

New formulation of leading-order anisotropic hydrodynamics

Outline

- Second order viscous hydrodynamics
- Expansion around an anisotropic background
- Leading order in anisotropic hydrodynamics
- Extension of the model



**HOT
QUARKS '14**

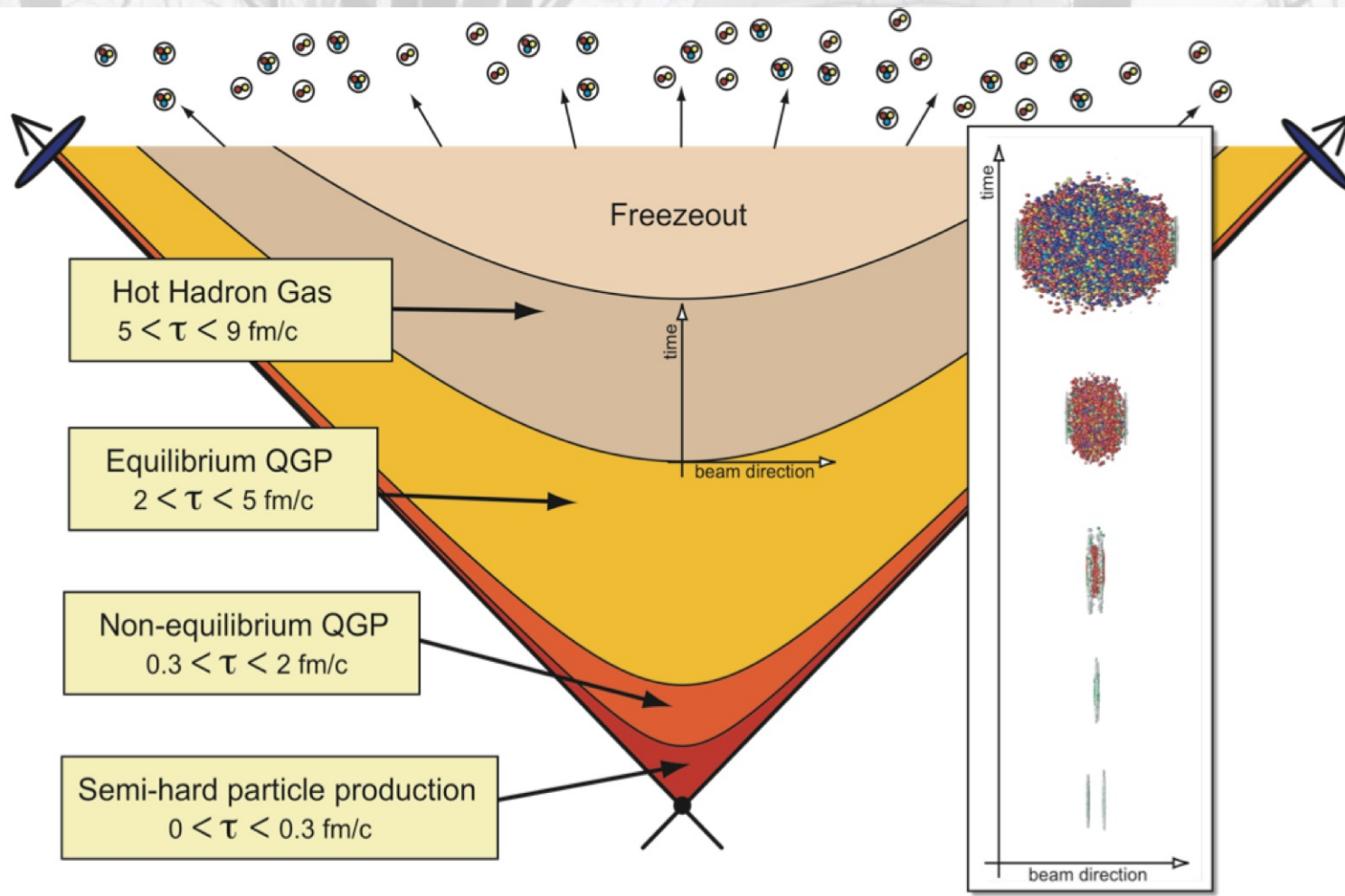
Workshop for young scientists on the physics
of ultrarelativistic nucleus-nucleus collisions

Leonardo Tinti

Motivations

Hydrodynamic modeling of heavy ion collisions (small viscosity)

Large gradients \rightarrow viscous corrections



Strong longitudinal expansion,

pressure anisotropy (also AdS/CFT)

Large momentum anisotropy from microscopic models (pQCCD, CGC)

From kinetic theory to hydrodynamics

Relativistic kinetic theory

**Boltzmann
equation**

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f = \mathcal{C}[f]$$

$$m \partial_t f(t, \mathbf{x}, \mathbf{p}) + \mathbf{p} \cdot \nabla_{\mathbf{x}} f(t, \mathbf{x}, \mathbf{p}) = \mathcal{C}[f]$$

and the relativistic generalization

$$p^\mu \partial_\mu f(x, p) = \mathcal{C}[f]$$

From kinetic theory to hydrodynamics

Relativistic hydrodynamics

Energy-momentum tensor

$T^{\mu\nu}$

$$\int_V d^3\mathbf{x} T^{0\mu} = P^\mu$$

Local four-momentum conservation

$$\partial_\mu T^{\mu\nu} = 0$$

Leading to global conservation if

$$\int_{\partial V} dS_i T^{i\mu} = 0$$

From kinetic theory to hydrodynamics

Relativistic Boltzmann equation:

$$p^\mu \partial_\mu f(x, p) = \mathcal{C}[f]$$

First moment:

$$\begin{aligned} \int dP p^\mu p^\nu \partial_\mu f &= \partial_\mu T^{\mu\nu} = \\ &= \int dP p^\nu \mathcal{C}[f] \quad \left[= 0 \right] \end{aligned}$$

• S. R. De Groot, W. A. van Leeuwen, Ch. G. van Weert, *Relativistic kinetic theory*, North Holland (1980).

