SEMI-ANALYTICAL APPROACHES TO MAGNET DESIGN

F. Borgnolutti, P. Fessia, F. Regis, L. Rossi, E. Todesco, CERN, Geneva, Switzerland

Abstract

We summarize the equations that have been derived in the past three years to evaluate the short sample field, stresses and forces, and the magnetic energy in superconducting quadrupoles and dipoles. We present the equations, which are based on simplified sector coils and on empirical corrective factors. The agreement with realistic coil lay-outs and the validity limits of this approach are discussed.

INTRODUCTION

During the conceptual phase of design of an accelerator or of part of it, it can be useful to have equations providing the expected level of field, coil width, stress and stored energy in an accelerator magnet using a given technology. In this paper we summarize the results of several works that have been carried out along this direction in the past years [1-5]. With respect to previous works [6,7] we focus ourselves on a sector coil rather than on a cosθ or cos2θ lay-out.

These equations are usually precise within 10% in a rather wide validity range, and can be used for a fast exploration of different solutions without the need of going for the complete magnet design. A relevant spin-off of these equations is also that they provide a benchmark for judging the efficiency of a coil design.

In the first section we present the usual linear fit for the Nb-Ti critical surface, and a novel hyperbolic fit for the Nb_3Sn that, contrary to the Summer law, allows to derive explicit equations for the short sample quantities. In the second section we derive the short sample field, gradient and current for dipoles and quadrupoles. We then give an estimate of the stress, and finally of the magnetic energy.

CRITICAL SURFACES

Nb-Ti critical surface is usually described by the Fitz-Webb [8] fit

$$j_{sc,c}(B) = C(T,\varepsilon)B^{\alpha-1}\left(1-\frac{B}{B^*_{c2}(T)}\right)^{\beta}. \qquad (1)$$

A linear fit

$$j_{sc,c}(B) = c(b-B), \qquad (2)$$

corresponding to a pinning force parabolic in B, is accurate within a few percent in most of the range which is currently used in superconducting magnets (5 T to 9 T at 4.2 K, 6 T to 12 T at 1.9 K, see Fig. 1). Typical values of the fitting constants are b=10 T at 4.2 K, 13 T at 1.9 K, and c=6×10^8 A/(Tm^2). The linear fit has the advantage of providing explicit solutions to the short sample field and current, which are given by the intersection of the loadline with the critical surface.

Nb_3Sn critical surface is usually described by the Kramer [9] fit

$$j_{sc,c}(B) = \frac{C(T,\varepsilon)}{\sqrt{B}}\left(1-\frac{B}{B^*_{c2}(T,\varepsilon)}\right)^2, \qquad (3)$$

which corresponds to the Fietz-Webb fit with α=0.5 and β=2. An hyperbolic fit

$$j_{sc,c}(B) = c\left(\frac{b}{B}-1\right) \qquad (4)$$

works well (within a few percent) in the interesting region for superconducting accelerator magnets, namely in the range of 8-17 T. see Fig. 1. Typical values of the fitting constants for a high current density conductor carrying 3000 A/mm^2 at 4.2 K and 12 T are b=21 T at 4.2 K, 23 T at 1.9 K, and c=4×10^9 A/m^2. The hyperbolic fit corresponds to a linear pinning force

$$j_{sc,c}(B)B = c(b-B). \qquad (5)$$

Also in this case, the hyperbolic fit has the advantage of providing explicit solutions to the short sample values for current and fields.

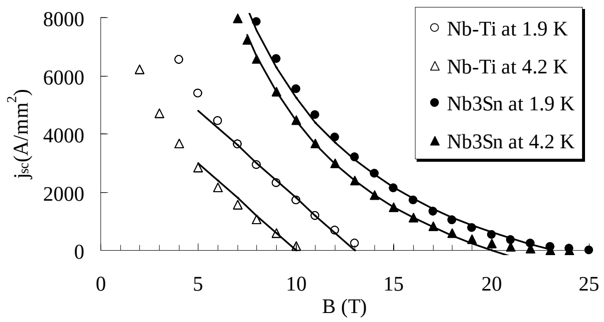

Fig. 1 Critical surface of Nb-Ti and Nb_3Sn according to Fitz-Webb and Kramer fit (markers) and linear and hyperbolic fits (solid lines).

