UNIVERSAL SUM RULES FOR EFFECTIVE FIELD THEORIES

Benasque Workshop
After the discovery: Hunting for a non-standard Higgs sector

Riccardo Torre

SISSA & Padova U. & INFN Padova

Benasque, Spain - 17 April 2014

B. Bellazzini, L. Martucci, RT, to appear

supported by the ERC Advanced Grant "DaMeSyFla" (Electroweak Symmetry Breaking, Flavour and Dark Matter: One Solution for Three Mysteries)

Analyticity and Unitarity

Motivations

Dispersion relations and UV-IR connection

Examples

INTRODUCTION AND MOTIVATIONS

- Effective field theory is a powerful tool to describe the dynamics of low-energy degrees of freedom

$$\mathcal{L}_{\mathrm{EFT}} = \sum_{i,n} \frac{c_i^{(n)}}{\Lambda^n} \mathcal{O}_i^{(n)}$$

- The operator dimension expansion is organised in powers of p/Λ and only a finite set of low energy coefficients (LECs) is relevant at any given order
- Approximate/exact invariance under an internal symmetry group H in the IR constrains the structure of the operators $\mathcal{O}_i^{(n)}$
- The LECs must also satisfy additional constraints when the underlying UV theory is Lorentz invariant and possesses a unitary, analytic and crossing symmetric S-matrix
- E.g. consider the theory of a single Goldstone boson with a shift symmetry $\pi \to \pi + c$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \pi)^2 + \frac{c}{\Lambda^4}(\partial_\mu \pi \partial^\mu \pi)^2$$

- This theory has sensible UV completion only for $c \geq 0$ since the forward elastic scattering amplitude should satisfy

$$\mathcal{A}''(s)\Big|_{s=0} = \int_0^\infty ds \frac{4\sigma(s)}{\pi s^2} \geq 0$$

Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi, hep-th/0602178

UV-IR CONNECTION AND TWICE SUBTRACTED DRS

$$\mathcal{A}''(s)\big|_{s=0} = \int_0^\infty ds \frac{4\sigma(s)}{\pi s^2} \geq 0$$

- The left-hand side can be computed in the low-energy EFT, while the right hand side corresponds to the integral over all energy scales of the $2 \to 2$ total cross section
- This UV-IR connection has been widely used to derive "twice subtracted" dispersion relations DRs, e.g.
  1. The proof of the a-theorem is based on such a DR for the dilation scattering which gives *Komargodski, Shwimmer, 1107.3987*
  $$c \sim a_{\mathrm{UV}} - a_{\mathrm{IR}} \geq 0$$
  2. In the $O(p^4)$ $SU(2)$ chiral Lagrangian one can derive *Pham, Truong, PRD 31 (1985)*
  $$l_{4,5} \geq 0$$
  3. In the $\pi\pi$ scattering in QCD one can even go beyond the forward scattering and derive general twice subtracted DRs knows as Roy equations *Roy, PLB 36 (1971)*
- All these dispersion relations (and many others) are derived for the LECs at $O(p^4)$ since the Froissart bound $\sigma(s) \sim (\log s)^2$ ensures the convergence of the integrals

FROM TWICE TO ONCE SUBTRACTED DISPERSION RELATIONS

- Under some assumptions, one can go beyond the twice subtracted DRs and can derive constraints on the coefficients of the $O(p^2)$ LECs
- Certain linear combinations of amplitudes still give convergent integrals
- This approach has been recently applied to the $SU(2)_L \times SU(2)_R/SU(2)_V$ EW chiral Lagrangian with a Higgs-like singlet

$$\mathcal{L} = \partial_\mu \pi^i \partial^\mu \pi^i (1 + a\frac{h}{v})$$

- This led to the sum-rule (more on this later)

Falkowski, Rychkov, Urbano, 1202.1532

$$1 - a^2 = \frac{v^2}{6\pi} \int_0^\infty \frac{ds}{s} \left(2\sigma_{I=0}(s) + 3\sigma_{I=1}(s) - 5\sigma_{I=2}(s)\right)$$

- In QCD within ChPT and with $a = 0$ this is known as the Olsson sum-rule and has been proven to be convergent since the combination of cross sections does not couple to the Pomeron (the piece saturating the Froissart bound drops in the combination)

Olsson, PR 162 (1967)

- More recently the same approach was applied to derive a sum rule for the scattering of a 4-plet of $SO(4)$ in CHM *(Urbano, 1310.5733)* and to study weakly coupled models with several Higgs bosons *(Grinstein, Murphy, Pirtskhalava, Uttayarat, 1401.0070)*