Hydrodynamics

Ansatz for the relativistic Boltzmann distribution

Perfect fluid:

$$f \simeq f_{\text{eq.}} = k \exp \left[-\frac{p \cdot U(x)}{T(x)} \right]$$



$$T^{\mu\nu} = k \int dP p^\mu p^\nu \exp \left[-\frac{p \cdot U}{T} \right] = (\varepsilon + P) U^\mu U^\nu - g^{\mu\nu} P$$

$$T_{\text{L.R.F.}} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

Dissipative hydrodynamics

$$f = f_{\text{eq.}} + \delta f \quad \longrightarrow \quad T^{\mu\nu} = T_{\text{eq.}}^{\mu\nu} + \delta T^{\mu\nu}$$

$\delta f \Rightarrow \delta T^{\mu\nu}$ treated as, small, perturbations

*Landau frame,
massless particles*

$$\delta T^{\mu\nu} = \pi^{\mu\nu}$$

$$U^\mu \pi_{\mu\nu} = 0 \quad g_{\mu\nu} \pi^{\mu\nu} = 0$$

Four equations, five more degrees of freedom!

Entropy current and entropy source

Obtaining the remaining equations from the second principle of thermodynamics

$$\mathcal{S}^\mu = \mathcal{S}_{\text{eq.}}^\mu + \delta\mathcal{S}^\mu \simeq \frac{p}{T}U^\mu + \frac{1}{T}T^{\mu\nu}U_\nu - \frac{\tau_\pi}{4\eta T}\pi^{\alpha\beta}\pi_{\alpha\beta}U^\mu$$

$$\partial_\mu \mathcal{S}^\mu \geq 0$$



$$\partial_\mu \mathcal{S}^\mu \simeq \frac{1}{T}\pi^{\mu\nu} \left[\sigma_{\mu\nu} - \frac{\tau_\pi}{2\eta}\Delta_\mu^\alpha \Delta_\nu^\beta D\pi_{\alpha\beta} - \frac{1}{2}T\pi_{\mu\nu}\partial \cdot \left(\frac{\tau_\pi}{2\eta T}U \right) \right]$$

$$\pi^{\mu\nu}\pi_{\mu\nu} \geq 0$$



$$\tau_\pi \Delta_\mu^\alpha \Delta_\nu^\beta D\pi_{\alpha\beta} + \pi_{\mu\nu} = 2\eta\sigma_{\mu\nu} - \pi_{\mu\nu}T\eta\partial \cdot \left(\frac{\tau_\pi}{2\eta T}U \right)$$

Anisotropic hydrodynamics

The background of the slide is a classical painting depicting a chaotic battle scene. In the center, a chariot is being pulled by horses, with soldiers engaged in combat. The scene is filled with smoke and dramatic lighting, suggesting a moment of intense action. The overall style is reminiscent of 17th or 18th-century European battle paintings.

Simply going to higher orders?

Why should we change?

Example, Bjorken flow

*0+1 dimensions: boost invariant in the longitudinal direction,
homogeneous in the transverse plane*

$$U = (\cosh \eta_{\parallel}, 0, 0, \sinh \eta_{\parallel}) \quad \eta_{\parallel} = \frac{1}{2} \ln \left(\frac{t+z}{t-z} \right) \quad \tau = \sqrt{t^2 - z^2}$$

Gradients of the four velocity are proportional to $1/\tau$

In the Navier-Stokes limit

$$\pi_{\mu\nu} \simeq 2\eta\sigma_{\mu\nu}$$

therefore

$$\frac{P_{\parallel}}{P_{\perp}} = \frac{P_{\text{eq.}} + \pi_{ZZ}}{P_{\text{eq.}} + \pi_{XX}} \simeq \frac{3T\tau - 16\bar{\eta}}{3T\tau + 8\bar{\eta}}$$

Anisotropic hydrodynamics

Reorganization of the hydrodynamic expansion

$$f = f_{\text{eq.}} + \delta f$$

around an anisotropic background instead of the local equilibrium

$$f = f_{\text{aniso.}} + \delta \tilde{f}$$

“Romatschke-Strickland” form:

$$f_{\text{aniso.}} = k \exp \left[- \frac{\sqrt{(p \cdot U(x))^2 + \xi(x) (p \cdot Z(x))^2}}{\Lambda(x)} \right]$$

0+1 dimensions

Boost invariant in the longitudinal direction, homogeneous in the transverse plane

$$T_{\text{L.R.F.}} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & P_{\perp} & 0 & 0 \\ 0 & 0 & P_{\perp} & 0 \\ 0 & 0 & 0 & P_{\parallel} \end{pmatrix}$$

No negative pressure already at the leading order

$$\left(P_{\perp}, P_{\parallel} \right) = \int dP \left(\frac{1}{2} p_{\perp}^2, p_{\parallel}^2 \right) f_{\text{RS}} > 0$$