In the following we will use sector coil models, where the coil is made up of insulated cables; therefore we have to rely on the usual definition of engineering current density

$$j_c(B) = \kappa c(b-B) \qquad (6)$$

$$j_c(B) = \kappa c\left(\frac{b}{B}-1\right), \qquad (7)$$

where the filling factor κ takes into account of the dilution of the superconductor present in the insulated coil. This is mainly given by the Cu/non-Cu ratio, plus the voids in the cable, and the contribution of the insulation. The factor κ ranges in between 0.25 to 0.35 in most of the cables used in accelerator magnets.

COIL LAY-OUTS AND EQUIVALENT WIDTH

Throughout the paper, all the equations will be derived for a lay-out based on a simple sector coil, in most cases of 60° angular width for a dipole (30° for a quadrupole). Indeed, the coil lay-outs feature several sectors and possibly several layers, with wedges to optimize field quality (see Fig. 2, left). In order to be able to apply our results to a generic coil made up of blocks and layers, we convert the total surface of the coil A to an equivalent width according to the equation

$$w_{eq} \equiv \left(\sqrt{1+\frac{3A}{2\pi r^2}} - 1 \right) r , \qquad (8)$$

where r is the aperture radius. The equivalent coil width is the width of the coil assuming that all the cables would fill a 60° sector of radial width w_{eq} (30° for a quadrupole), see Fig. 2. All lengths are given in meters.

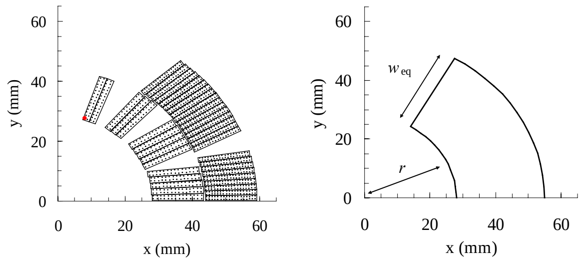

Fig. 2 Coil cross-section of the LHC dipole (one quarter, left side) and equivalent coil with the same area and inner radius (one quarter, right side).

SHORT SAMPLE FIELD AND GRADIENT

Dipoles

According to the approach developed in [1], the short sample field in T for a Nb-Ti dipole is given by

$$B_{ss} \sim \frac{\kappa c b \gamma_0 w_{eq}}{1+\kappa c \gamma_0 \left(w_{eq} + a r \right)} , \qquad (9)$$

where the constants values are γ_0=6.63×10^{-7} Tm/A and a=0.04. The short sample current in A/m^2 is given by

$$j_{ss} \sim \frac{\kappa c b}{1+\kappa c \gamma_0 \left(w_{eq} + a r \right)} . \qquad (10)$$

In the Nb_3Sn case one has

$$B_{ss} \sim \frac{\kappa c \gamma_0 w_{eq}}{2} \left(\sqrt{\frac{4b}{\kappa c \gamma_0 (w_{eq} + a r)} + 1} - 1 \right) \qquad (11)$$

$$j_{ss} \sim \frac{\kappa c}{2} \left(\sqrt{\frac{4b}{\kappa c \gamma_0 (w_{eq} + a r)} + 1} - 1 \right) , \qquad (12)$$

where the values of the constant γ_0 and a are the same as for Eq. (9), whereas b and c are the parameters of the Nb_3Sn hyperbolic fit (4).

The above equations rely on an empirical fit of the ratio λ between peak field and central field

$$\lambda = 1 + a \frac{w}{r} , \qquad (13)$$

which has been derived for a sector coil. The fitting value a=0.04 implies that for a coil width equal to the aperture radius, the field in the coil is 4% larger than the central field. This estimate is valid for w_{eq}/r>0.15. For very thin coils or very large apertures, giving rise to a relevant difference between peak field in the coil and central field, the estimate is not applicable.

The above estimates are valid for the coil without grading and without iron. Iron has the main effect of allowing to get the same field with a lower current density; this effect can be estimated according to the standard formulas [1]. Moreover, iron has some impact on the short sample field. This effect becomes small (a few percent) for w_{eq}/r>0.5. Graded coil mainly allow to reach the same field with less coil. They also allow a larger short sample; for accelerator magnets a gain of up to 5% has been obtained [1].

