Exact solution of the (0+1)-dimensional Boltzmann equation for a massive gas

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Hot Quarks 2014

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Outline

- Motivation
  - Successes of viscous hydrodynamics in description of relativistic heavy-ion collisions — intensive studies of transport coefficients
  - Our idea is to perform comparisons of exact solutions of simple kinetic equations with hydro approaches, which allows us to select correct forms of these coefficients
- Kinetic equation
  - Boltzmann equation
  - Boost-invariant variables
  - Moments of the equation $\rightarrow$ Landau matching
  - Numerical method
- Results
  - Time dependence of thermodynamics-like variables
  - Bulk viscousity
  - Shear viscosity
- Quantum statistics
- Conclusions

Motivation

- Experimental and theoretical studies of heavy-ion collisions showed that the behavior of matter produced in such collisions is very well described by viscous hydrodynamics, with a very small viscosity to entropy density ratio
- These results brought a lot of attention to the studies of kinetic coefficients whose values determine the magnitude of important observables such as the elliptic flow
- Interestingly, different theoretical methods lead to different values of the kinetic coefficients
- Moreover, the form of the second order hydrodynamic equations depends on the specific values of the kinetic coefficients

Motivation

- Our idea is to perform comparisons of exact solutions of simple kinetic equations with hydrodynamic approaches — this allows for numerical determination of the kinetic coefficients
- Instead of performing complicated simulations based on the Boltzmann equation we analyze its simple form which can be solved exactly (Baym, Phys. Lett. **B138**, 18 (1984).)
- We extend here some of the recent results obtained for massles particles:

  W. Florkowski, R. Ryblewski, M. Strickland, Phys. Rev. **C88** (2013) 024903

  W. Florkowski, R. Ryblewski, M. Strickland, Nucl. Phys. **A916** (2013) 249

Motivation

LIMITATIONS OF OUR MODEL:

- Collision term treated in the relaxation time approximation (RTA) with a constant equilibration time
- Only longitudinal expansion included (along the z-axis) — justified for early stages of the evolution (1–2 fm/c)
- Boost invariance — justified in the central region ($z \approx 0$)
- All particles have the same mass m

Motivation

ADVANTAGES OF OUR MODEL:

- We find exact solutions of the kinetic equation numerically
- We find the proper forms of shear and bulk viscosities by studying the system's approach towards equilibrium

Kinetic equation
General setup

- Boltzmann equation (BE) in the relaxation-time approximation (RTA)

$$p^\mu \partial_\mu G(x,p) = C[G(x,p)] \qquad C[G] = p \cdot u \, \frac{G^{\rm eq} - G}{\tau_{\rm eq}}$$

  background thermal distribution

$$G^{\rm eq} = \frac{2}{(2\pi)^3} \exp(-p \cdot u / T)$$

- boost-invariant variables (Bialas, Czyz)

$$w = tp_\parallel - zE \qquad v = tE - zp_\parallel = \sqrt{w^2 + \left(m^2 + \vec{p}_\perp^{\,2}\right)\tau^2}$$

$$E = \frac{vt + wz}{\tau^2} \qquad p_\parallel = \frac{wt + vz}{\tau^2}$$

- boost-invariant form of the kinetic equation

$$\frac{\partial G}{\partial \tau} = \frac{G^{\rm eq} - G}{\tau_{\rm eq}}$$

$$G^{\rm eq}(\tau, w, p_\perp) = \frac{2}{(2\pi)^3} \exp\left[-\frac{\sqrt{w^2 + (m^2 + p_\perp^2)\tau^2}}{T(\tau)\tau}\right]$$

Kinetic equation
Moments

- zeroth moment (describes particle production in our case)

$$\partial_\mu \int dP\, p^\mu G = \int dP\, C \qquad \frac{dn}{d\tau} + \frac{n}{\tau} = \frac{n^{\mathrm{eq}} - n}{\tau_{\mathrm{eq}}}$$

