Second Quantization of Mathieu Moonshine

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Talk based on:
[arXiv:1312.0622] (w/ R. Volpato)
[arXiv:1302.5425] (w/ M. Gaberdiel, & R. Volpato)
[arXiv:1211.7074] (w/ M. Gaberdiel, H. Ronellenfitsch, R. Volpato)

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The most famous example is **Monstrous Moonshine**.

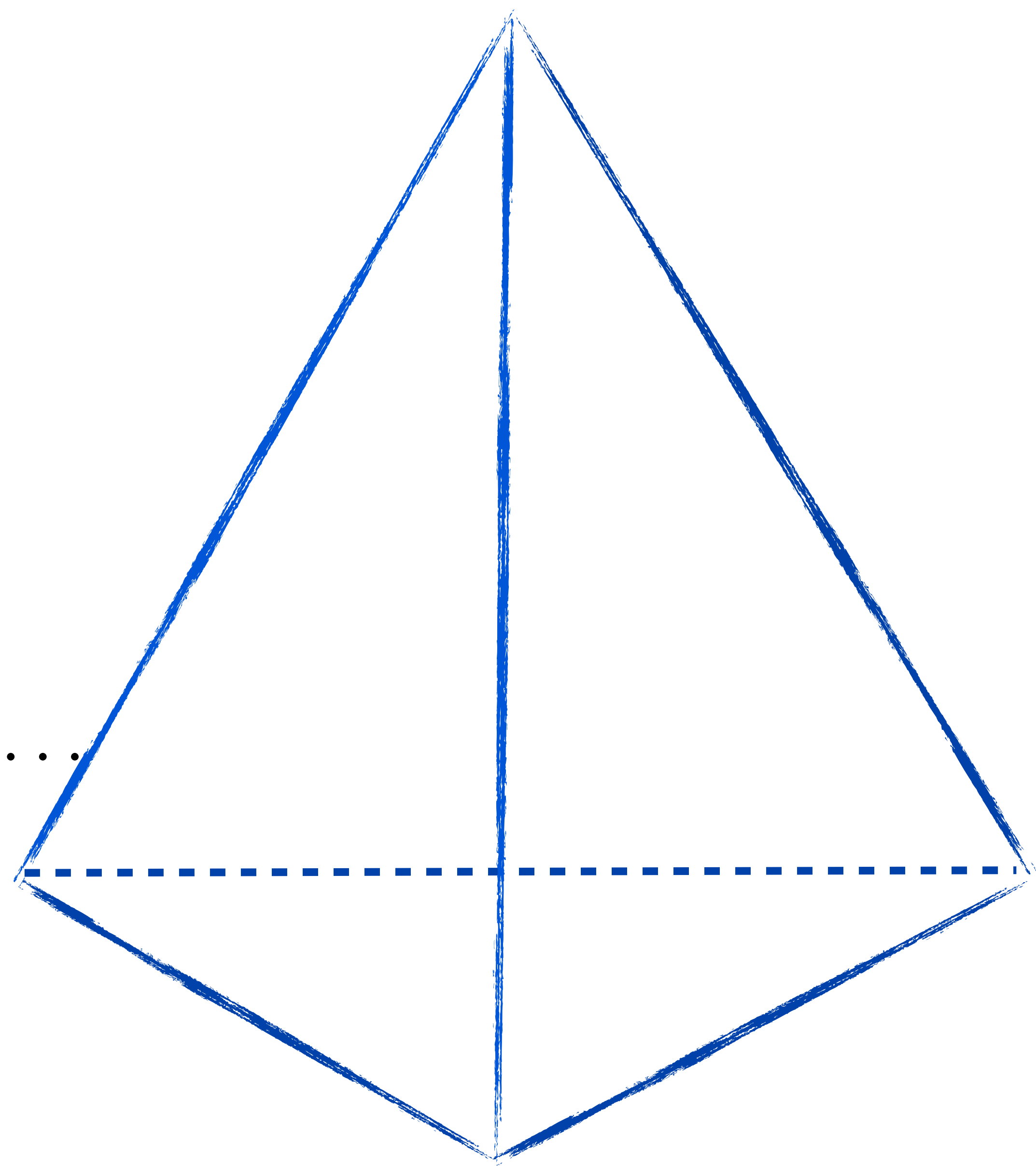
Monstrous Moonshine

monster group

$\mathbb{M}$

$J(\tau) = q^{-1} + 196884q + \cdots$

modular function



bosonic string theory on

$(T^{24}/\Lambda_{\mathrm{Leech}})/\mathbb{Z}_2$

(holomorphic VOA $V^\natural$)