Quadrupoles

According to the approach developed in [2], the short sample gradient for a Nb-Ti quadrupole is given by

$$G_{ss} \sim \frac{\kappa c b \gamma_0 \log\left(1+\frac{w_{eq}}{r}\right)}{1+\kappa c r \left(a_{-1} \frac{r}{w_{eq}} + 1 + a_1 \frac{w_{eq}}{r} \right) \gamma_0 \log\left(1+\frac{w_{eq}}{r}\right)} \qquad (14)$$

where the constants values are γ_0=6.63×10^{-7} Tm/A as for the dipoles, and a_{-1}=0.04 and a_1=0.11. The short sample current in (A/m^2) is given by

$$j_{ss} \sim \frac{\kappa c b}{1+\kappa c r \left(a_{-1} \frac{r}{w_{eq}} + 1 + a_1 \frac{w_{eq}}{r} \right) \gamma_0 \log\left(1+\frac{w_{eq}}{r}\right)} \qquad (15)$$

In the Nb_3Sn case one has

$$G_{ss} \sim \frac{\kappa c \gamma_0 \log\left(1+\frac{w_{eq}}{r}\right)}{2} \left(\sqrt{\frac{4b}{\left(a_{-1} \frac{r}{w_{eq}} + 1 + a_1 \frac{w_{eq}}{r} \right) \gamma_0 \log\left(1+\frac{w_{eq}}{r}\right) r \kappa c} + 1} - 1 \right) \qquad (16)$$

$$\boldsymbol{j_{ss}} \sim \frac{\kappa \boldsymbol{c} \gamma_0}{2} \left(\sqrt{\frac{4\boldsymbol{b}}{\left(\boldsymbol{a}_{-1} \frac{\boldsymbol{r}}{\boldsymbol{w_{eq}}} + 1 + \boldsymbol{a}_1 \frac{\boldsymbol{w_{eq}}}{\boldsymbol{r}} \right) \gamma_0 \log\left(1+\frac{\boldsymbol{w_{eq}}}{\boldsymbol{r}}\right) \boldsymbol{r} \kappa \boldsymbol{c}} + 1} - 1 \right) \qquad (17)$$

where the values of the constant γ_0 and a_1, a_{-1} are the same as for Eq. (14), and b and c are the parameters of the Nb_3Sn hyperbolic fit (4).

The above equations rely on an empirical fit of the ratio λ between peak field and gradient times aperture

$$\lambda = \boldsymbol{a}_{-1} \frac{\boldsymbol{r}}{\boldsymbol{w}} + 1 + \boldsymbol{a}_1 \frac{\boldsymbol{w}}{\boldsymbol{r}} , \qquad (18)$$

which has been derived for a sector coil. The fitting value a_1=0.04 and a_{-1}=0.11 implies that for a coil width equal to the aperture radius, the field in the coil is 15% larger than the central field. As in the dipole case, the estimate is valid for w_{eq}/r>0.15. For very large coils w_{eq}/r>1 the sector coil estimate can be pessimist, i.e. one can find other lay-outs where λ does not diverge (see [2] for more details).

The above equations neglect both iron and grading. Analysis of several lay-outs has shown that graded coils can give short sample gradient up to 10% larger than what given in (14) and (16). For the iron influence, one can apply the same considerations made for the dipoles.

FORCES AND STRESSES

Dipoles

According to the approach developed in [3], the stress in the midplane for a dipole made up of a sector coil is estimated by

$$\sigma_\varphi \sim \frac{\sqrt{3}}{6\pi} j^2 \mu_0 \mathrm{Max}_{\rho\in[r,r+w_{eq}]}\left[\frac{3\rho^2(r+w_{eq})-r^3-2\rho^3}{\rho}\right] \tag{19}$$

where the variable ρ spans over the coil midplane, i.e. from r to $r+w_{eq}$. The expression accounts for the usual part dependent on the square of the current, plus a geometric part which is the argument of the Max function. To get the stress in short sample conditions one has to substitute the expressions (10) for Nb-Ti and (12) for Nb_3Sn. The equation is derived by keeping the first order terms of the expression for a 60° sector coil. In general, coil lay-outs with a relevant difference between the angular width of the first and the second layer feature a larger stress. The equation is derived for a coil without iron and without grading.

Quadrupoles

A similar method has been developed in [4], thus leading to the estimate of the forces in the midplane of a quadrupole

$$\sigma_\varphi \sim \frac{\sqrt{3}}{16\pi} j^2 \mu_0 \mathrm{Max}_{\rho\in[r,r+w_{eq}]}\frac{1}{\rho^2}\left[\rho^4 - r^4 + 4\rho^4 \ln\left(\frac{r+w_{eq}}{\rho}\right)\right] \tag{20}$$

Also in this case, the variable ρ spans over the coil midplane, i.e. from r to $r+w_{eq}$, and the expression accounts for a part dependent on the square of the current and a geometric part. To get the stress in short sample conditions one has to substitute the expressions (14) for Nb-Ti and (16) for Nb_3Sn. The same considerations given for the dipoles apply to the quadrupole case.