Positive by construction

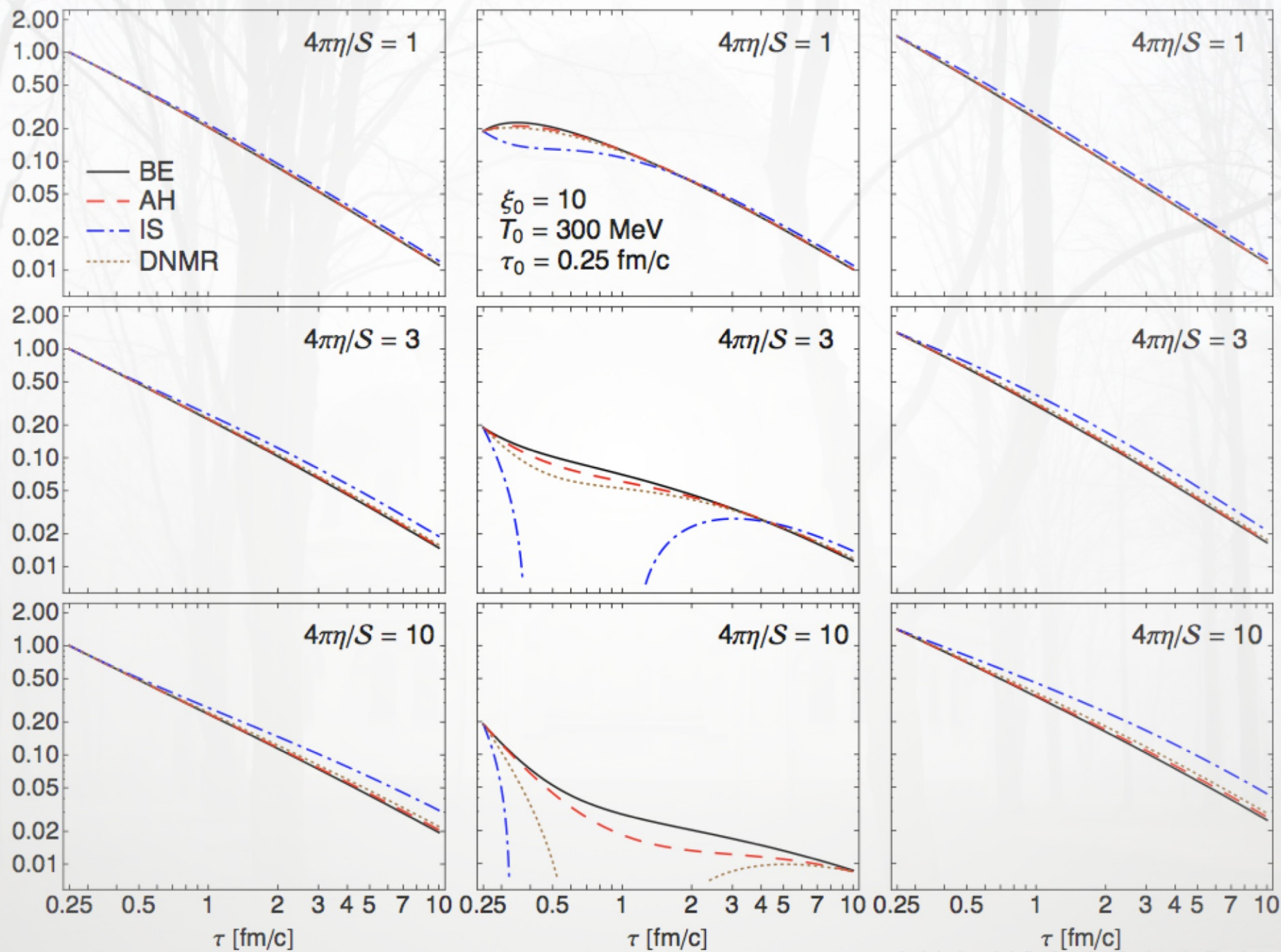
Exact solution! Test for viscous and anisotropic hydrodynamics

Some plots

$$\varepsilon(\tau)/\varepsilon(\tau_0)$$

$$3P_{\parallel}(\tau)/\varepsilon(\tau_0)$$

$$3P_{\perp}(\tau)/\varepsilon(\tau_0)$$



Radial flow

1+1 dimensions, boost invariance in the longitudinal direction, rotation invariance in the transverse plane

Four-velocity

$$U = \gamma \begin{pmatrix} 1 \\ v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} \cosh \theta_{\perp} \cosh \eta_{\parallel} \\ \sinh \theta_{\perp} \cos \phi \\ \sinh \theta_{\perp} \sin \phi \\ \cosh \theta_{\perp} \sinh \eta_{\parallel} \end{pmatrix}$$

Another unknown function

$$\tanh \theta_{\perp}(\tau, r) = \cosh \eta_{\parallel} \sqrt{v_x^2 + v_y^2}$$

Orthonormal basis

$$X = \begin{pmatrix} \sinh \theta_{\perp} \cosh \eta_{\parallel} \\ \cosh \theta_{\perp} \cos \phi \\ \cosh \theta_{\perp} \sin \phi \\ \sinh \theta_{\perp} \sinh \eta_{\parallel} \end{pmatrix} \quad Y = \begin{pmatrix} 0 \\ -\sin \phi \\ \cos \phi \\ 0 \end{pmatrix} \quad Z = \begin{pmatrix} \sinh \eta_{\parallel} \\ 0 \\ 0 \\ \cosh \eta_{\parallel} \end{pmatrix}$$

• W Florkowski and R Ryblewski, Phys. Rev. C 85, 044902 (2012)

