Progress and challenges of high frequency diffraction limit in wakefield theory

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Outline of the talk

- Approximations in the theory of wakefields
- Optical model
- Parabolic equation (PE) for calculation of wakefields
- Scaling properties of the impedance in PE
- How to calculate wakefield of point charge in simulations?

I will talk about geometrical impedance assuming perfectly conducting walls.

Motivation: short bunches

RMS bunch lengths in future lepton accelerators

PEP-X	5 mm
CEPC	3 mm
TLEP-W	2.2 mm
ILC	300 μm
LCLS-II	25 μm

We have to compute wakefields for shorter and shorter bunches!

Calculation of wakefields is more difficult for long, small-angle tapers.

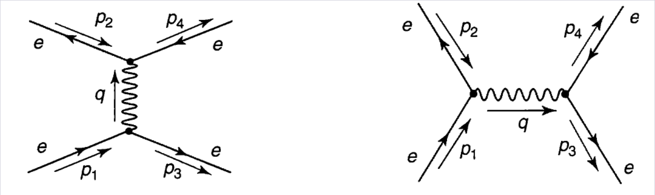
The difficulty is associated with a small parameter σ_z/b, where b is the typical size of the structure (in the vacuum chamber) that generates the impedance. On the other hand the small parameter allows us to develop approximate analytical theories and use them for numerical calculations.

Perturbation theory and precise calculations

There is nothing bad in using approximate equations!

Perturbation theory and precise calculations

There is nothing bad in using approximate equations!
Theory: QED
Small parameter: $\alpha \approx 1/137$
Relative precision of calculations: 10^{-11}



Optical approximation

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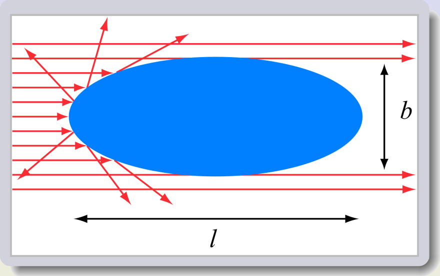
Optical approximation

The wake in bunch of length σ_z is formed by wavelengths $k \sim 1/\sigma_z$.



In electromagnetic theory the limit $k \to \infty$ corresponds to optics (the wavelength is much smaller than the size of the objects). Hence in the limit $\sigma_z \to 0$ there should be an analog of optical theory for wakefields.

A general theory of wakefields in optical approximation was developed in[1]. The advantage of this approach is that it allows to easily calculate the wakes for even 3D, non-axisymmetric geometries.

[1] Stupakov, Bane, Zagorodnov, PRST-AB **10**, 054401 (2007); Bane, Stupakov, Zagorodnov, PRST-AB **10**, 074401 (2007).

Impedance and wake in optical approximation

In the optical regime:

$Z_{\|}$ is real and independent of frequency; wake of a point charge $w_{\|} \propto \delta(z)$ and wake of a bunch with distribution $\lambda(z)$:

$$W_{\|}(z) \propto \lambda(z)$$

$Z_{\perp}$ is also real and depends on frequency as ω^{-1}; point charge wake $w_{\perp} \propto h(z)$, and wake of bunch distribution is

$$W_{\perp}(z) \propto \int^{z} \lambda(z')\, dz'$$

(Figure from [2]).

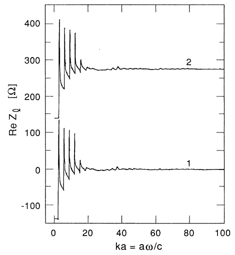


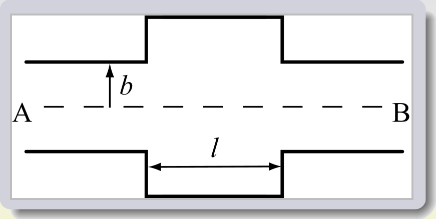
FIG. 10. The real part of the longitudinal coupling impedance of a cross section step as a function of parameter $ka = a\omega/c$; $b/a = 0.1$; the matrix size is 60×60; (1) $\mathrm{Re}Z_{\mathrm{in}}$, (2) $\mathrm{Re}Z_{\mathrm{out}}$.

The longitudinal impedance of a step transition does not depend on ω at high frequencies.