- first moment (describes energy-momentum conservation)

$$\partial_\mu \underbrace{\int dP\, p^\nu p^\mu G}_{} = \int dP\, p^\nu C = 0 \qquad \frac{d\mathcal{E}}{d\tau} = -\frac{\mathcal{E} + \mathcal{P}_\parallel}{\tau}$$

$$T^{\mu\nu} = (\mathcal{E} + \mathcal{P}_\perp) u^\mu u^\nu - \mathcal{P}_\perp g^{\mu\nu} + (\mathcal{P}_\parallel - \mathcal{P}_\perp) V^\mu V^\nu$$

$$u^\mu = \left(\frac{t}{\tau}, 0, 0, \frac{z}{\tau}\right) \qquad V^\mu = \left(\frac{z}{\tau}, 0, 0, \frac{t}{\tau}\right)$$

- Landau matching

$$\int dP\, p^\nu C = 0$$

- 0th and 1st moments are fulfilled automatically for the exact solution of BE

Kinetic equation
Landau matching

- Landau matching allows us to find effective temperature T

$$\begin{aligned}\mathcal{E}(\tau) &= \mathcal{E}^{\rm eq}(\tau)\\ \mathcal{E}(\tau) &= \frac{g_0}{\tau^2}\int dP\, v^2\, G(\tau, w, p_\perp)\\ &= \frac{g_0}{\tau^2}\int dP\, v^2\, G^{\rm eq}(\tau, w, p_\perp)\\ &= \frac{g_0 T m^2}{\pi^2}\left[3TK_2\left(\frac{m}{T}\right) + mK_1\left(\frac{m}{T}\right)\right]\end{aligned}$$

- In the limit of vanishing particle masses:

$$\frac{g_0 T m^2}{\pi^2}\left[3TK_2\left(\frac{m}{T}\right) + mK_1\left(\frac{m}{T}\right)\right] \xrightarrow[m=0]{} \frac{6 g_0 T^4}{\pi^2}$$

Kinetic equation
Formal solution

- formal structure of the solutions (Baym, Phys. Lett. **B138**, 18 (1984).)

$$G(\tau, w, p_\perp) \quad = \quad D(\tau, \tau_0) G_0(\tau, w, p_\perp) + \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau_{\rm eq}(\tau')} D(\tau, \tau') \, G^{\rm eq}(\tau', w, p_\perp)$$

$$D(\tau_2, \tau_1) = \exp\left[- \int_{\tau_1}^{\tau_2} \frac{d\tau''}{\tau_{\rm eq}(\tau'')} \right]$$

- equilibration time in our calculations is constant

$$\tau_{eq} = 0.5 \ \mathrm{fm/c}$$

- Romatschke-Strickland (RS) form of the initial condition

$$G_0(w, p_\perp) = \frac{1}{4\pi^3} \exp\left[- \frac{\sqrt{(1+\xi_0) w^2 + (m^2 + p_\perp^2) \tau_0^2}}{\Lambda_0 \, \tau_0} \right]$$

- $1 + \xi_0 = x_0$ - initial value of the anisotropy parameter, Λ_0 defines initial transverse-momentum scale (transverse temperature)

Kinetic equation
Numerical method

$$\frac{g_0 T m^2}{\pi^2}\left[3TK_2\left(\frac{m}{T}\right)+mK_1\left(\frac{m}{T}\right)\right] = \frac{g_0}{2\pi^2}\left[D(\tau,\tau_0)\Lambda_0^4\tilde{\mathcal{H}}_2\left(\frac{\tau_0}{\tau\sqrt{1+\xi_0}},\frac{m}{\Lambda_0}\right) + \int_{\tau_0}^{\tau}\frac{d\tau'}{\tau_{\rm eq}(\tau')}D(\tau,\tau')T'^4\tilde{\mathcal{H}}_2\left(\frac{\tau'}{\tau},\frac{m}{T'}\right)\right].$$