General approach

First principles

Analytic structure

Crossing matrix

GENERAL APPROACH

- We consider the $2 \to 2$ forward ($t = 0$) scattering of (in principle reducible) representations $\mathbf{r}$ of a global symmetry H and decompose them in irreducible representations

$$\mathbf{r} \otimes \mathbf{r} = \bigoplus_{I,\xi} \mathbf{r}_{I(\xi)}$$

- The eigen-amplitudes can be expanded in powers of the energy

$$A_I(s) \sim a_I^{(0)} + a_I^{(1)} s + a_I^{(2)} s^2 + \ldots$$

- The scattering among different irreps vanishes for the Wigner-Eckart theorem

$$\mathcal{A}(\mathbf{r}^{\alpha}_{I(\xi)} \to \mathbf{r}^{\beta}_{J(\xi')}) = \delta_{\alpha\beta}\delta_{IJ}\mathcal{A}_{I(\xi,\xi')}(s)$$

- Analyticity of the amplitude in the complex-plane implies (Cauchy theorem)

$$a_I^{(n)} = \frac{1}{2\pi i}\oint \frac{A_I(s)}{s^{n+1}} ds$$

FIRST PRINCIPLES

- Analyticity

$$\frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{\mathcal{A}(s)}{(s-\mu^2)^{n+1}}=\sum_{s_*}\mathrm{Res}\left[\frac{\mathcal{A}(s)}{(s_*-\mu^2)^{n+1}}\right]+\frac{1}{n!}\mathcal{A}^{(n)}(\mu^2)$$

- Unitarity (optical theorem)

$$\mathrm{Im}(\mathcal{A}_{I(\xi,\xi)}(s))=s\sigma^{\mathrm{tot}}_{I(\xi,\xi)}(s)$$

- Crossing symmetry

$$\mathcal{A}_{ab\to cd}(s)=\mathcal{A}_{a\bar{d}\to c\bar{b}}(u)\quad\Longrightarrow\quad\mathcal{A}(u)_{I(\xi,\xi')}=X_{I(\xi,\xi')J(\zeta,\zeta')}\mathcal{A}(s)_{J(\zeta,\zeta')}\qquad X^2=1$$

- The involutive matrix X is called the crossing matrix, is built from the Clebsch-Gordan coefficients and implements crossing symmetry in the space of irreps (more to follow)

Foldy, Kottles, AoP 48 (1968)

- This matrix can be used to construct two projectors

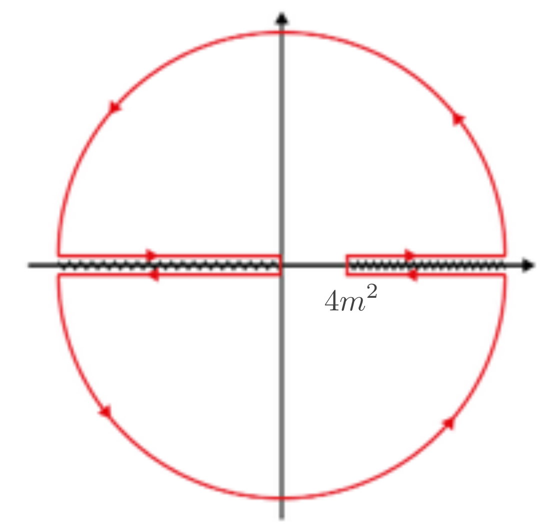
$$P_{(\pm)}\equiv\frac{1}{2}(1\pm X)\ ,\qquad P^2_{(\pm)}=P_{(\pm)}\ ,\qquad P_{(+)}P_{(-)}=0$$

- These projectors and the properties of the matrix X are the key ingredient to guarantee convergence of the once-subtracted sum rules

SUM RULES

$$\frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{\mathcal{A}(s)}{(s-\mu^2)^{n+1}} = \sum_{s_i} \mathrm{Res}\left[\frac{\mathcal{A}(s)}{(s_i-\mu^2)^{n+1}}\right] + \frac{1}{n!}\mathcal{A}^{(n)}(\mu^2)$$