[2] Heifets, Kheifets, RMP, **63**, 631, 1991.

Limitations of the optical model

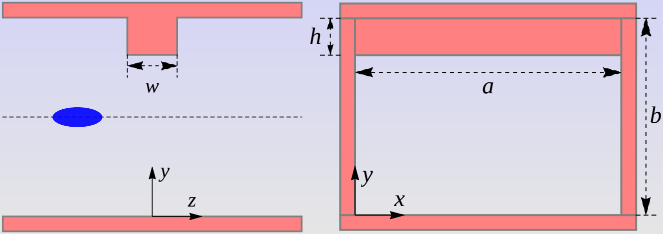
The optical theory neglects the diffraction effects. It predicts zero impedance for the pillbox cavity.



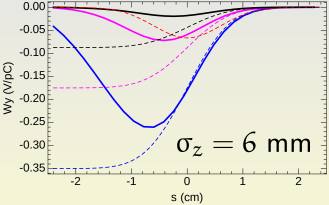
Pillbox cavity. Diffraction theory gives the longitudinal impedance for a cavity

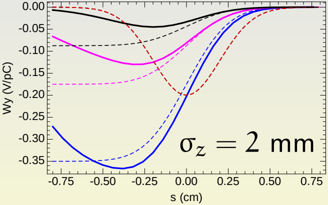
$$Z_{\parallel,\mathrm{diff}} = \frac{2(1+i)}{\pi^{1/2}}\sqrt{\frac{l}{cb^2\omega}}$$

Example: $\sigma_z = 6$ and 2 mm, transverse wake



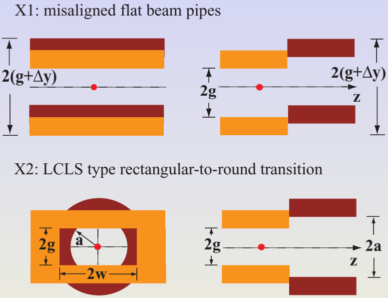
$a = 10$ cm
$b = 5$ cm
$w = 1$ cm

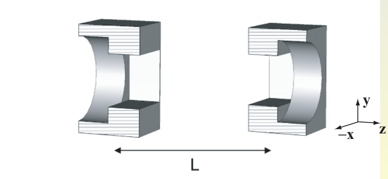




Depth h: 1 cm (blue), 0.5 cm (magenta) and 0.25 cm (blue). The dashed lines of the same color show the corresponding wakes computed with the optical model.

More Complicated Transitions





Cases considered:

- misaligned flat beam pipes
- LCLS rectangular-to-round transition

Cross-section view (left) and longitudinal view (right) of rectangular-to-round transition.

A pair of LCLS transitions in perspective view.

Parabolic equation

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Parabolic equation

The parabolic equation is used:

- In diffraction theory. Proposed by M. A. Leontovich in 1944. Applied to various diffraction problems by V. Fock in 40-50.
- In the FEL theory.
- To compute synchrotron radiation of relativistic particles in toroidal pipe[3].

Synchrotron radiation of relativistic particles can be treated using the parabolic equation [4].

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[3] Stupakov, Kotelnikov, PRST-AB **6**, 034401 (2003); Agoh, Yokoya, PRST-AB **7**, 054403 (2004).

[4] Geloni et al., DESY Report 05-032, (2005).

Parabolic equation

The Fourier transformed electric field $\hat{\mathbf{E}}$ and the longitudinal component of the current $\hat{j}_s$ are written with the additional factor e^{-iks}:

$$\hat{\mathbf{E}}(x, y, s, \omega) = e^{-iks} \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, \mathbf{E}(x, y, s, t)$$

$$\hat{j}_s(x, y, s, \omega) = e^{-iks} \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, j_s(x, y, s, t)$$

where $k \equiv \omega/c$. One also introduces the transverse component of the electric field $\hat{\mathbf{E}}_\perp$ as a two-dimensional vector $\hat{\mathbf{E}}_\perp = (\hat{E}_x, \hat{E}_y)$, and the longitudinal component of the electric field $\hat{E}_s$.

