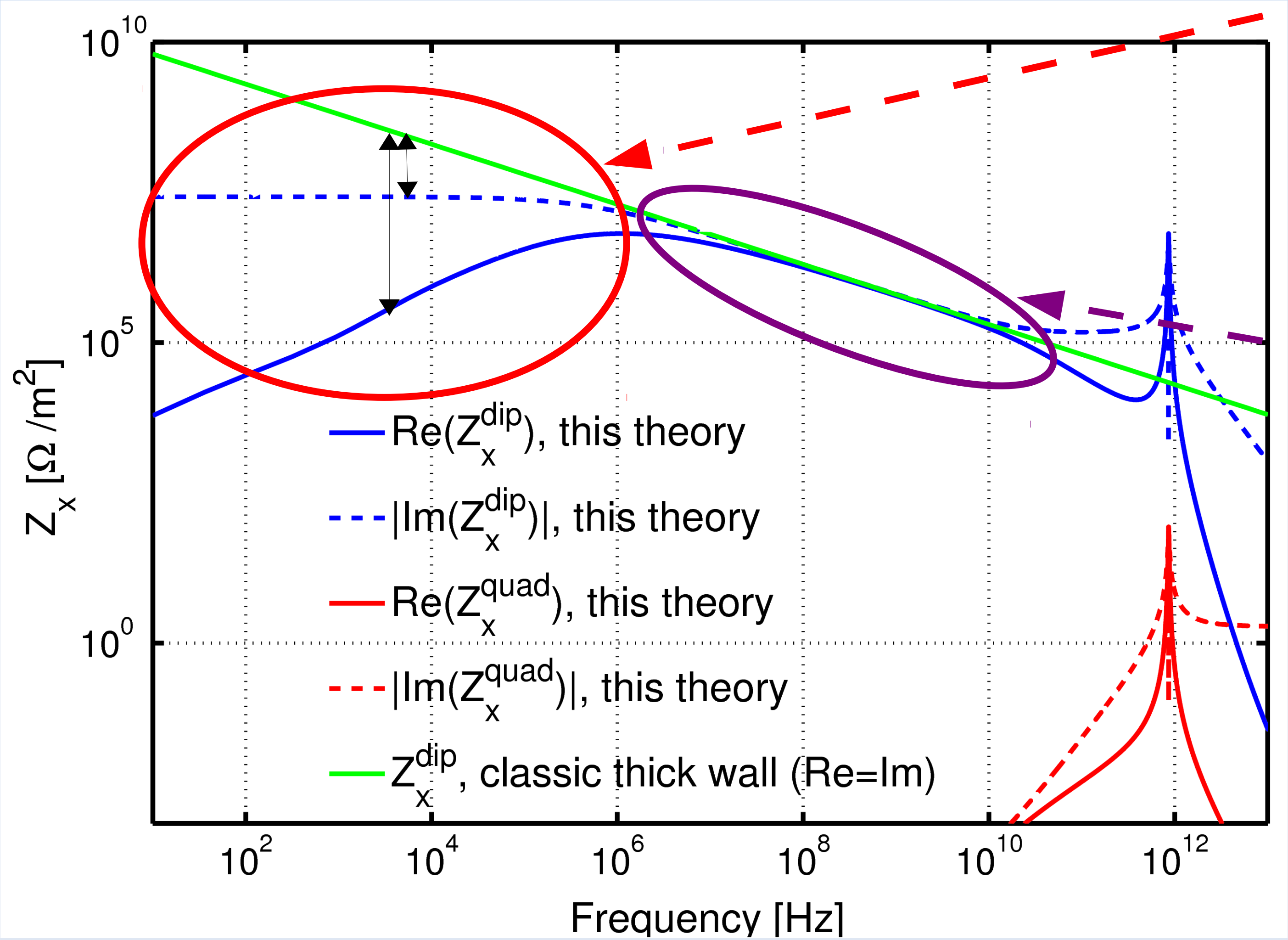

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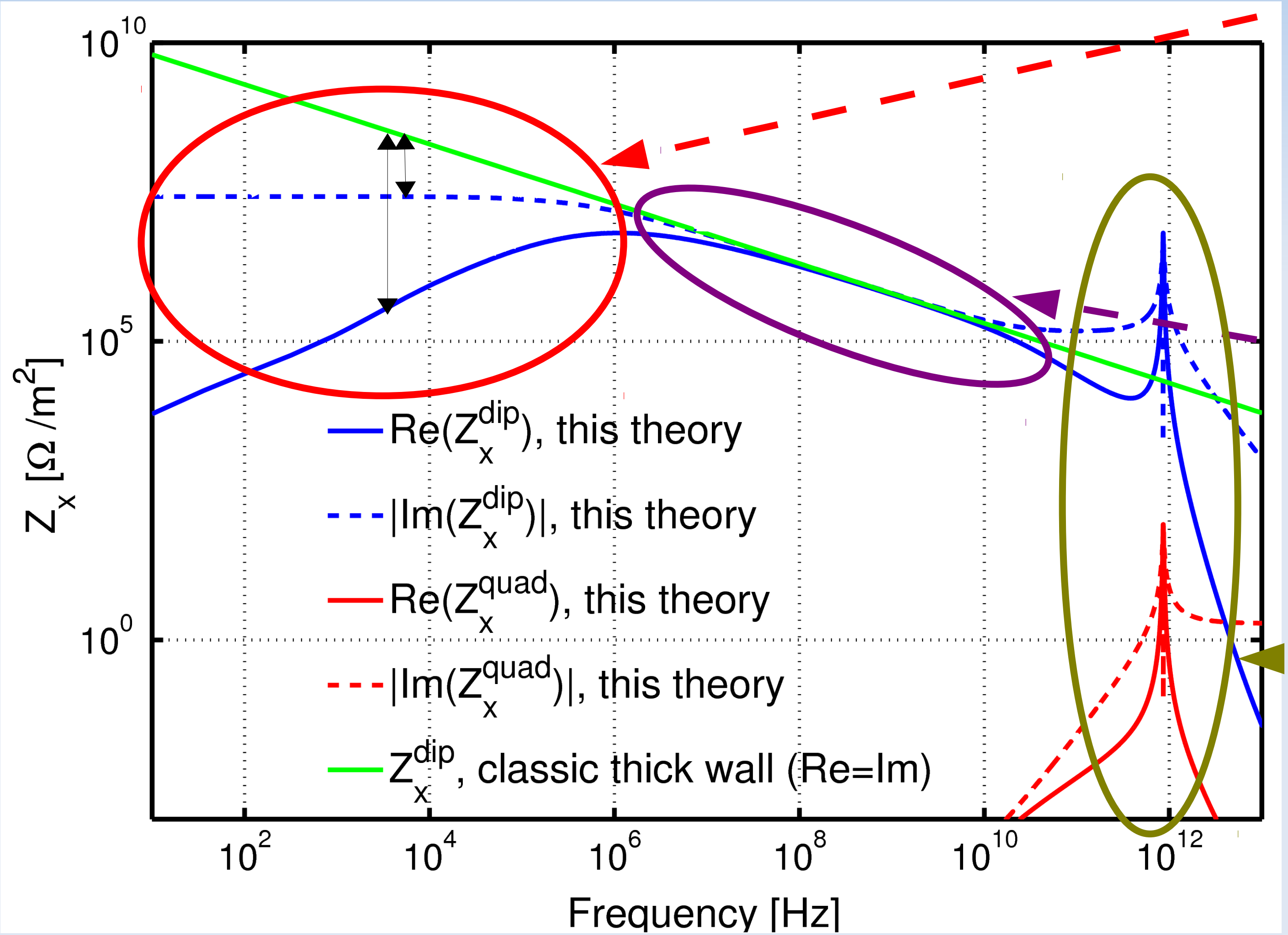