monster Lie algebra $\mathfrak{m}$

Monstrous Moonshine

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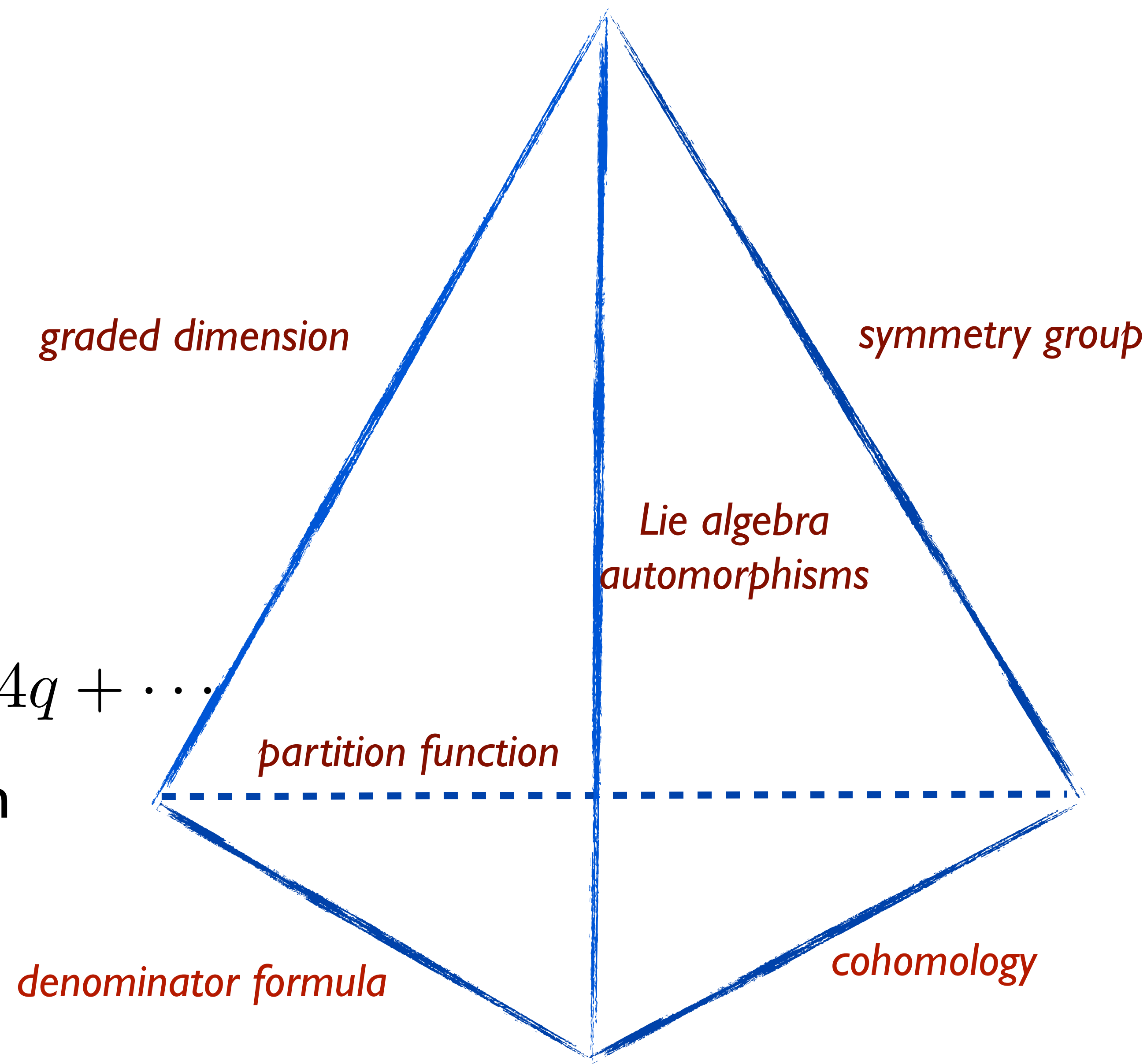