It is assumed that $\hat{\mathbf{E}}_\perp$ $\hat{j}_s$ are "slow" functions of s, such that $\partial/\partial s \ll k$. It means that we are interested in components of the field propagating in the positive direction of s at small angles to the axis. In particular, we neglect a part of the field propagating in the negative direction of s.

Parabolic equation

From the wave equation for the field it follows that [5]

$$\frac{\partial}{\partial s}\hat{E}_\perp = \frac{i}{2k}\left(\nabla_\perp^2 \hat{E}_\perp + \frac{2k^2 x}{R}\hat{E}_\perp - \frac{4\pi}{c}\nabla_\perp \hat{j}_s\right)$$

where $\nabla_\perp = (\partial/\partial x, \partial/\partial y)$, R is the radius of curvature (for a straight pipe $R^{-1} \to 0$, $s \to z$). The longitudinal electric field can be expressed through the transverse one and the current

$$\hat{E}_s = \frac{i}{k}\left(\nabla_\perp \cdot \hat{E}_\perp - \frac{4\pi}{c}\hat{j}_s\right)$$

A remarkable feature of this equation is that $\hat{E}_\perp$ varies in s over the distance much larger than $\lambda = k^{-1}$.
In contrast to the optical approximation PE takes into account diffraction effects (the pillbox impedance is derivable from PE). It is valid for high frequencies, and especially good for small-angle transitions.

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[5] G. Stupakov, New Journal of Physics **8**, 280 (2006); G. Stupakov, Reviews of Accelerator Science and Technology **3**, 3956 (2010).

Impedance scaling in PE

Analysis shows that the longitudinal impedance $Z_L(\omega)$ in a small-angle geometry (3D, in general), with characteristic length L in z-direction is

$$Z_L(\omega) = F\left(\frac{\omega}{L}\right)$$

Compute impedance for a short structure, $Z_{\frac{1}{n}L}$, and use the scaling law

$$Z_L(\omega) = Z_{L/n}\left(\frac{\omega}{n}\right)$$

Translating the impedance into the longitudinal wake we find

$$w_{L,\sigma_z}(s) = n w_{L/n, n\sigma_z}(ns)$$

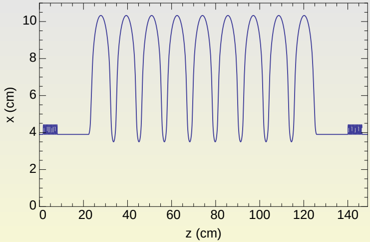
For the transverse wake

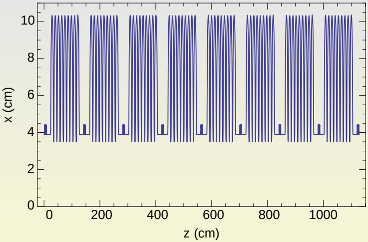
$$\boldsymbol{w}^{(t)}_{L,\sigma_z}(s) = \boldsymbol{w}^{(t)}_{L/n, n\sigma_z}(ns)$$

The computational time in 2D reduces by n^3.

Practical example of using the scaling property

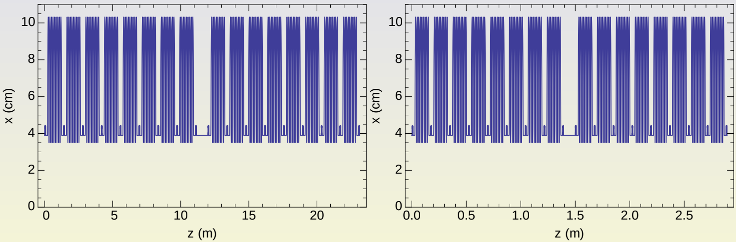
The nominal LCLS-II bunch length is $\sigma_z = 25\ \mu\text{m}$. The beam is accelerated in SC RF cavities, with a cryomodule housing 8 nine-cell cavities. The length of the cryomodule is ~12 m. It is important to calculate the cavity heating due to the energy deposited by the beam through the wakefield.





Practical example of using the scaling property

One wake was calculated with $\sigma_z = 25$ µm for two cryomodules (3.5 days run time), the other was calculated for $\sigma_z = 200$ µm in the cryomodule geometry shrunk 8 times longitudinally (40 min run time).