Low frequencies: importance of general theory w.r.t classic formula (factor ~10 for imag. part, >100 for real part)

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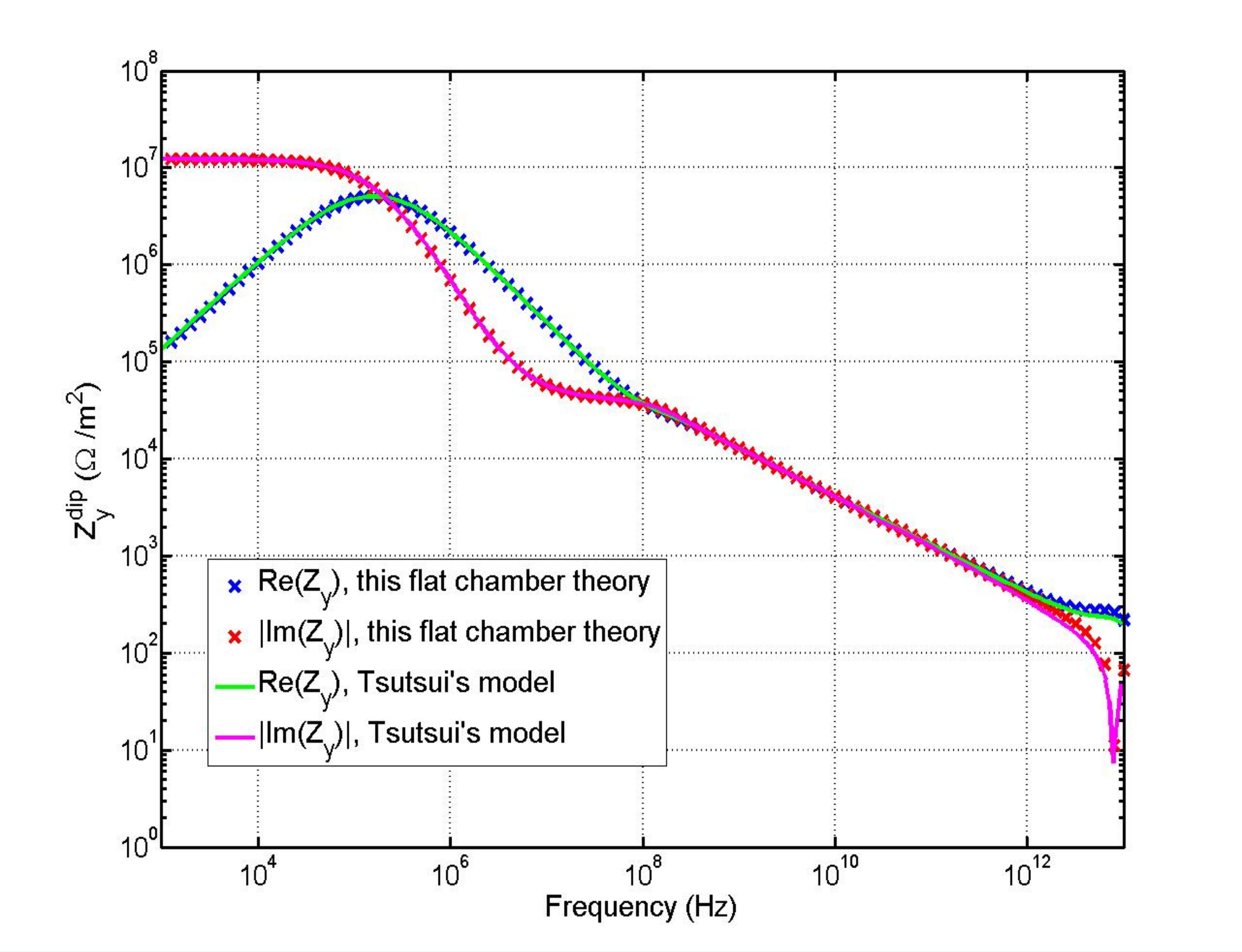
High frequencies: resonance + new quadrupolar term (in this theory only).