- iterative method (Banerjee, Bhalerao, Ravishankar):
  1) use a trial function $T' = T(\tau')$ on the RHS of the dynamic equation
  2) the LHS of the dynamic equation determines the new $T = T(\tau)$
  3) use the new $T(\tau)$ as the trial one
  4) repeat steps 1-3 until the stable $T(\tau)$ is found
- particle density, transverse and longitudinal pressure

$$n(\tau) = \frac{g_0}{\tau}\int dP\, v\, G(\tau,w,p_\perp)$$

$$\mathcal{P}_{\parallel}(\tau) = \frac{g_0}{\tau^2}\int dP\, w^2\, G(\tau,w,p_\perp)$$

$$\mathcal{P}_{\perp}(\tau) = \frac{g_0}{2}\int dP\, p_T^2\, G(\tau,w,p_\perp)$$

$\widetilde{\mathcal{H}}$ functions

- $\widetilde{\mathcal{H}}_2$, $\widetilde{\mathcal{H}}_{2\parallel}$, and $\widetilde{\mathcal{H}}_{2\perp}$ functions are defined as integrals:

$$\begin{aligned}
\widetilde{\mathcal{H}}_2(y,z) &= \int_0^\infty dr\, r^3 \mathrm{e}^{-\sqrt{r^2+z^2}} \mathcal{H}_2\left(y, \frac{z}{r}\right), \\
\widetilde{\mathcal{H}}_{2\parallel}(y,z) &= \int_0^\infty dr\, r^3 \mathrm{e}^{-\sqrt{r^2+z^2}} \mathcal{H}_{2\parallel}\left(y, \frac{z}{r}\right), \\
\widetilde{\mathcal{H}}_{2\perp}(y,z) &= \int_0^\infty dr\, r^3 \mathrm{e}^{-\sqrt{r^2+z^2}} \mathcal{H}_{2\perp}\left(y, \frac{z}{r}\right)
\end{aligned}$$

$\mathcal{H}$ functions

- $\mathcal{H}_2$, $\mathcal{H}_{2\parallel}$, and $\mathcal{H}_{2\perp}$ functions are defined similarly as:

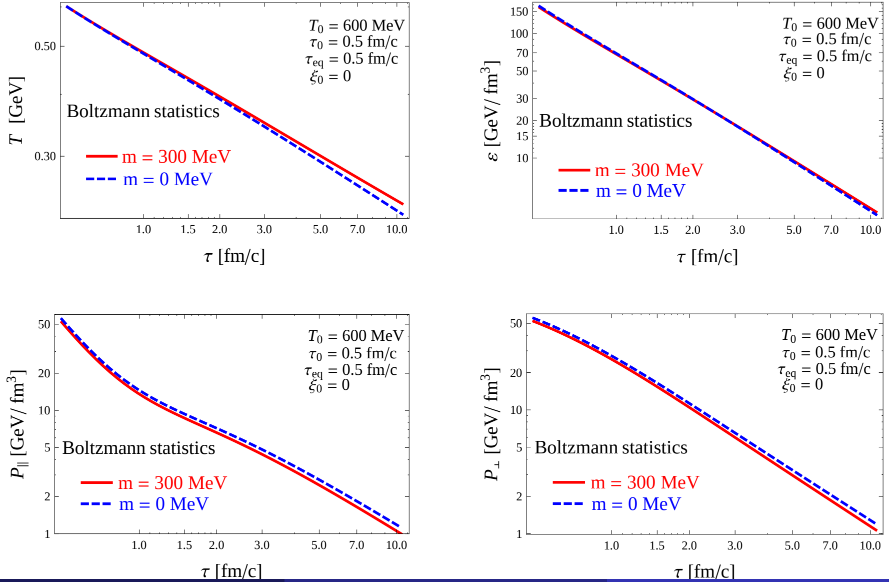
$$\mathcal{H}_2\left(y, \frac{z}{r}\right) = y\int_0^\pi d\phi\,\sin\phi\sqrt{y^2\cos^2\phi + \sin^2\phi + \left(\frac{z}{r}\right)^2},$$

$$\mathcal{H}_{2\parallel}\left(y, \frac{z}{r}\right) = y^3\int_0^\pi d\phi\,\frac{\sin\phi\cos^2\phi}{\sqrt{y^2\cos^2\phi + \sin^2\phi + \left(\frac{z}{r}\right)^2}},$$

$$\mathcal{H}_{2\perp}\left(y, \frac{z}{r}\right) = y\int_0^\pi d\phi\,\frac{\sin^3\phi}{\sqrt{y^2\cos^2\phi + \sin^2\phi + \left(\frac{z}{r}\right)^2}}$$

These integrals are analytic but the results are rather lengthy and not shown here.