Improving the starting point of the expansion

For a conformal system

$$f_{\text{aniso.}} = k \exp \left[- \frac{\sqrt{(1 + \xi_X) (p \cdot X)^2 + (1 + \xi_Y) (p \cdot Y)^2 + (1 + \xi_Z) (p \cdot Z)^2}}{\lambda(x)} \right]$$

Pressure not isotropic in the transverse plane

$$\sum_I \xi_I = 0$$

$$I \in \{X, Y, Z\}$$

$$T_{\text{L.R.F.}} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & P_X & 0 & 0 \\ 0 & 0 & P_Y & 0 \\ 0 & 0 & 0 & P_Z \end{pmatrix}$$

Still two trivial equations from the local four-momentum conservation and four unknown scalar function: θ_{\perp}, λ and two independent anisotropy parameter

How to match with second order hydrodynamics

Using a close to equilibrium expansion

$$f \simeq f_{\text{eq}} \left(1 + \frac{\lambda - T}{T^2} (p \cdot U) - \frac{\xi_X (p \cdot X)^2 + \xi_Y (p \cdot Y)^2 + \xi_Z (p \cdot Z)^2}{2T(p \cdot U)} \right)$$

We can find the pressure corrections

$$\left(T^{\mu\nu} = \int dP p^\mu p^\nu f \right) \pi^{\mu\nu} = \sum_I \pi_I I^\mu I^\nu$$

$$T^{\mu\nu} \simeq T_{\text{eq}}^{\mu\nu} - \frac{32\pi}{5} k T^4 \left(\xi_X X^\mu X^\nu + \xi_Y Y^\mu Y^\nu + \xi_Z Z^\mu Z^\nu \right)$$

We can verify the matching if the equations for the anisotropy parameters correspond to the second order hydrodynamics equations for the pressure corrections

$$\xi_I \simeq -\frac{15}{4} \frac{\pi_I}{\varepsilon}$$

The remaining equations

Looking at the second moment of the Boltzmann equation, in relaxation time approximation, to close the system.

$$p^\lambda \partial_\lambda f = (p \cdot U) \frac{f_{\text{eq.}} - f}{\tau_{\text{eq.}}} \Rightarrow \int dP p^\mu p^\nu p^\lambda \partial_\lambda f = \frac{U_\lambda}{\tau_{\text{eq.}}} \int dP (p^\mu p^\nu p^\lambda f_{\text{eq.}} - p^\mu p^\nu p^\lambda f)$$

$$\partial_\lambda \Theta^{\lambda\mu\nu} = \frac{1}{\tau_{\text{eq}}} (U_\lambda \Theta_{\text{eq}}^{\lambda\mu\nu} - U_\lambda \Theta^{\lambda\mu\nu})$$

$$\Theta^{\lambda\mu\nu} = \int dP p^\lambda p^\mu p^\nu f \quad \Theta_{\text{eq}}^{\lambda\mu\nu} = \int dP p^\lambda p^\mu p^\nu f_{\text{eq.}}$$

Many of the projections of these equations along the vectors of the basis are trivial, but there is a useful combination of the remaining ones

$$\frac{D\xi_I}{1 + \xi_I} + 2\sigma_I + \frac{\xi_I}{\tau_{\text{eq.}}} \left(\frac{T}{\lambda}\right)^5 \sqrt{\prod_J (1 + \xi_J)} - \frac{1}{3} \sum_J \frac{D\xi_J}{1 + \xi_J} = 0$$

Only two of the three equations are independent, if two of them are fulfilled all of them are

Small anisotropy limit

$$D\xi_I + 2\sigma_I + \frac{\xi_I}{\tau_{\text{eq}}} \simeq 0$$

Using the first moment equations
and after some calculations

$$\tau_{\text{eq.}} D\pi_I + \pi_I = 2 \left(\frac{4}{15} \varepsilon \tau_{\text{eq.}} \right) \sigma_I - \pi_I \left[\frac{4}{15} \varepsilon \tau_{\text{eq.}} T \partial \cdot \left(\frac{15}{8\varepsilon T} U \right) \right]$$