MAGNETIC ENERGY

Quadrupoles

According to [5], the magnetic energy in a quadrupole of aperture radius r and equivalent coil width w, with a current density j is given by

$$U \sim \frac{3\mu_0 j^2}{16} r^4 \left(\left[\left(1+\frac{w_{eq}}{r}\right)^4 - 1\right] - 4\ln\left(1+\frac{w_{eq}}{r}\right)\right) \tag{21}$$

This expression is independent of the conductor type, which only affects j. The current density at short sample can be computed using the estimates (12) or (14). The equation is derived for an ironless case with no grading. It is based on a Fourier analysis of the current density, keeping the first harmonics, i.e. the $\cos 2\theta$ component. For graded coils one can apply the corrective term

$$U_{grad} = U\left[d\left(\left(1+\frac{A_g}{A}(g-1)\right)^2 - 1\right)+1\right] \tag{22}$$

where g is the grading, i.e. the ratio between the current densities in the two layers, A_g is the area of the graded coil, A the total area of the coil, and d=1.16 is a fitting constant. In presence of unsaturated iron, the energy is enhanced by the following factor

$$\frac{U_{iron}}{U} = 1+\left(\frac{r}{r_i}\right)^4 \frac{(1+t)^2\left[(1+t)^4-1\right]\ln(1+t)}{\frac{1}{2}\left[(1+t)^4-1\right]-2\ln(1+t)} \tag{42}$$

where $t=w_{eq}/r$, and r_i is the iron radius. The equation (21) with the two corrective factors (22) and (23) allow estimating the magnetic energy (and the inductance) with a precision of the order of 10% (calculations done on a set of lay-outs found in the literature, see [7]).

SUMMARY OF PARAMETERS AND CONSTANTS

The set of equations given in the previous sections include constants and parameters. We give a summary of the notations to ease the reader in their implementation.

Magnet parameters: aperture radius r, in meters.

Coil parameters:

- Area A: surface of the coil in m^2.
- Equivalent width w_{eq}, in meters, defined according to Eq. (8) as the width of the coil which has the same area A of a 60° (30° for quadrupoles) sector.
- Graded area A_g: surface of the graded coil in m^2 (only for magnetic energy).
- Iron radius r_i, in meters (only for magnetic energy equations).

Cable parameters:

- Filling ratio κ. It is the fraction of non-Cu present in the area of the insulated coil.
- Grading g: ratio between current density in the outer and in the inner layer (only for magnetic energy equations).

Superconductor parameters:

- c: is related to the slope of the critical surface, $c=6\times10^8$ A/(Tm^2) for the Nb-Ti and $c=4\times10^9$ A/m^2 for the Nb_3Sn. Please note that units are different for the two materials.

- b: extrapolation of the critical field, in T, according to the fit, b=10 or 13 T at 4.2 or 1.9 K for Nb-Ti, b=21 or 23 T at 4.2 or 1.9 K for Nb_3Sn

Constants:

- $\gamma_0=6.63\times10^{-7}$ Tm/A is a constant related to the field given by a sector coil through an integration of Biot-Savart equations
- a, a_{-1}, a_1 are constants used for the empirical fitting of the ratio peak field/central field (or gradient times aperture for quadrupoles). For dipoles a=0.04, for quadrupoles a_{-1}=0.04 and a_1=0.11.
- d=1.16 is a constant derived in the empirical fitting to take into account for the effect of grading in magnetic energy (only for magnetic energy equations).
- $\mu_0=4\pi\ 10^{-7}$ is the permeability constant.

CONCLUSION

In this paper we summarized equations giving an estimate of the short sample current, field and gradient, midplane stress and magnetic energy in superconducting dipoles and quadrupoles. The equations are based on sector coils and make use of a semi-analytical approach. Empirical fittings are used in some case as well as corrective factors to include grading and iron. The validity limits are outlined.

These equations can be used both for a fast exploration of the parameter space in the phase of a conceptual design, or for benchmarking realistic coil lay-outs based on Rutherford cables.

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