Thermodynamics-like variables



Bulk viscous pressure

- Bulk viscosity (Redlich and Sasaki, PRC **79** (2009) 055207):

$$\zeta(T) = \frac{g_0 m^2}{3\pi^2 T} \int_0^\infty p^2 e^{-\frac{\sqrt{m^2+p^2}}{T}} \left[c_s^2(T) - \frac{p^2}{3(m^2+p^2)} \right] dp.$$

- Bulk viscous pressure in the kinetic theory

$$\Pi_\zeta^k = \frac{1}{3}\left[\mathcal{P}_\parallel(\tau) + 2\mathcal{P}_\perp(\tau) - 3\mathcal{P}_{\rm eq}(\tau)\right]. \quad (1)$$

- First order hydrodynamics

$$\Pi_\zeta(\tau) = -\frac{\zeta(T(\tau))}{\tau}, \quad (2)$$

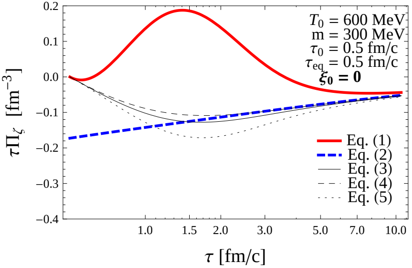
- Second order hydrodynamics

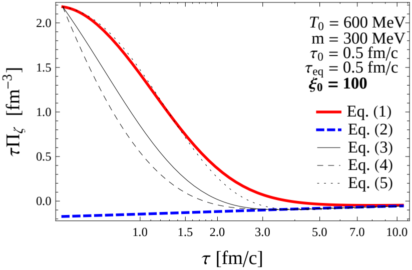
$$\tau_{\rm eq}\frac{d\Pi_\zeta^{\rm h}}{d\tau} + \Pi_\zeta^{\rm h} = -\frac{\zeta}{\tau} - \frac{\tau_{\rm eq}\Pi_\zeta^{\rm h}}{2}\left(\frac{1}{\tau} - \frac{1}{\zeta}\frac{d\zeta}{d\tau} - \frac{1}{T}\frac{dT}{d\tau}\right), \quad (3)$$

$$\tau_{\rm eq}\frac{d\Pi_\zeta^{\rm h}}{d\tau} + \Pi_\zeta^{\rm h} = -\frac{\zeta}{\tau} - \frac{4\tau_{\rm eq}\Pi_\zeta^{\rm h}}{3\tau}, \quad (4)$$

$$\tau_{\rm eq}\frac{d\Pi_\zeta^{\rm h}}{d\tau} + \Pi_\zeta^{\rm h} = -\frac{\zeta}{\tau}. \quad (5)$$

Comparison with exact solutions - bulk viscosity





More details – 17:40 **R. Ryblewski** *Bulk viscous evolution within anisotropic hydrodynamics*

Shear viscosity

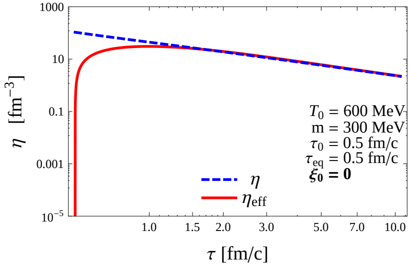
- The calculation by Anderson and Witting, Physica **74** (1974) 466, gives the shear viscosity coefficient in the form

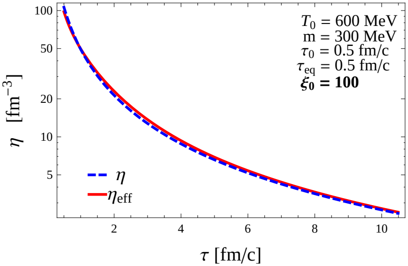
$$\eta = \frac{\tau_{\rm eq} p}{15} \left(\frac{m}{T}\right)^3 \left[\frac{3T^2}{m^2}\frac{K_3}{K_2} - \frac{T}{m} + \frac{K_1}{K_2} - \frac{K_{i\,1}}{K_2}\right]$$