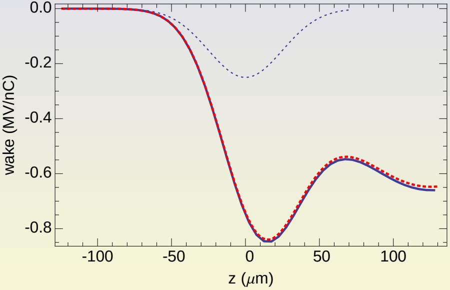


Real geometry (left) and scaled geometry (right).

Practical example of using the scaling property

Surprisingly, the scaling works very well for the cavities.

$$w_{\mathrm{L}}(s) = 8 w_{\frac{1}{8}\mathrm{L}}(8s)$$



After rescaling the results agree very well!

How to calculate wakefield of point charge in simulations?

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How to calculate wakefield of point charge in simulations?

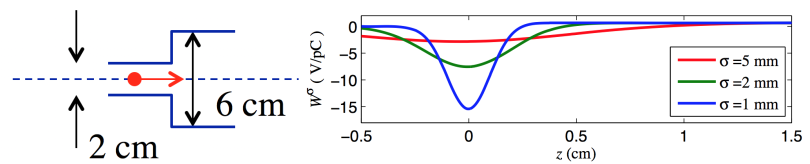
In paper[6] we suggested a method how to calculate short bunch wake-potentials, and even point-charge wakefields, running an EM solver for a relatively long bunch. This approach can save greatly on calculation speed and provides physics insights.

The idea behind the method is to use a combination of computer simulations with an analytical form of the wakefield for a given geometry in the limit $\sigma_z \to 0$.

[6] Podobedov, Stupakov, PRST-AB **16**, 024401 (2013)

Basic idea illustrated on step-out transition

Consider a particular example of the wake-potential of a short bunch passing through a step-out transition from radius r_{min} to r_{max}.



The plot of the wake-potential $W^\sigma(z)$ in this case, for several values of σ, is shown. With decreasing σ, the wake-potential becomes larger inside the bunch; in the limit $\sigma \to 0$, it diverges as $1/\sigma$. The singular part of the wake in the limit $\sigma \to 0$ is provided by the optical model.

Basic idea

In the limit $\sigma \to 0$ the optical model gives

$$W_s^\delta(z) = -\frac{Z_0 c}{\pi}\delta(z)\ln\frac{r_{max}}{r_{min}}$$

Using this as a Green function,

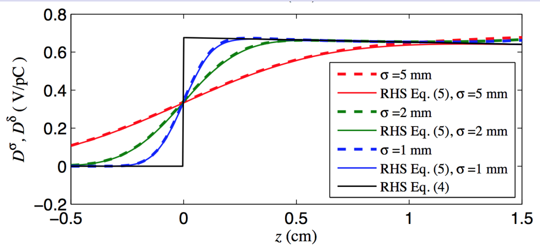
$$W_s^\sigma = -\frac{Z_0 c}{2^{1/2}\pi^{3/2}\sigma}\ln\frac{r_{max}}{r_{min}}e^{-z^2/2\sigma^2}$$

Subtracting it from the wake we introduce the difference

$$D^\sigma(z) = W^\sigma(z) - W_s^\sigma$$

Basic idea

Plot of $D^{\sigma}(z)$



When $\sigma \to 0$ this function approaches a well defined limit shown by the solid line. We denote this limit by $D^{\delta}(z)$,
$D^{\delta}(z) = \lim_{\sigma \to 0} D^{\sigma}(z)$.

Basic idea

In the vicinity of point $z = 0$ D^δ can be approximated

$$\mathrm{D}^\delta(z) \approx (\alpha + \beta z)h(z),$$

where $\mathrm{H}(z)$ is the step function ($h = 1$ for $z > 0$ and $h = 0$ otherwise). Then

$$\begin{aligned}
\mathrm{D}^\sigma(z) &= \int dz' \lambda(z + z')\mathrm{D}^\delta(z') \\
&= \frac{\alpha + \beta z}{2}\left(1 + \mathrm{erf}\left(\frac{z}{\sqrt{2}\sigma}\right)\right) + \frac{\beta\sigma}{\sqrt{2\pi}} e^{-z^2/2\sigma^2}
\end{aligned}$$

The crucial element of the method is that α and β can be obtained from simulations running a relatively long bunch through the system and fitting to the formula above.