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In more detail, monstrous moonshine involves the following reasoning

Conjecture: there exists $V^\natural = \bigoplus_{n=-1}^{\infty} V_n^\natural$ graded $\mathbb{M}$-module (VOA)

graded dimension: $\operatorname{gdim} V^\natural = \sum_{n=-1}^{\infty} (\dim V_n^\natural) q^n = q^{-1} + 196884q + \cdots = J(\tau)$

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McKay-Thompson series $T_g(\tau) = \mathrm{Tr}_{V^\natural}(g\, q^{L_0 - 1}) = \sum_{n=-1}^{\infty} \mathrm{Tr}_{V_n^\natural}(g) q^n \qquad \forall g \in \mathbb{M}$

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Moonshine conjecture (Conway-Norton): The McKay-Thompson series are modular-invariant under some genus zero $\Gamma_g \subset SL(2, \mathbb{R})$

Γ_g genus zero $\longleftrightarrow$ $\Gamma_g \backslash \mathbb{H} \sim$ 

The module $V^\natural$ was **constructed** by Frenkel-Lepowsky-Meurman and the Conway-Norton conjecture was **proven** by Borcherds in 1992.

In 2010, Eguchi, Ooguri, Tachikawa conjectured that there is Moonshine in the elliptic genus of K3 connected to the finite sporadic group $M_{24} \subset S_{24}$

EOT observation: Fourier coefficients of K3-elliptic genus are (sums of) dimensions of irreps of M_{24}

A completely new moonshine phenomenon to explore!

Monstrous Moonshine	Mathieu Moonshine
monster group $\mathbb{M}$	Mathieu group M_{24}
bosonic CFT	superconformal field theory
Virasoro algebra	$\mathcal{N}=(4,4)$ superconformal algebra
J-function	elliptic genus of K3
McKay-Thompson series	twining genera
monster module $V^\natural$	**?**
monster Lie algebra $\mathfrak{m}$	**?**

[Eguchi, Ooguri, Tachikawa][Cheng][Gaberdiel, Hohenegger, Volpato][Eguchi, Hikami][Gannon]

Despite this amazing progress, we still don't understand ***why Mathieu moonshine holds***. More precisely, we cannot answer the question:

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The aim of this talk is to describe a "two-step generalization" of Mathieu moonshine that will hopefully help to shed light on this question.

Mathieu moonshine → **generalized Mathieu moonshine** → **second-quantized Mathieu moonshine**

Outline

1. (Generalized) Mathieu moonshine

2. Second quantization & black hole counting

3. Connection with umbral moonshine

4. Summary and outlook

1. (Generalized) Mathieu moonshine

Mathieu Moonshine

Non-linear sigma models with target space $K3$

$\longrightarrow$ $\mathcal{N}=(4,4)$ superconformal algebra with $c=6$

$\longrightarrow$ Large moduli space of such theories:

$$\mathcal{M}=O(4,20;\mathbb{Z})\backslash O(4,20;\mathbb{R})/(O(4)\times O(20))$$

The physical spectrum varies over moduli space, but there is a graded "partition function", the **elliptic genus**, which is constant.

The elliptic genus of $K3$ only receives contributions from the right-moving RR-ground states.