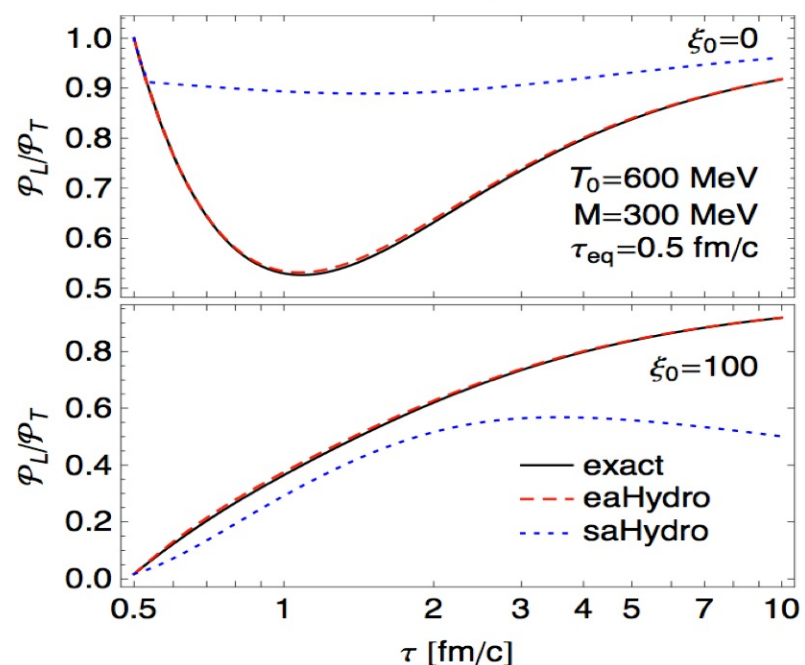
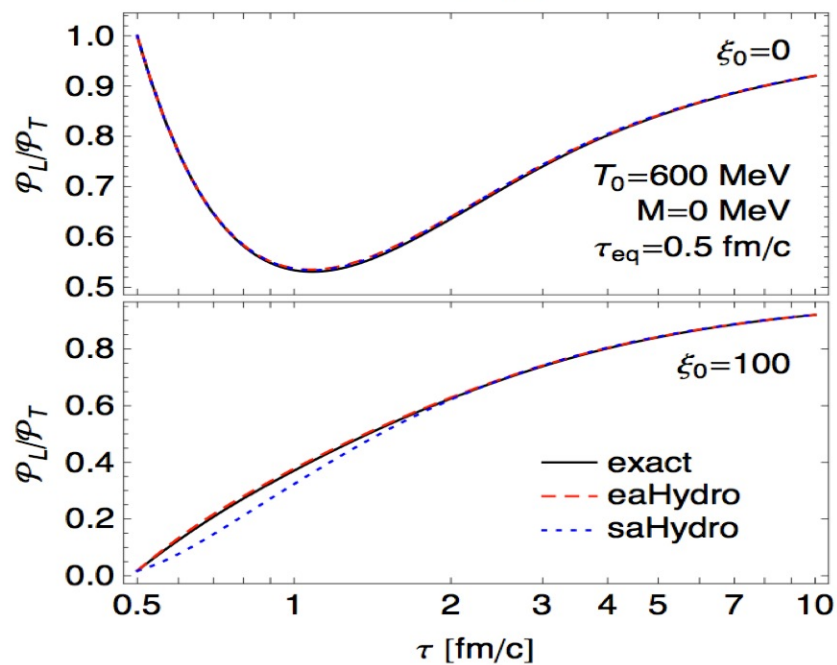
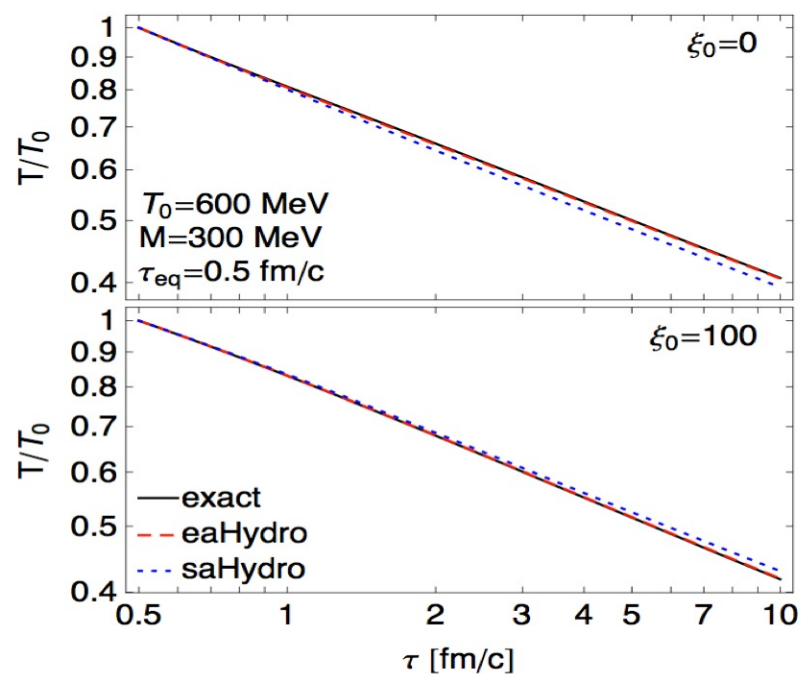
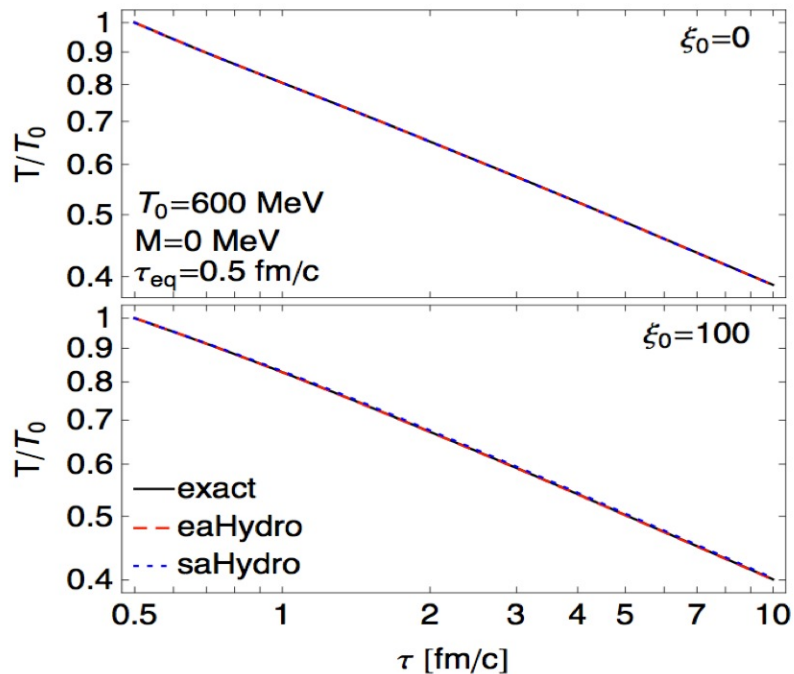
Consistent with Israel-Stewart

$$\tau_\pi D\pi_I + \pi_I = 2\eta\sigma_I - \pi_I \left[\eta T \partial_\lambda \left(\frac{\tau_\pi}{2\eta T} U^\lambda \right) \right]$$