Basic idea

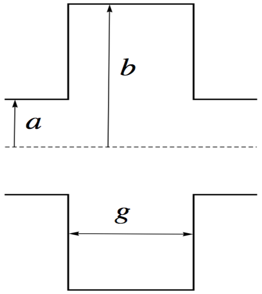
Comparing this with the simulated $\mathrm{D}^{\sigma}(z)$ in the region $z < 3\sigma$ one can find the parameters α and β and thus to establish the dependence of $\mathrm{D}^{\delta}(z)$ in this region. After $\mathrm{D}^{\delta}(z)$ is found, we have the wakefield of the point charge

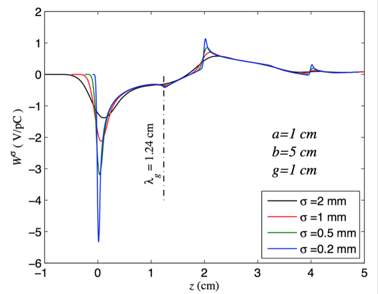
$$W^{\delta}(z) = W_{\mathrm{s}}^{\delta} + \mathrm{D}^{\delta}(z),$$

[note that W_{s}^{δ} is a delta-function].

The particular form of the singular part of the wake-potential, W_{s}^{δ}, is determined by the high-frequency limit of the impedance for a given geometry; in most cases it can be found in the literature.

Pill-box cavity





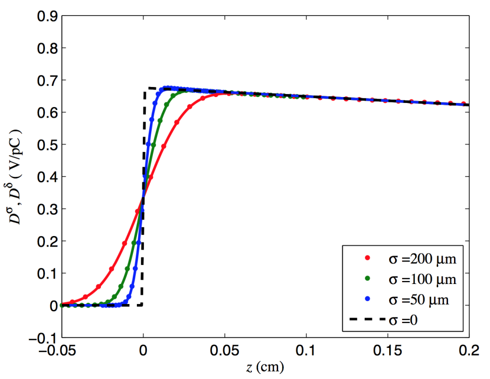
From the diffraction model, the singular part of the wake in the limit $\sigma \to 0$

$$W_s^\delta = \kappa z^{-1/2} h(z), \ \kappa = -\frac{Z_0 c}{\pi^2 a}\sqrt{\frac{g}{2}}$$

$$W_s^\sigma = \frac{\kappa}{\sigma^{\frac{1}{2}}} f\left(\frac{z}{\sigma}\right), \ f(s) = e^{-\frac{s^2}{4}}\sqrt{\frac{\pi|s|}{8}}\left[I_{-\frac{1}{4}}\left(\frac{s^2}{4}\right) + \mathrm{sign}(s) I_{\frac{1}{4}}\left(\frac{s^2}{4}\right)\right]$$

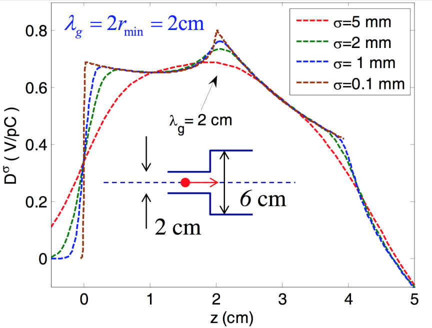
Pill-box cavity

Cavity wake with the singular part subtracted, $D^\sigma(z)$. One can see the same step-like structure $D^\delta(z) \approx (\alpha + \beta z)h(z)$ in the limit $\sigma \to 0$.

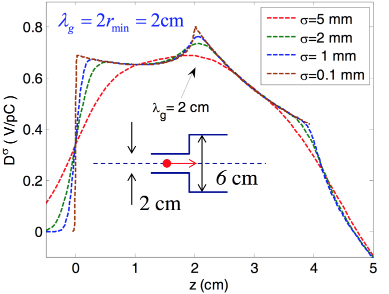


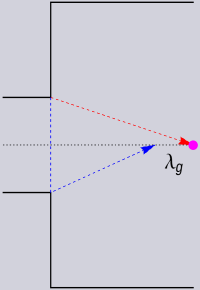
How to choose σ for simulations? λ_g parameter

The representation $D^{\delta}(z) \approx (\alpha + \beta z)H(z)$ is valid in some vicinity of $z = 0$. For many geometries there exists a limiting length, which we denote by λ_g, beyond which this representation cannot be extended. In some cases the wake has another singularity at $z = \lambda_g$, in other cases it can be a discontinuity of the wake or a singularity of its derivative with respect to z.