- The amplitude $\mathcal{A}(s)$ has a cut on the real axis running from $4m^2$ to $+\infty$ and by crossing symmetry also another cut from $-\infty$ to 0
- There can be poles at s_i on the real axis below $4m^2$ from light particles propagating
- Heavier particles don't give poles on the physical Riemann sheet
- By deforming the contour of the integral from a small circle around $2m^2$ to a big contour we get



$$\frac{1}{n!}\mathcal{A}^{(n)}(\mu^2) = c^{\Lambda\,(n)} + \int_{4m^2}^{\Lambda^2} \frac{ds}{2\pi i}\left[\frac{1}{(s-\mu^2)^{n+1}} + (-1)^n \frac{X}{(s+\mu^2)^{n+1}}\right][\mathcal{A}(s+i\epsilon) - \mathcal{A}(s-i\epsilon)]$$

$$A(s) = A_{I(\xi,\xi')} \qquad\qquad X = X_{I(\xi,\xi')J(\zeta,\zeta')}$$

- Even if we started with purely elastic scattering, crossing in the space of irreps can generate inelastic transitions between degenerate irreps, so we cannot naively conclude

$$[\mathcal{A}(s+i\epsilon) - \mathcal{A}(s-i\epsilon)]/2i = \mathrm{Im}\mathcal{A}(s) = s\sigma(s)$$

SUM RULES

$$\frac{1}{n!}\mathcal{A}^{(n)}(\mu^2) = c^{\Lambda\,(n)} + \int_{4m^2}^{\infty}\frac{ds}{\pi}\left[\frac{1}{(s-\mu^2)^{n+1}} + (-1)^n\frac{X}{(s+\mu^2)^{n+1}}\right]s\sigma(s)$$

- For $n \geq 2$ all the integrals converge for $\Lambda \to \infty$ (guaranteed by the Froissart bound)

Froissart, PR 123 (1961)

- For $n = 1$ one needs more ingredients to ensure the convergence of the integrals
  - For strongly coupled theories the Froissart bound can only be reached through the t-channel exchange of singlet objects and the corresponding amplitudes are "universal" (see e.g. Regge theory and the Pomeranchuk's theorem)
  - For weakly coupled theories the Froissart bound is not expected to be saturated
- Therefore the $c^{\Lambda\,(n)}$ coefficient drops and using the projectors $P_{(\pm)}$ one gets the sum rules

$$\frac{1}{n!}P_{(\pm)}\mathcal{A}^{(n)}(\mu^2) = \frac{2}{\pi}\int_0^{\infty}\frac{ds}{s^n}P_{(\pm)}\sigma(s) \qquad \begin{array}{l} P_+ \text{ for } n = 2N \\ P_- \text{ for } n = 2N+1 \end{array} \qquad N \in \mathbb{N}$$

- This is a matrix equation, and the number and the structure of the sum rules is entirely determined by the symmetry group and the properties of the crossing matrix
- The presence of $P_{(-)}$ implies that in this case no combination of cross sections with a definite sign can appear in the sum rule
- It is therefore in general very difficult to constrain the sign of the left-hand side from once subtracted DRs

THE CROSSING MATRIX

- The crossing matrix X is given in terms of Clebsch-Gordan coefficients by

$$X_{I(\xi,\xi')J(\zeta,\zeta')} = \frac{1}{\dim \mathbf{r}_{I(\xi,\xi')}} \sum_{\alpha,\beta} \sum_{a,b,c,d} C^{a\bar{d}}_{I(\xi,\xi')\alpha} \bar{C}^{c\bar{b}}_{I(\xi,\xi')\alpha} \bar{C}^{ab}_{J(\zeta,\zeta')\beta} C^{cd}_{J(\zeta,\zeta')\beta}$$

- This matrix has two key properties (in the non-degenerate case) that are relevant for the derivation of the sum rules

Foldy, Kottles, AoP 48 (1968)

$$X^2 = 1\,, \qquad \sum_J X_{IJ} = 1$$

- The first property means that it is involutive and can be diagonalised with ± 1 eigenvalues
- The second property says that the vector with all entries equal to 1 is eigenvector with eigenvalue 1
- The number of independent sum rules is given by the dimension of the -1-eigenspace of the crossing matrix, which equals the number antisymmetric irreps in the decomposition
- In the degenerate case it's more complicated to give a proof of these properties, but they still hold with few modifications
- Degeneracy can be broken by other quantum numbers (spin, charge, etc.) and in this case the matrix X can be consistently reduced (I will not discuss this more technical case here)

Sum rules: examples

Fundamentals of SO(N)

EXAMPLE: N OF SO(N)

- We consider the scattering of fundamental irreps of $SO(N)$ for $N \neq 4$

$$\mathbf{N} \otimes \mathbf{N} = \mathbf{1} \oplus \mathbf{A} \oplus \mathbf{S}$$