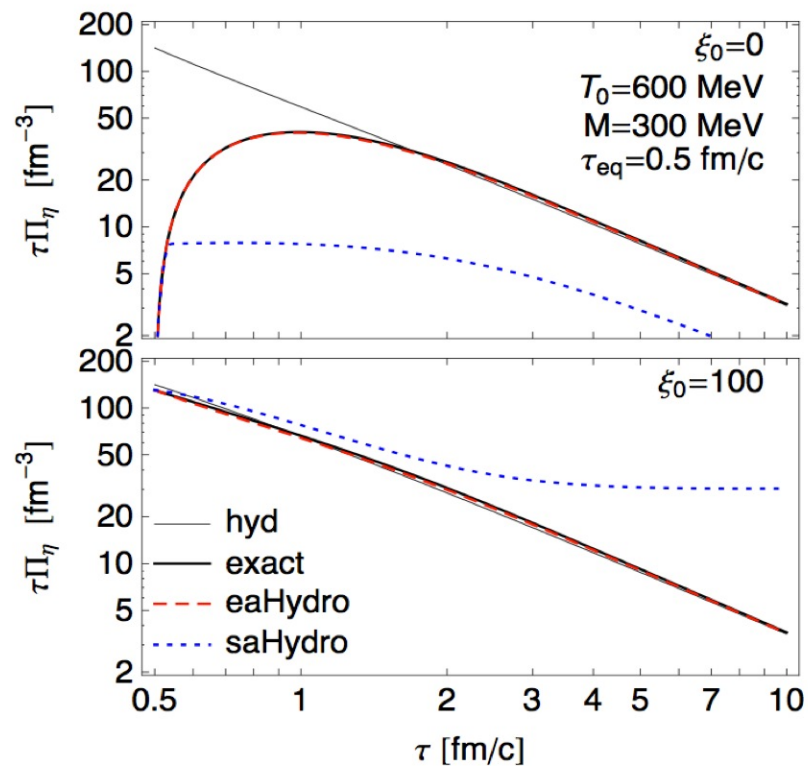
provided

$$\tau_\pi = \tau_{\text{eq.}} \quad \eta = \frac{4}{15} \varepsilon \tau_{\text{eq.}} \Rightarrow \tau_\pi = \frac{5\bar{\eta}}{T}$$

How does it work in practice?

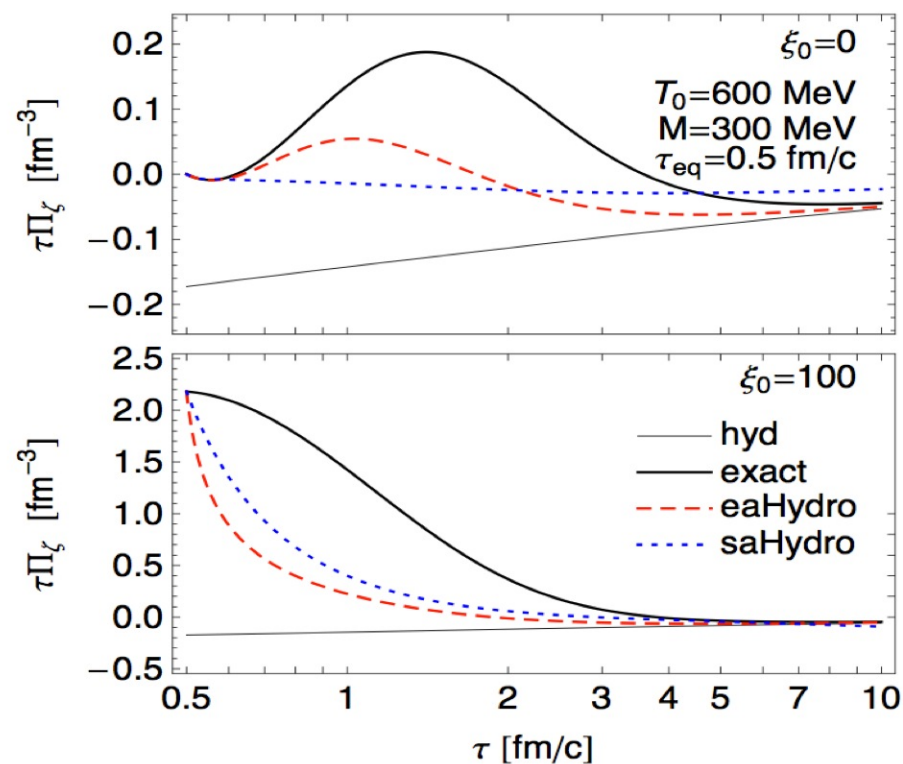


How does it work in practice?



So far so good, but...