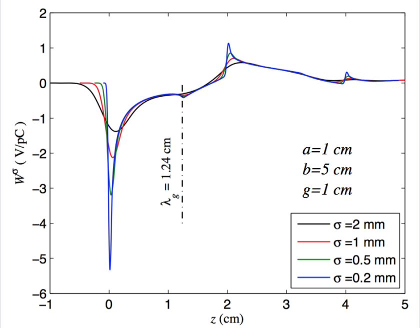
Explanation of spikes on the wake

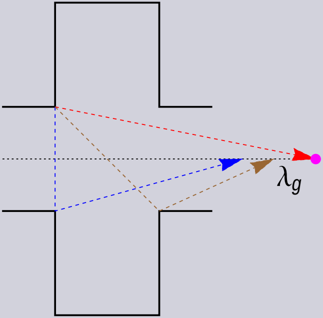




Multiple reflections from the angles of the structure lead to secondary wake spikes.

Pill-box cavity

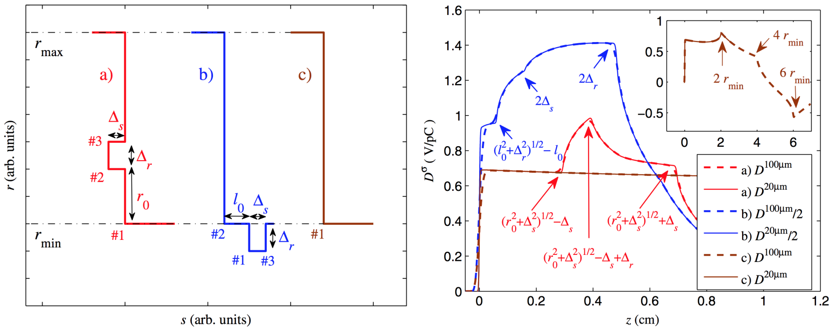




For this example $\lambda_g = \sqrt{(2a)^2 + g^2} - g = 1.24$ cm.

More complicated geometries

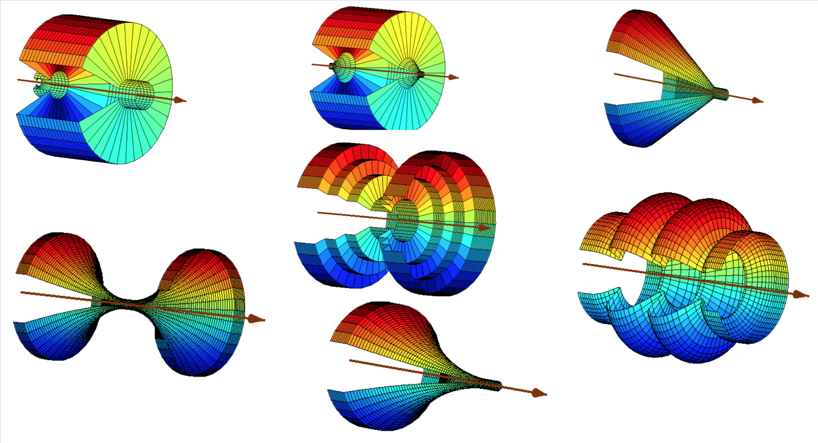
More complicated structures can be analyzed in a similar way.