- The once subtracted sum rule gives in this case

$$2a_{\mathbf{1}}^{(1)} + N a_{\mathbf{A}}^{(1)} - (N+2) a_{\mathbf{S}}^{(1)} = \frac{2}{\pi} \int_0^\infty \frac{ds}{s} \left[2\sigma_0^{\text{tot}} + N\sigma_{\mathbf{A}}^{\text{tot}} - (N+2)\sigma_{\mathbf{S}}^{\text{tot}}\right]$$

- The LECs $a_i^{(1)}$ can be explicitly computed once a low energy effective theory is assumed
- Taking e.g. the theory of the GBs of $SO(N+1) \to SO(N)$ (sphere) or $SO(N,1) \to SO(N)$ (hyperboloid) and couple it to Higgs like states transforming as $\mathbf{1}$ and $\mathbf{S}$

$$\partial_\mu \pi^i \partial^\mu \pi^i \left(1 + a\frac{h}{f_\pi} + b\frac{h^{ab}}{f_\pi}\right)$$

- The once subtracted sum rule gives in this case

$$\left(\pm 1 - a^2 + \frac{N+2}{2N} b^2\right) = \frac{f_\pi^2}{2\pi N} \int_0^\infty \frac{ds}{s} \left[2\sigma_{\mathbf{1}}^{\text{tot}} + N\sigma_{\mathbf{A}}^{\text{tot}} - (N+2)\sigma_{\mathbf{S}}^{\text{tot}}\right]$$

- For the case $SO(4) \to SO(3)$ one recovers the sum rule we quoted in the beginning

$$1 - a^2 = \frac{v^2}{6\pi} \int_0^\infty \frac{ds}{s} \left(2\sigma_{I=0}(s) + 3\sigma_{I=1}(s) - 5\sigma_{I=2}(s)\right)$$

Massive gauge theories

Sum rules for gauge boson scattering

On the validity of the Equivalence Theorem

MASSIVE GAUGE THEORIES

$$1 - a^2 = \frac{v^2}{6\pi} \int_0^\infty \frac{ds}{s} \left(2\sigma_{I=0}(s) + 3\sigma_{I=1}(s) - 5\sigma_{I=2}(s)\right)$$

- This sum rule was interpreted in terms of the $SO(4) \to SO(3)$ EWCh Lagrangian to constrain the couplings of a Higgs-like scalar singlet *Falkowski, Rychkov, Urbano, 1202.1532*
- The left hand-side was computed using the Equivalence Theorem (ET) starting from the Lagrangian (gauge-less limit $g = g' = 0$, i.e. $m_W = 0$)

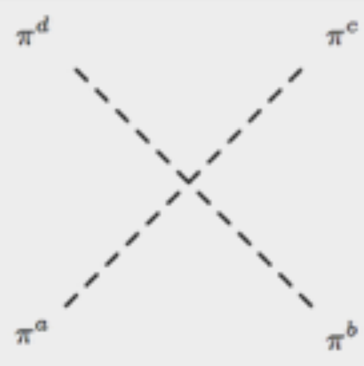
$$\partial_\mu \pi^i \partial^\mu \pi^i \left(1 + a\frac{h}{v}\right)$$

- Very recently the calculation was repeated using the unitary gauge and the authors claimed that the left-hand side of the sum rule changes to *Espriu, Mescia, 1403.7386*

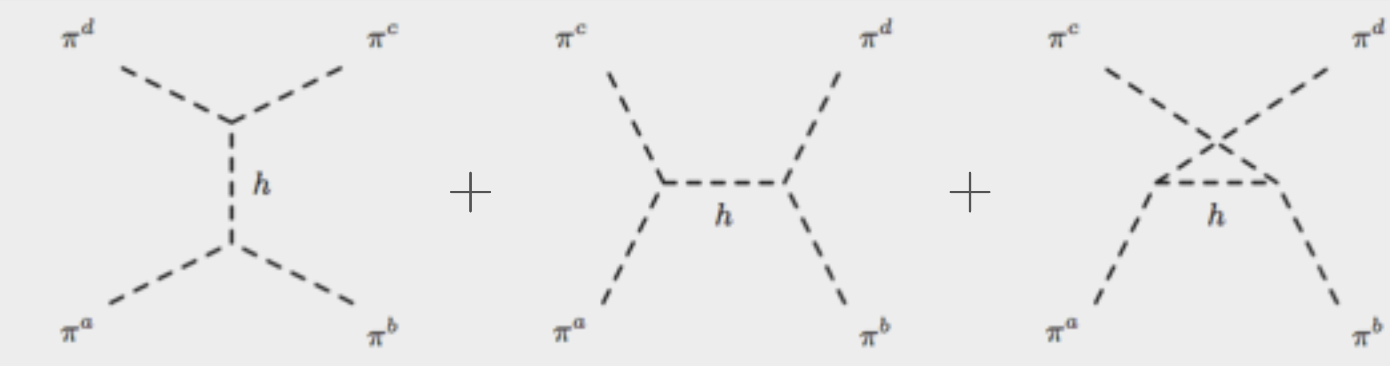
$$1 - a^2 \quad \Longrightarrow \quad 3 - a^2$$