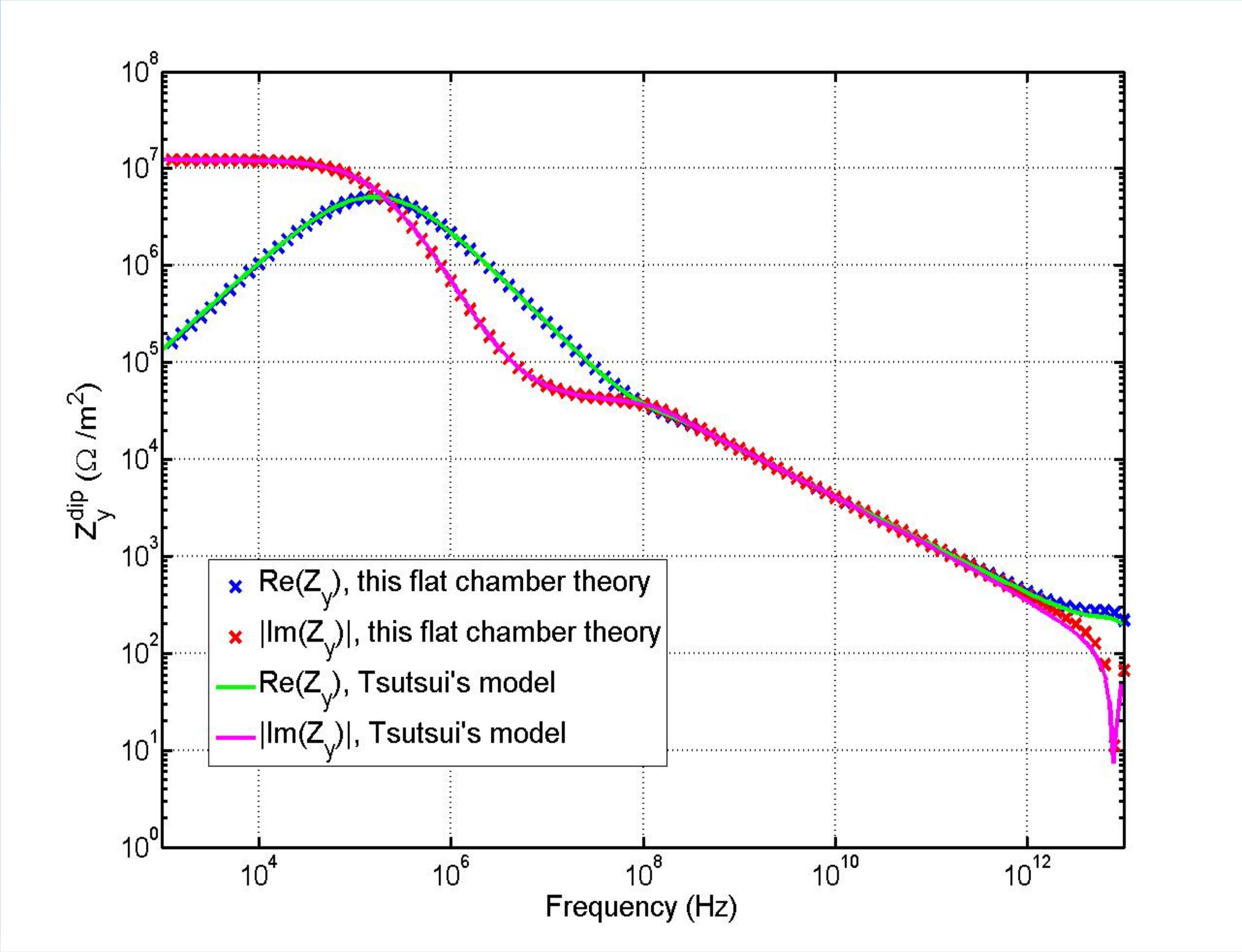
Flat chamber results: comparison to Tsutsui's formalism

- For 3 layers (LHC copper-coated graphite collimator - 10μm Cu, half-gap 2mm), comparison with Tsuitsui's model (LHC project note 318) on a rectangular geometry, the two other sides being taken far enough apart:



Flat chamber results: comparison to Tsutsui’s formalism

- For 3 layers (LHC copper-coated graphite collimator - 10µm Cu, half-gap 2mm), comparison with Tsuitsui’s model (LHC project note 318) on a rectangular geometry, the two other sides being taken far enough apart:



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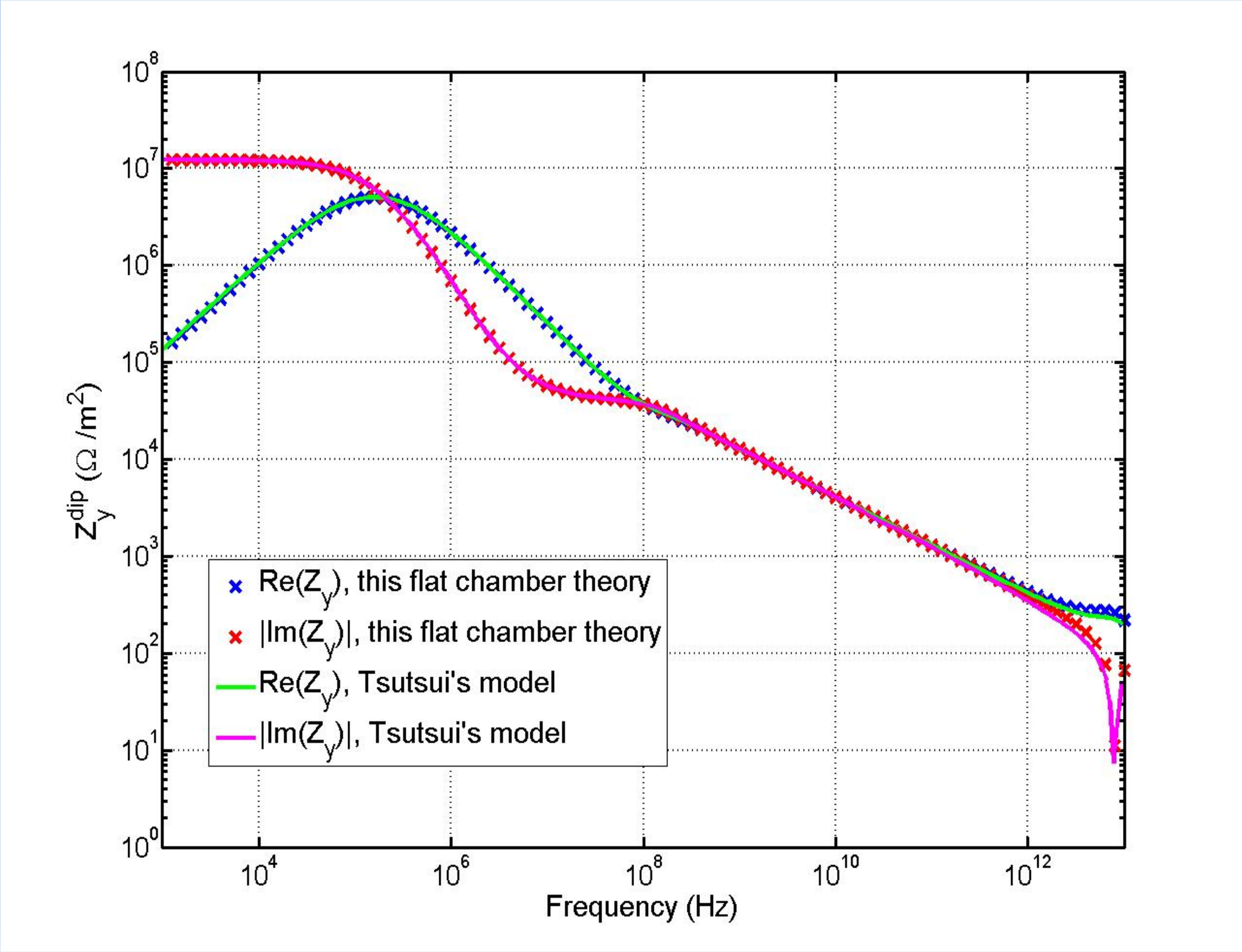
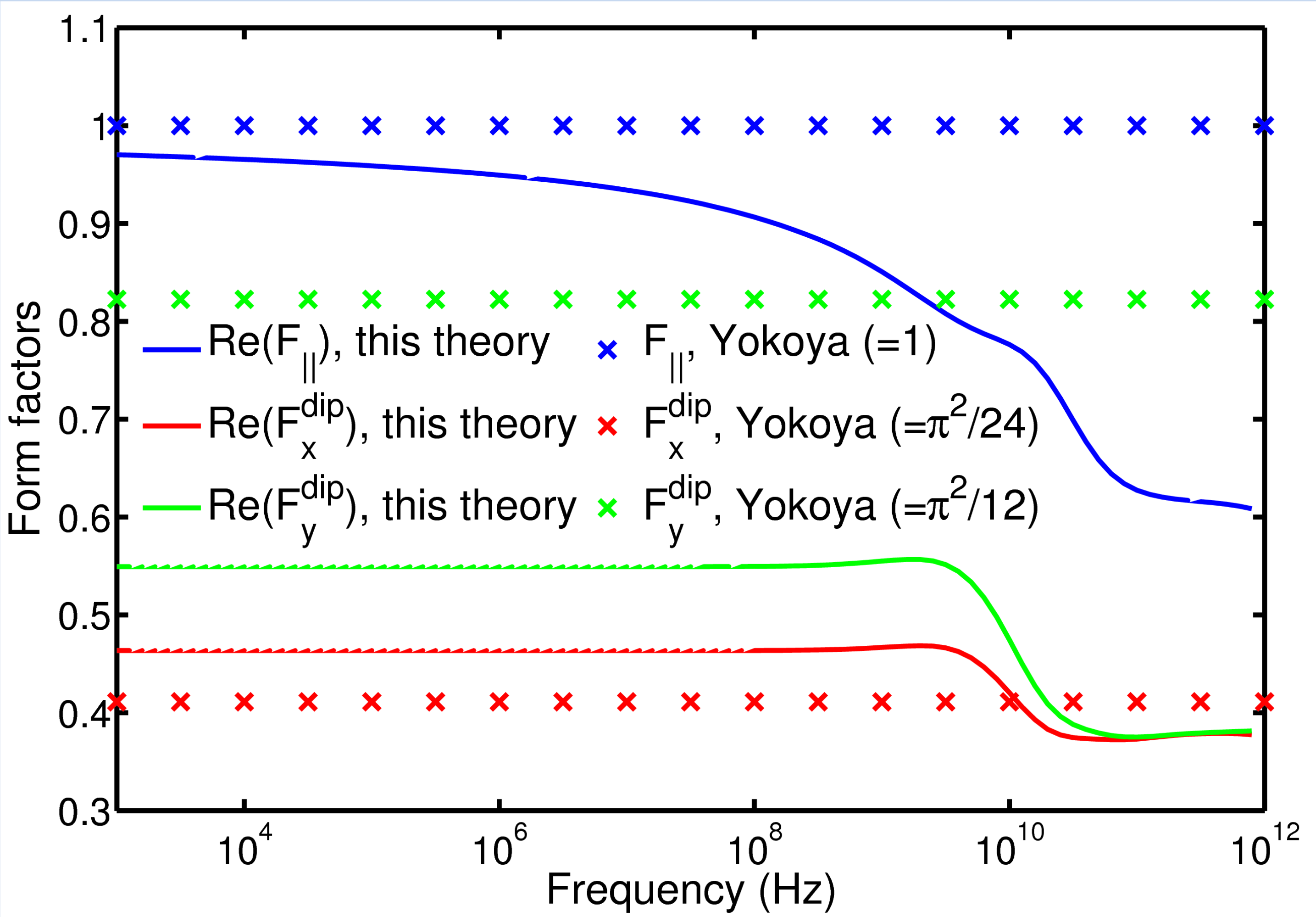


⇒ Very good agreement between the two approaches.

⇒ Can compute impedance for many layers (particularly interesting for coating – difficult to simulate in 3D EM codes)

Form factors **flat / axisymmetric**

- Ratio of flat chamber impedances w.r.t longitudinal and transverse dipolar axisymmetric ones → generalize Yokoya factors (Part. Acc., 1993, p. 511). In the case of a single-layer ceramic (hBN) at 450 GeV:



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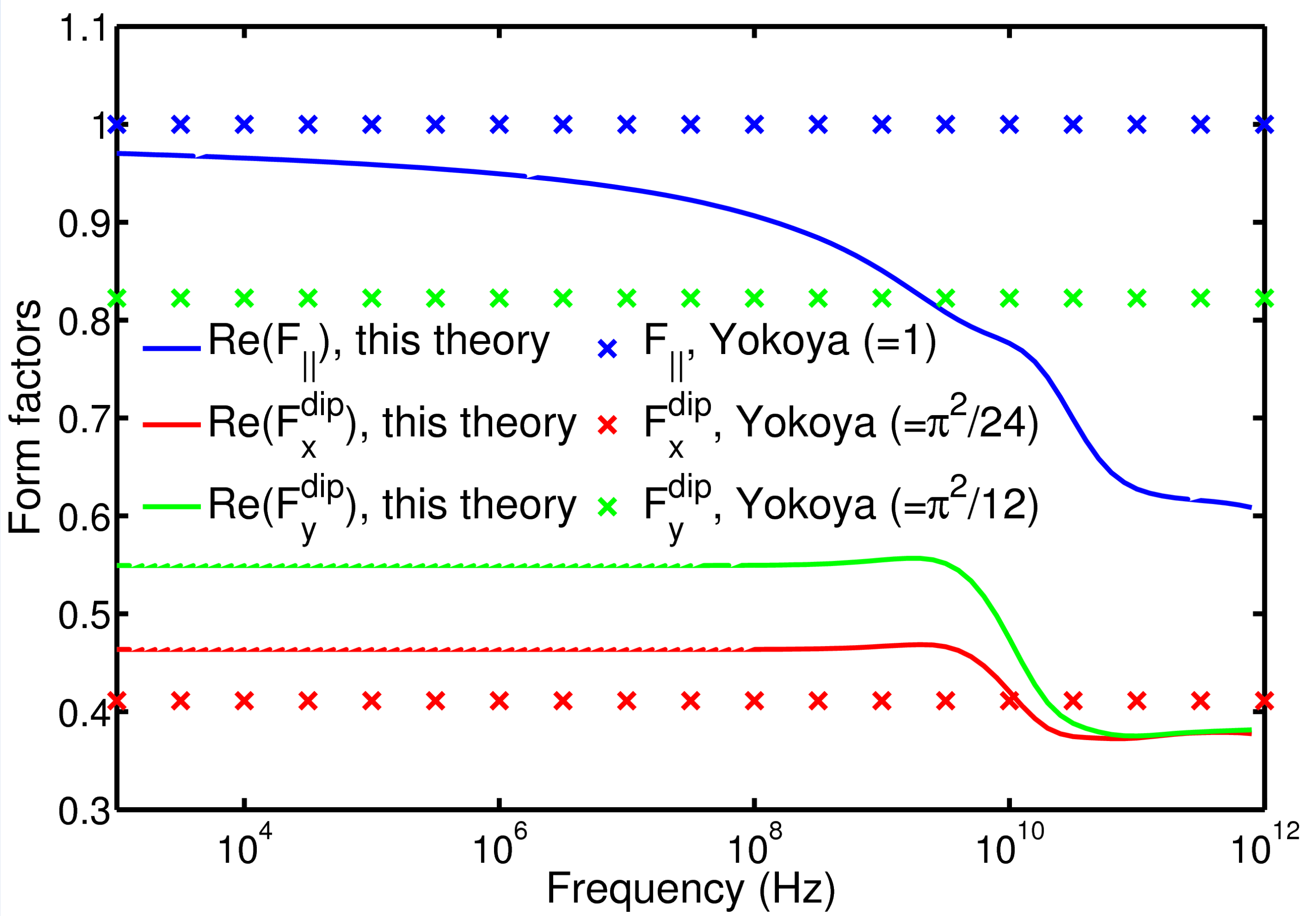

⇒In this particular case, frequency dependent form factors quite ≠ from the Yokoya factors (was first observed with other means by B. Salvant et al *IPAC'10*, p. 2054).

⇒We can get such form factors for any material or material combination (i.e. several layers).

Wake functions

- Wake functions are the Fourier transforms of the impedances, e.g.

$$W_x(\tau) = -\frac{j}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega e^{j\omega\tau} Z_x(\omega),$$ for a test particle at τ seconds behind the source

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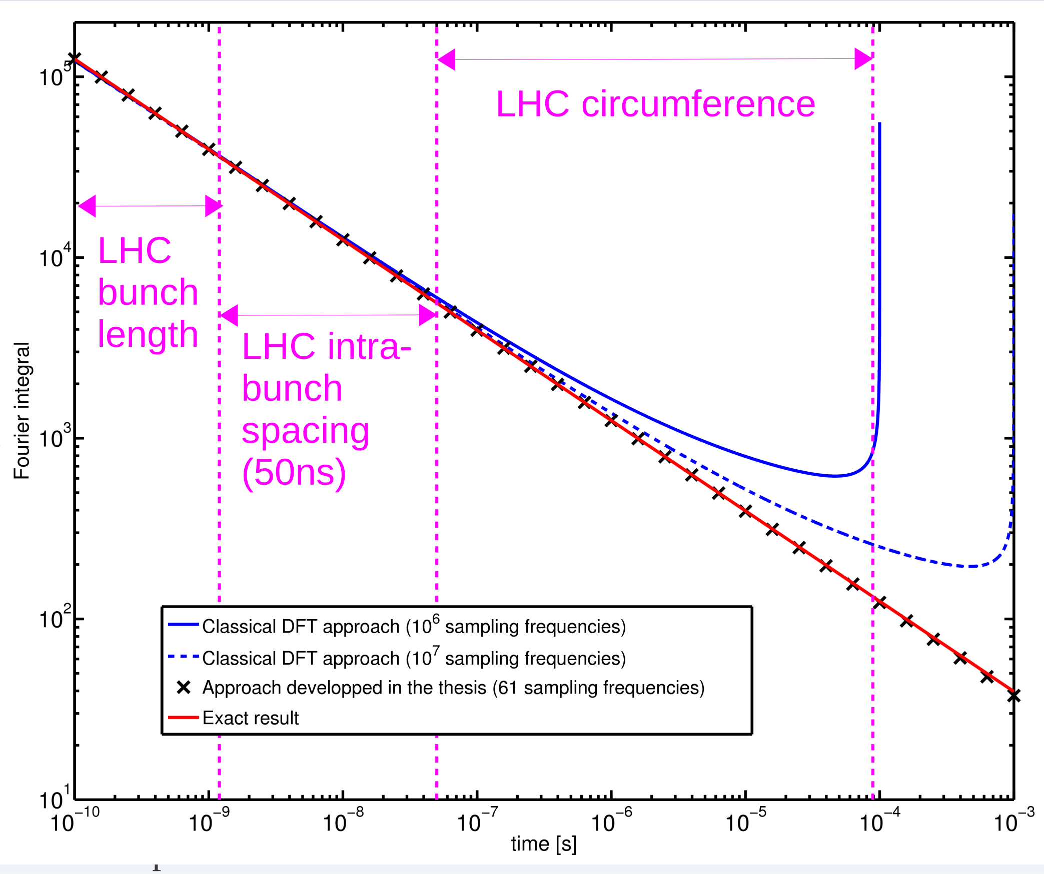
⇒ developped a ”new” method (based on idea from 1928): given any frequency sampling, on each subinterval replace the impedance by its cubic interpolation, and integrate it analytically.