- From the kinetic equation we obtain the effective shear viscosity as

$$\eta_{\rm eff} = \frac{\left(\mathcal{P}_\perp - \mathcal{P}_\parallel\right)\tau}{2}$$

Comparison with exact solutions - shear viscosity





Quantum statistics

New formulas

- New formulas for the background distribution function and the initial condition

$$G^{\mathrm{eq}\,\pm}(\tau, w, p_\perp) = \frac{2}{(2\pi)^3} \frac{1}{\exp\left[\frac{\sqrt{w^2+(m^2+p_\perp^2)\tau^2}}{T(\tau)\tau}\right] \pm 1}$$

$$G_0^{\pm}(w, p_\perp) = \frac{1}{4\pi^3} \frac{1}{\exp\left[\frac{\sqrt{(1+\xi_0)w^2+(m^2+p_\perp^2)\tau_0^2}}{\Lambda_0\,\tau_0}\right] \pm 1}$$

$$\widetilde{\mathcal{H}}_2^{\pm}(y, z) = \int\limits_0^\infty dr\, r^3 \frac{1}{e^{\sqrt{r^2+z^2}} \pm 1} \mathcal{H}_2\left(y, \frac{z}{r}\right)$$

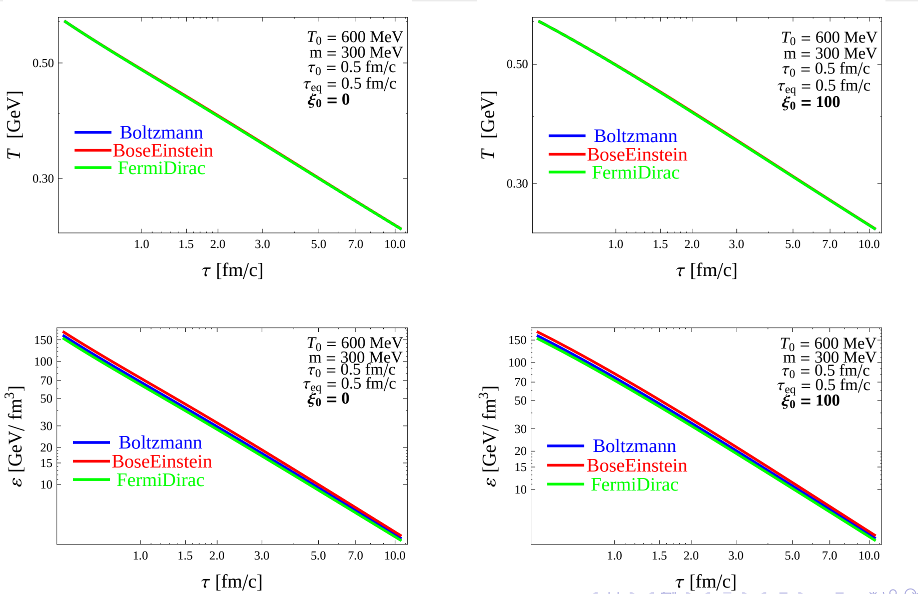
- Shear viscosity

$$\eta = \frac{2 g_0 \tau_{\mathrm{eq}}}{15 T} \int \frac{d^3p}{(2\pi)^3} \frac{p^4}{E^2} G_0^{\pm}(1 \mp G_0^{\pm})$$