It still does not resolve problems with bulk viscosity



Summary & outlook

- **Anisotropic hydrodynamics is a reorganization of the hydrodynamic expansion, around an anisotropic distribution.**
- **Pressure anisotropies already at the leading order. Very good approximation in the exactly solvable $0+1$ case**
- **Generalized ansatz for the leading order. Well defined in the early stages and consistent with second order viscous hydrodynamics close to local equilibrium in presence of radial flow.**
- **NEXT: How much “physics” can we include in the leading order? Non boost invariant systems? Non cylindrical radial expansion?**

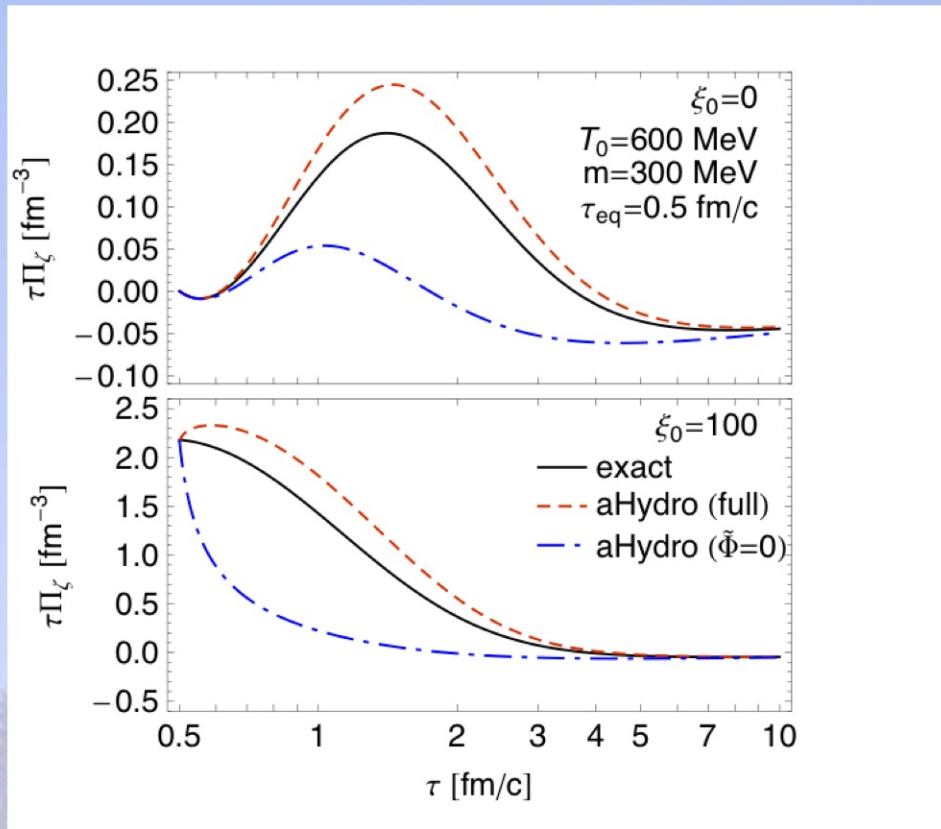


Backup slides

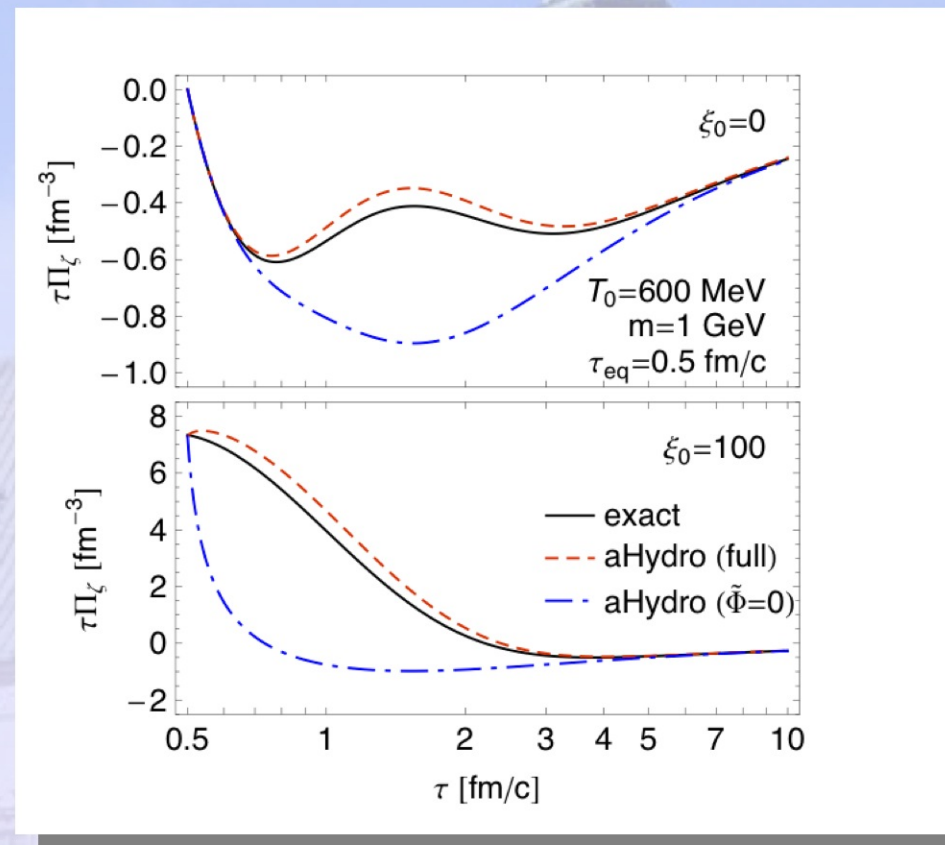
Explicit degree of freedom for the bulk viscosity

$$f = k \exp \left\{ -\frac{1}{\lambda} \sqrt{M^2 + \sum_I \left(1 + \phi + \xi_I \right) (p \cdot I)} \right\}$$