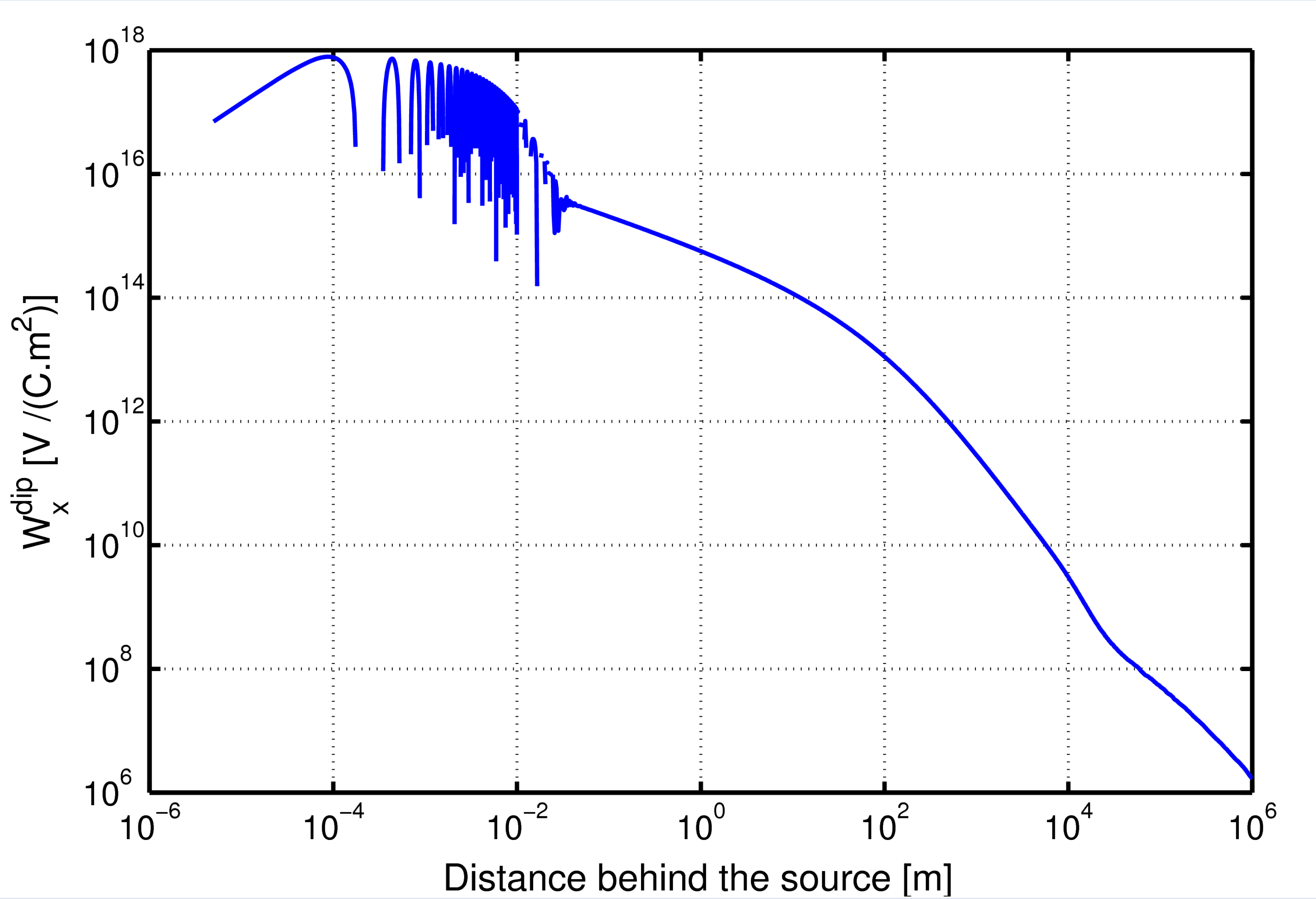
Example with $Z_x = 1/\sqrt{|\omega|}$

→ contrary to DFT, this method can correctly handle the different scales.



Wake function: results (axisymmetric)

- 25mm-thick graphite of radius 1.5 mm (surrounded by stainless steel):