- The discrepancy was attributed to the failure of the ET in the forward region $t = 0$
- In fact the ET is expected to hold only when all the kinematic variables are much larger than the mass of the particles, i.e. $s, t, u \gg m_W^2$
- However, we find that in this case the discrepancy can also be understood using the ET provided that one includes all the relevant contributions

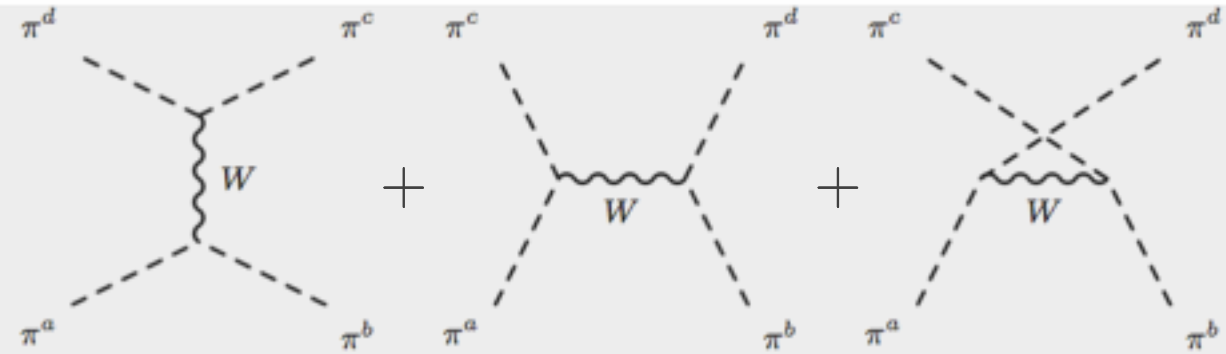
EQUIVALENCE THEOREM



$$\sim \frac{s}{v^2}$$



$$\sim -\frac{a^2}{v^2}\frac{s^2}{s-m_h^2}$$



$$\sim \frac{g^2}{4}\frac{u-s}{t-m_W^2} - \frac{g^2}{4}\frac{s-t}{u-m_W^2}$$

- For finite t one can drop m_W^2 everywhere getting the usual result

$$\mathcal{A} = \frac{s}{v^2}(1-a^2)$$

- However, for $t=0$ one cannot just set $m_W = 0$ due to the singularity in the t-channel
- Making the calculation with finite mass (not gauge-less) one gets (independent of g!)

$$\mathcal{A} = \frac{s}{v^2}(1-a^2) + \frac{g^2}{4}\frac{u-s}{m_W^2} = \frac{s}{v^2}(3-a^2)$$

- This is in agreement with the calculation performed in the unitary gauge but only in the Higgs-Kibble model (thanks to A.Falkowski for pointing this out)
- Further work is needed to get a constraint in the full SM

Conclusion and Prospects

CONCLUSIONS

- Dispersion relations are a useful tool to relate coefficients of a low energy effective theory to quantities that can be measured independently of the scale (IR/UV relations)
- Under some conditions they can be used also for the leading $O(p^2)$ LECs giving once subtracted DRs
- We developed a general approach to determine the number and the form of the sum rules for any symmetry group H and we applied it to compute (as an example) once subtracted sum rules for the fundamental of $SO(N)$ and the adjoint of $SU(N)$
- The general approach is mainly based on the mathematical properties of the crossing matrix that can be completely determined from group theory
- We find that in general, for once subtracted sum rules no-definite sign combination of cross sections can appear and one cannot constrain the sign of the LECs (as done for example in chiral QCD or in the proof of the a-theorem)
- We examined again the case of the $SO(4)/SO(3)$ EWCh Lagrangian and we confirm the result recently presented which disagrees with previous results
- The sum rule relating the Higgs coupling to W and Z bosons seems to become weaker in light of the new result and it seems difficult to constrain the sign of $1-a^2$ even in the absence of the negative contribution from a quintuplet exchange (more work is needed in this direction)

THANK YOU