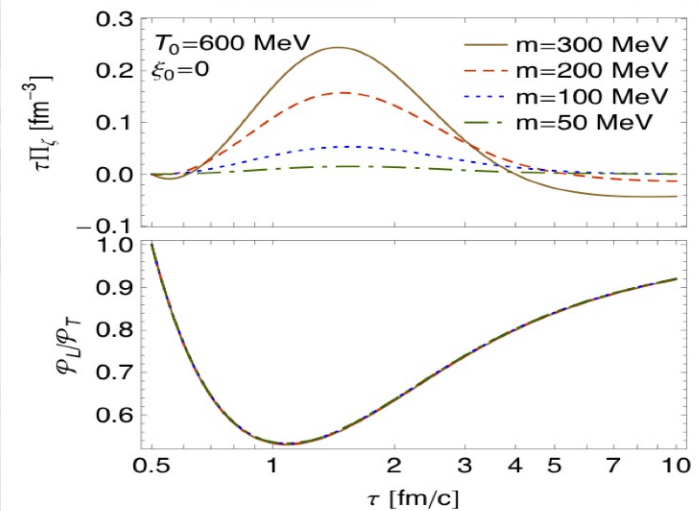
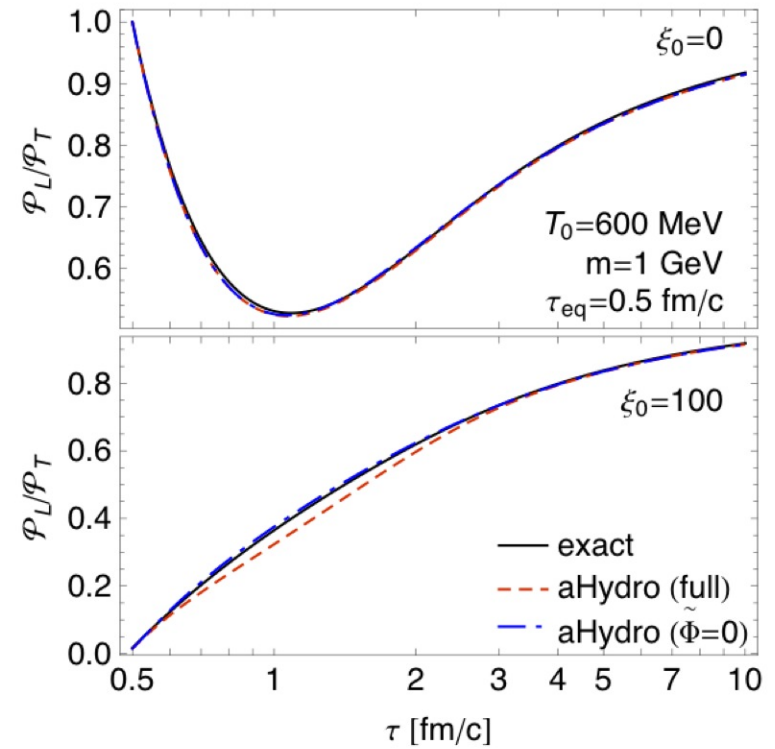
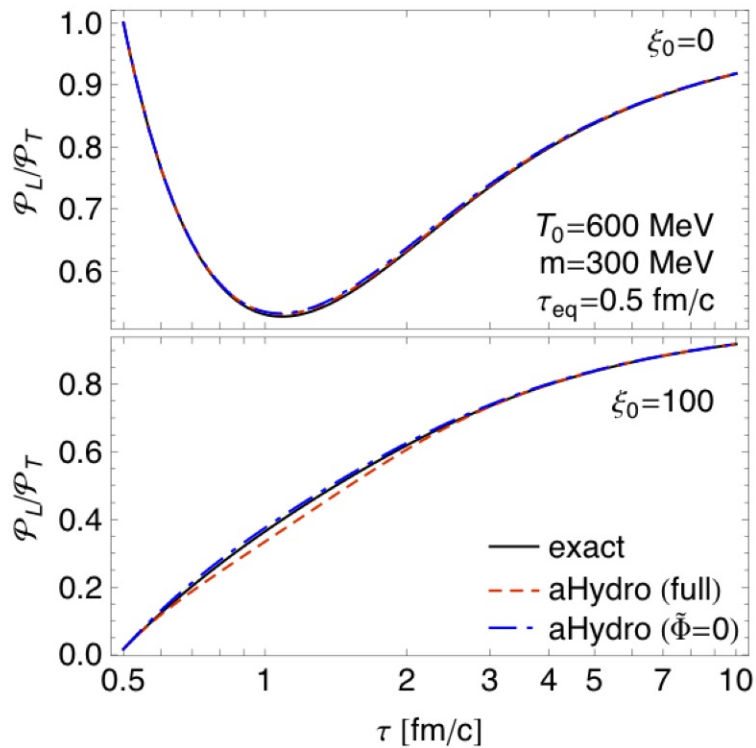
- *Equations from the second moment of the Boltzmann equation*
- *Additional equation from the zeroth moment*
- *Matching with second order viscous hydrodynamics*
- **M Nopoush, R Ryblewski, M Strickland, arXiv:1405.1355**



Better qualitative and quantitative agreement,



for the bulk evolution...



However, not always better...