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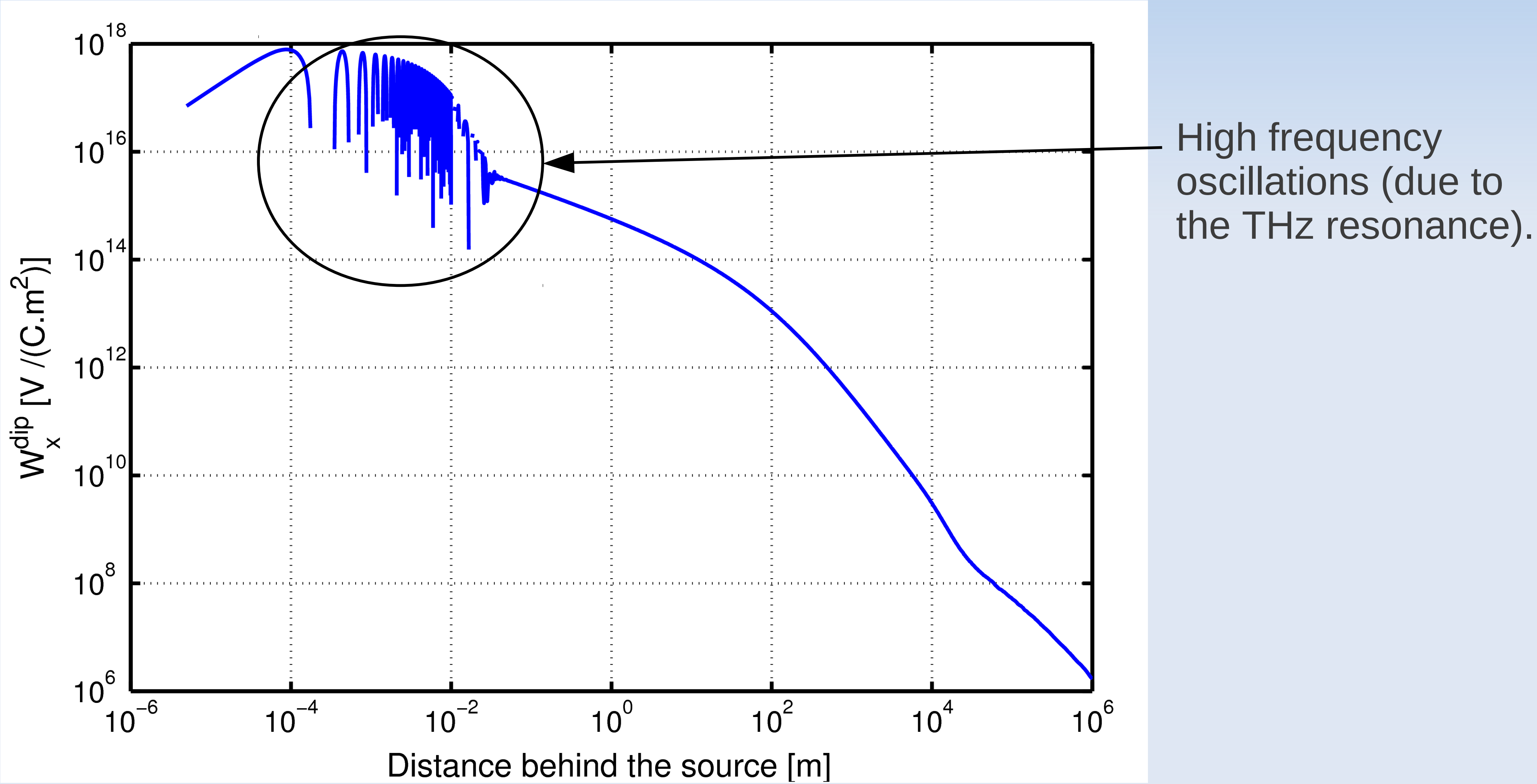

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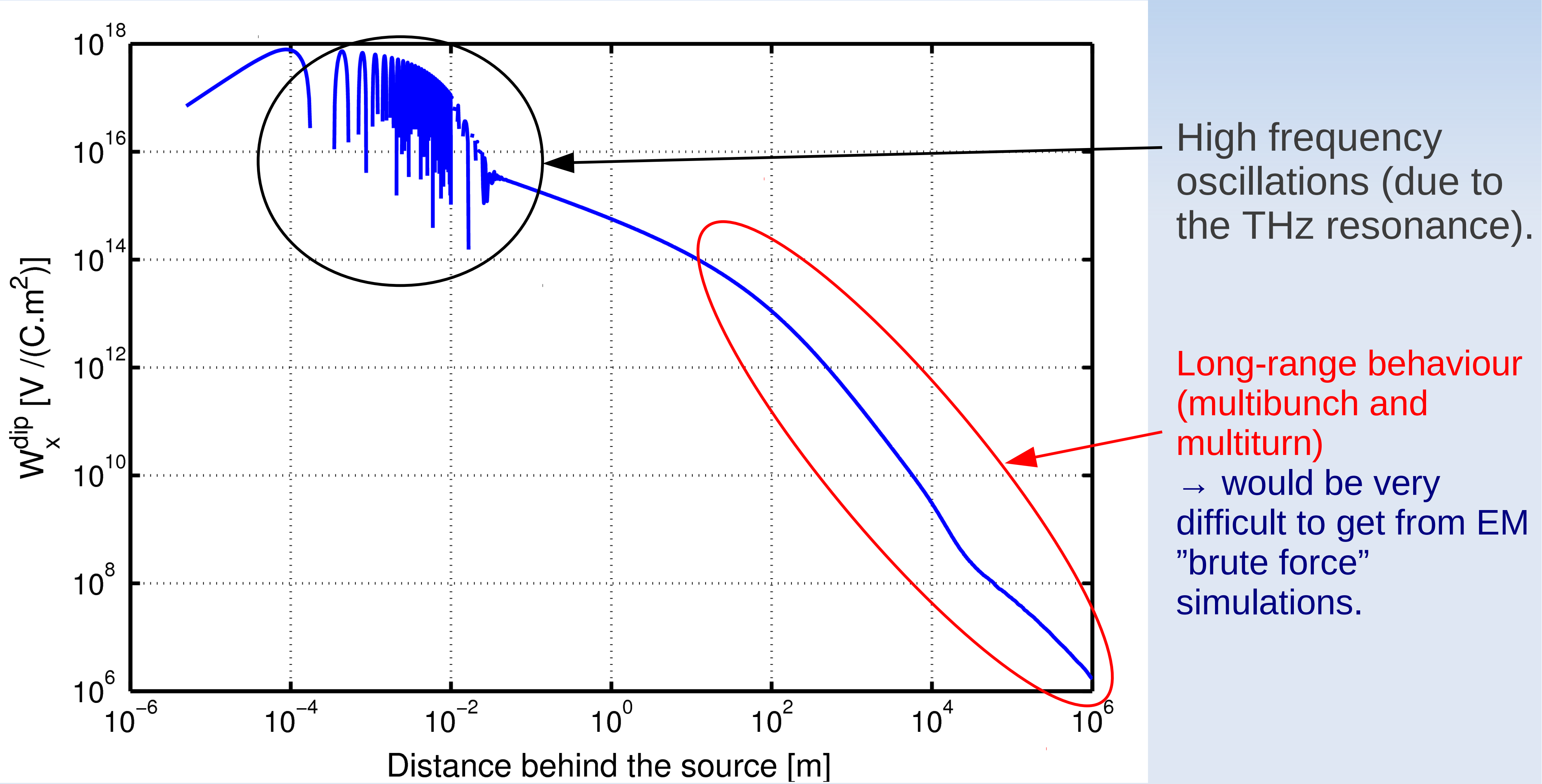