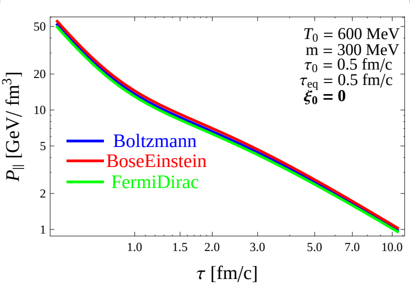
- Bulk viscosity

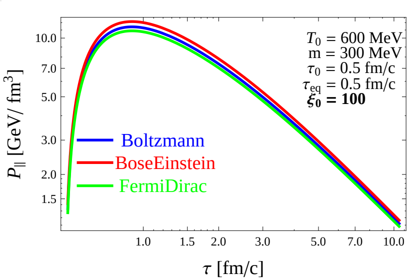
$$\zeta = \frac{2 g_0 \tau_{\mathrm{eq}}}{3 T} \int \frac{d^3p}{(2\pi)^3} \frac{m^2}{E^2} G_0^{\pm}(1 \mp G_0^{\pm}) \left(c_S^2 E - \frac{p^2}{3E}\right)$$

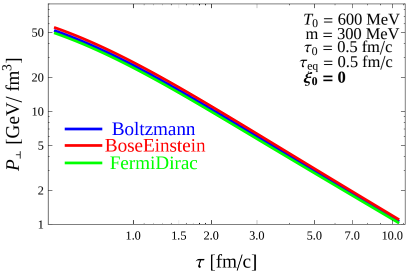
Thermodynamics-like variables - Quantum statistics