How it all works together

- Determine analytical singular wake model: $W_s^{\delta}(z) \& W_s^{\sigma}(z)$
- Determine λ_g
- Calculate the wake-potential with your favorite EM solver for $\sigma_0 \ll \lambda_g$: $W_{\mathrm{ECHO}}^{\sigma_0}(z)$
- Subtract the singular wake: $D^{\sigma_0}(z) = W_{\mathrm{ECHO}}^{\sigma_0}(z) - W_s^{\sigma_0}(z)$
- Fit the remainder, $D^{\sigma_0}(z)$, with the function (fit range $|z/\sigma_0| < 3$ works well): $\frac{\alpha+\beta z}{2}\left(1+\mathrm{erf}\left(\frac{z}{\sqrt{2}\sigma_0}\right)\right) + \frac{\beta\sigma_0}{\sqrt{2\pi}} e^{-z^2/2\sigma_0^2}$
- Short-bunch wake (for arb. $\sigma < \sigma_0$) is then $W^{\sigma}(|z| < 3\sigma_0) = \frac{\alpha+\beta z}{2}\left(1+\mathrm{erf}\left(\frac{z}{\sqrt{2}\sigma}\right)\right) + \frac{\beta\sigma}{\sqrt{2\pi}} e^{-z^2/2\sigma^2} + W_s^{\sigma}(z)$ and $W^{\sigma}(|z| > 3\sigma_0) = W_{\mathrm{ECHO}}^{\sigma_0}(z)$
- For point-charge: $W^{\delta}(|z| < 3\sigma_0) = (\alpha + \beta z)h(z) + W_s^{\delta}(z)$ and $W^{\sigma}(|z| > 3\sigma_0) = W_{\mathrm{ECHO}}^{\sigma_0}(z)$

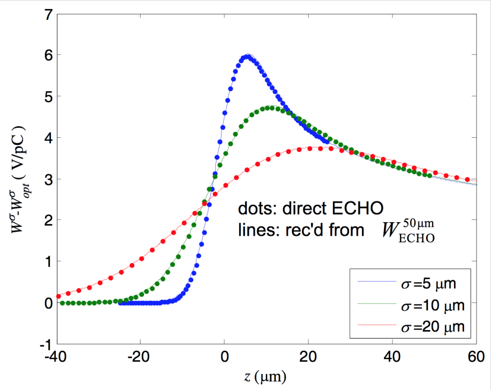
The method was applied to many geometries

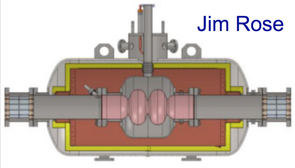


It worked well for all of them.

Practical example: NSLS-II Landau cavity

- 1.5 GHz dual cell cavity, $r_{\text{side pipe}} = 6$ cm
- Final results for the short-range wakes:



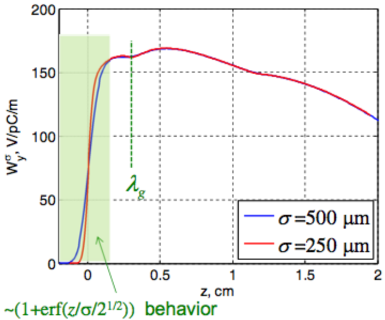


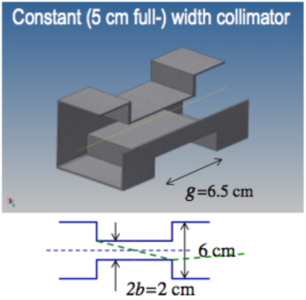
To find 10 μm bunch wake: Brute force: 480 hours of Intel(R) Xeon(R) 5570@2.93 Ghz CPU to $z_{max} = 1$ cm. Our method: uses only $\sigma = 50$ μm bunch, saves a factor of 5^3 on CPU time and 5^2 on memory. Gives a model of the point-charge wake as a bonus.

3D

3D Example

Wakes by I. Zagorodnov,
3D ECHO + CST mesher





$$\lambda_g = \sqrt{(2b)^2 + g^2} - g = 3\ \text{mm}$$

- Observe λ_g, where expected
- Expected behavior near the origin; can easily fit point-charge wake α & β
- Same applies for logitudinal (+ optical model), and for quadrupolar wakes

Conclusions

- For large and smooth accelerator structures, and short bunches, direct EM solver calculations can be extremely time and memory-consuming. Using approximate methods that employ the small geometric parameters greatly facilitates the numerical problem.
- We developed a new method to accurately obtain wakefields of short bunches, including that of a point-charge, by combining a (processed) long-bunch wake from an EM solver and a singular analytical wake model.
- We showed that this method often provides great savings in computing time required to calculate wake-potentials due to very short bunches. The method resolves an important practical question, as to how short of a bunch one needs to use in an EM solver, so that shortening this bunch further would not result in any new information about the wake.