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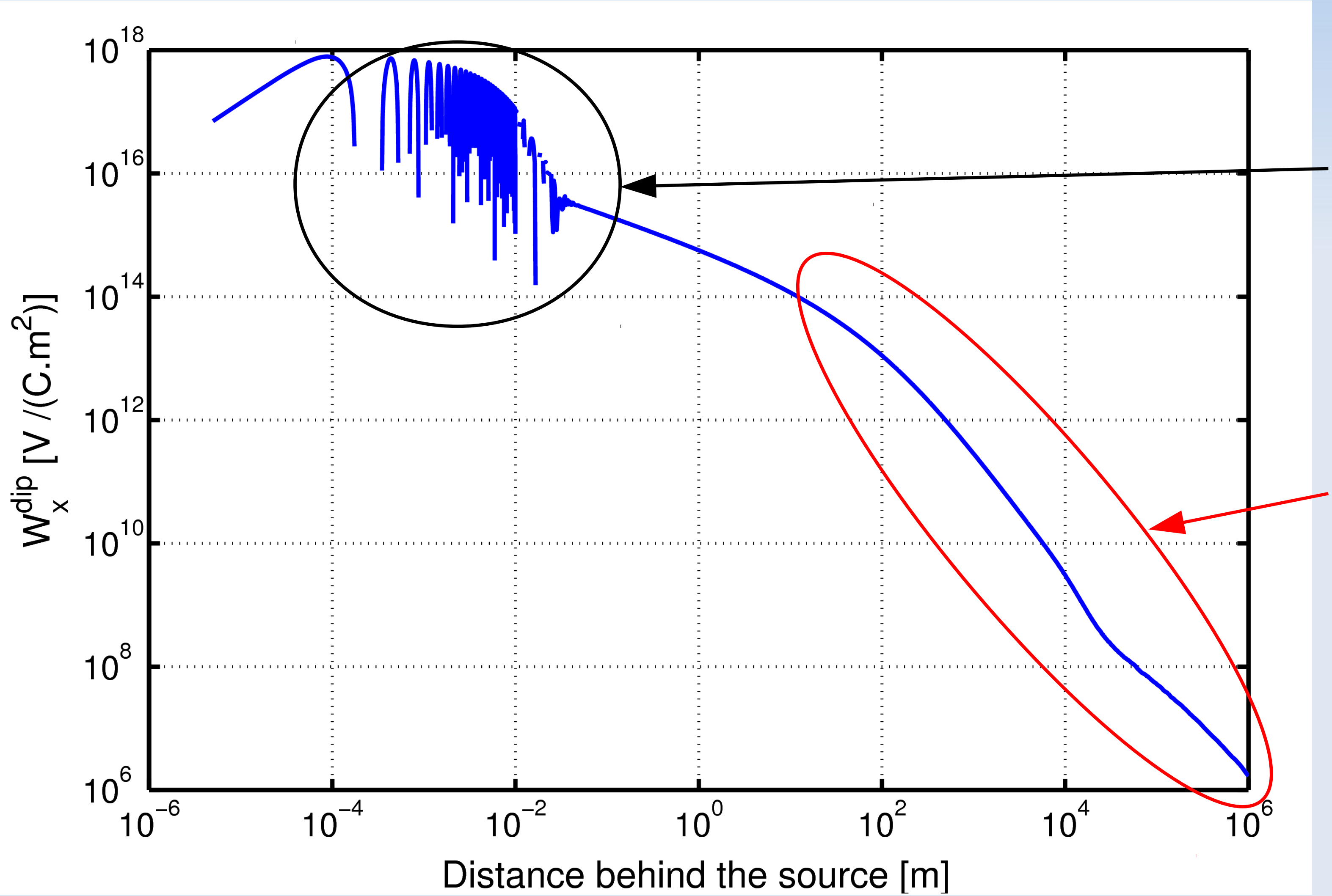

High frequency oscillations (due to the THz resonance).

Long-range behaviour (multibunch and multiturn)

→ would be very difficult to get from EM "brute force" simulations.

Wake in-front can also be computed.

Possible extensions

- What happens when source and test particles have different velocities ? e.g. let's look at radial force in axisymmetric chamber:

$$\mathrm{F}_r = q\left(E_r - v\mu_0 H_\theta\right) = \frac{jq\gamma^2}{k}\left(1-\beta^2\right)\frac{\partial E_s}{\partial r} = \frac{jq}{k}\frac{\partial E_s}{\partial r},$$

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⇒ Radial force now also depends on H_s and can be very different.

Example: AWAKE experiment at CERN (electron beam circulating in same chamber as proton beam)

Possible extensions

- What about multilayer elliptic chambers ?

  → in principle can be handled with the same ideas, using the following change of variables – or conformal map (Palumbo-Vaccaro *Il Nuevo Cimento* 89A:243, Gluckstern et al *Phys. Rev. E* 47:656, Piwinski *DESY* 94-068):

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⇒ provided all layer boundaries are ellipses with same eccentricities (confocal), we can in principle apply the same formalism as in cylindrical, replacing:

- cosines and sines by periodic Mathieu functions,
- modified Bessel functions by modified Mathieu functions,

and of course re-doing the decompositions & field matching accordingly...

Conclusions

- For multilayer axisymmetric chambers, Zotter formalism has been extended to all orders of non-linearity, and its implementation improved thanks to the matrix formalism for the field matching → the number of layers is no longer an issue.
- For multilayer flat chambers, a new theory similar to Zotter's has been derived, giving also impedances without any assumptions on the materials conductivity, on the frequency or on the beam velocity.
- Both these theories were benchmarked.
- An efficient algorithm to compute wake functions from impedances spreading over a large frequency range, using a non-equidistant discretization, has been developped.

⇒ Implemented in codes (Rewall, ImpedanceWake2D, available at http://impedance.web.cern.ch), used for various machines (LHC, HL-LHC, PS, SPS, CLIC, FCC,TLEP).

- Possible extensions could be investigated:
  - case with different source and test velocities (relatively easy),
  - multilayer elliptic chamber (more difficult).

Thank you for your attention !