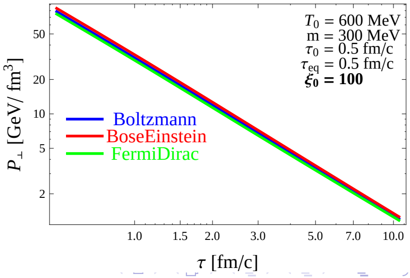


Thermodynamics-like variables







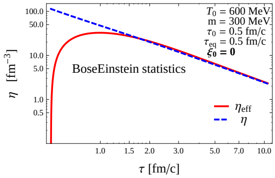


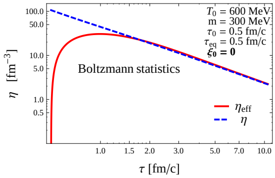
Shear viscosity

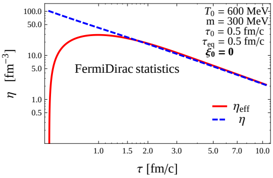
Bose–Einstein

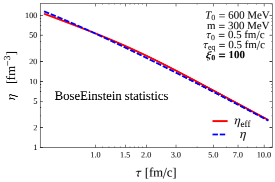
Boltzmann

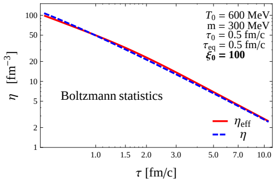
Fermi–Dirac

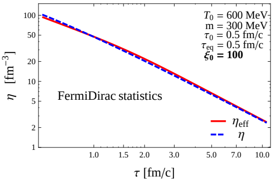










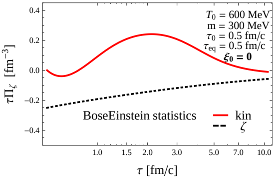


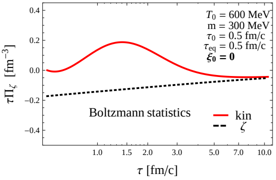
Bulk viscosity

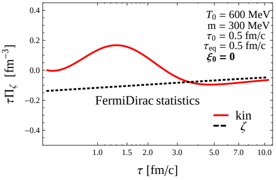
Bose–Einstein

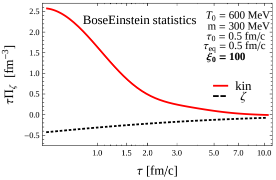
Boltzmann

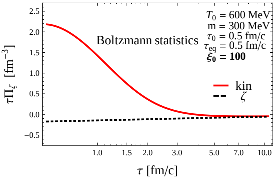
Fermi–Dirac

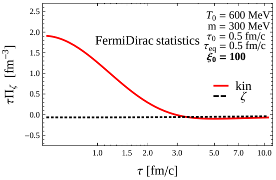




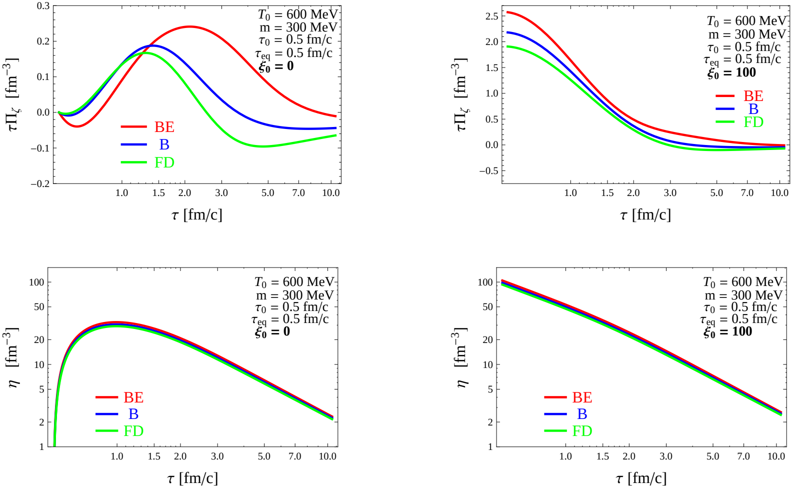








Bulk and shear viscosity - comparison - Quantum statistics



Conclusions

- We have constructed exact solutions of the one-dimensional boost-invariant kinetic equation treated in the relaxation time approximation.
- The previous approaches valid for massless particles have been generalized.
- We have established the correspondence between the late, near equilibrium evolution of the system described by the kinetic theory and by the viscous hydrodynamics.
- We have shown that the late time behavior of the bulk viscous pressure is determined by the bulk viscosity formula used before (e.g., by Redlich–Sasaki, Bożek).
- We have identified problems connected with the proper description of the bulk pressure.
- Anderson-Witting formula for the shear viscosity works well in the case of massive particles (and also for massless particles, as it was shown before).

Thank You

Backup Slides

Bulk viscosity - Anderson and Witting

- Anderson and Witting formula, Physica **74** (1974) 466, Physics for bulk viscosity:

$$\zeta = \frac{\tau p}{3}\frac{m}{T}\left[\frac{3\left(G^2\zeta - 5G - \zeta\right)}{\zeta^2 + 5G\zeta - G^2\zeta^2 - 1} + \frac{\zeta^2}{3}\left(\frac{3G}{\zeta^2} - \frac{1}{\zeta} + \frac{K_1}{K_2} - \frac{Ki_1}{K_2}\right)\right]$$

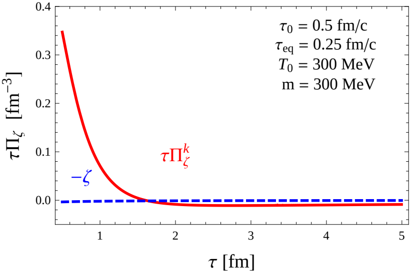
here

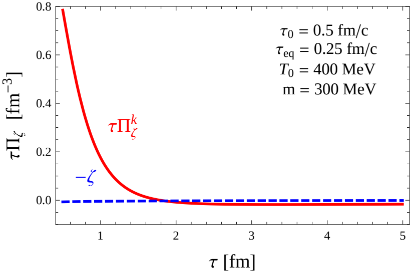
$$\zeta = \frac{m}{T},$$

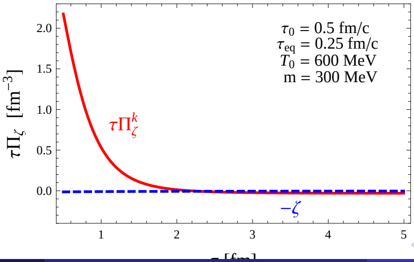
$$G = \frac{K_3}{K_2},$$

$$K_{i,n}(\zeta) = \int_{\zeta}^{\infty} K_{i,n-1}(t)dt = \int_{0}^{\infty} \frac{\mathrm{e}^{-\zeta \cosh t}}{\cosh^n t}dt$$

Comparison with exact solutions - Anderson-Witting formula for bulk viscosity







Comparison with exact solutions - bulk viscosity