The elliptic genus is defined by [Witten][Kawai, Yamada, Yang] $(q = e^{2\pi i\tau}, y = e^{2\pi iz})$

$$\phi_{K3} = \mathrm{Tr}_{\mathcal{H}_{RR}}\left((-1)^{J_0+\bar{J}_0} q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24} y^{J_0}\right)$$

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Cartan generator in the left $SU(2)$ of $\mathcal{N}=(4,4)$

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Virasoro generator

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→ **Holomorphic** in both τ and z

→ **Modular** and **elliptic** properties:

arbitrary left-movers, but only right-moving ground states contribute

$$\phi_{K3}\Big(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\Big) = e^{2\pi i \frac{cz^2}{c\tau+d}} \phi_{K3}(\tau, z) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

$$\phi_{K3}(\tau, z+\ell\tau+\ell') = e^{-2\pi i(\ell^2\tau+2\ell z)} \phi_{K3}(\tau, z) \qquad \ell, \ell' \in \mathbb{Z}$$

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follows from spectral flow

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This identifies ϕ_{K3} with a **weak Jacobi form** of weight 0 and index 1:

$$\phi_{K3}=\phi_{0,1}=8\left(\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2}+\frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2}+\frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2}\right)$$

[Eguchi, Ooguri, Taormina, Yang]

The elliptic genus further satisfies

$$\phi_{K3}(\tau, z=0) = \chi(K3) = 24$$

Now denote by $\mathcal{H}$ the space of states contributing to the elliptic genus.

We have a decomposition of $\mathcal{H}$ into irreps of the left superconformal algebra, which induces a decomposition of the elliptic genus: [Ooguri]

$$\phi_{K3}(\tau, z) = 20 \cdot \chi_{h=\frac{1}{4}, j=0} - 2 \cdot \chi_{h=\frac{1}{4}, j=\frac{1}{2}} + \sum_{n=1}^{\infty} A_n \cdot \chi_{h=\frac{1}{4}+n, j=\frac{1}{2}}$$

where $\chi_{h,j}(\tau, z) = \mathrm{Tr}_{\mathcal{H}_{h,j}}\left((-1)^{J_0} q^{L_0 - \frac{c}{24}} y^{J_0}\right)$

$\mathcal{N}=4$ *character*

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massive representations

massless representations

Eguchi, Ooguri, Tachikawa observed

$$A_1 = 90 = \mathbf{45} + \overline{\mathbf{45}}$$

$$A_2 = 462 = \mathbf{231} + \overline{\mathbf{231}}$$

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Dimensions of irreducible representations of the largest Mathieu group M_{24} !

EOT conjecture: $\mathcal{H} \circlearrowleft M_{24}$

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$\longrightarrow$ $M_{24} \subset S_{24}$ permutation group on 24 elements

$\longrightarrow$ $M_{24} = \mathrm{Aut}(A_1^{24})/\mathrm{Weyl}$

$\longrightarrow$ $\mathrm{order}(M_{24}) = 244823040$

26 irreducible representations

26 conjugacy classes

By now, the Mathieu Moonshine conjecture has been established in the sense that all the **twining genera**

$$\phi_h(\tau, z) = \mathrm{Tr}_{\mathcal{H}_{RR}}\left(h\,(-1)^{J_0+\bar{J}_0} q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24} y^{J_0}\right)$$

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However, this still does not resolve the issue of ***why*** this holds.

In fact, it has been shown that there is **no superconformal sigma model on K3 with M_{24} -symmetry**

[Gaberdiel, Hohenegger, Volpato] (see also [Huybrechts])

Generalized Mathieu Moonshine

[Gaberdiel, D.P., Ronellenfitsch, Volpato]

We shall now generalize this story by introducing a larger family of functions, the ***twisted twining genera***

$$\phi_{g,h} \ : \ \mathbb{H} \times \mathbb{C} \to \mathbb{C}$$

for each commuting pair

$$g, h \in M_{24}$$

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This is the analogue of Norton's **generalized monstrous moonshine**

$$Z_{g,h} \,:\, \mathbb{H} \to \mathbb{C} \qquad g, h \in \mathbb{M}$$

$$Z_{e,h}(\tau) = T_h(\tau)$$

McKay-Thompson series

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Partially explained by orbifolds of the FLM monster VOA $V^\natural$.

[Dixon, Ginsparg, Harvey]
[Tuite]

Proven in special cases but the full conjecture still open.

[Dong, Li, Mason][Höhn][Tuite]
[Carnahan]

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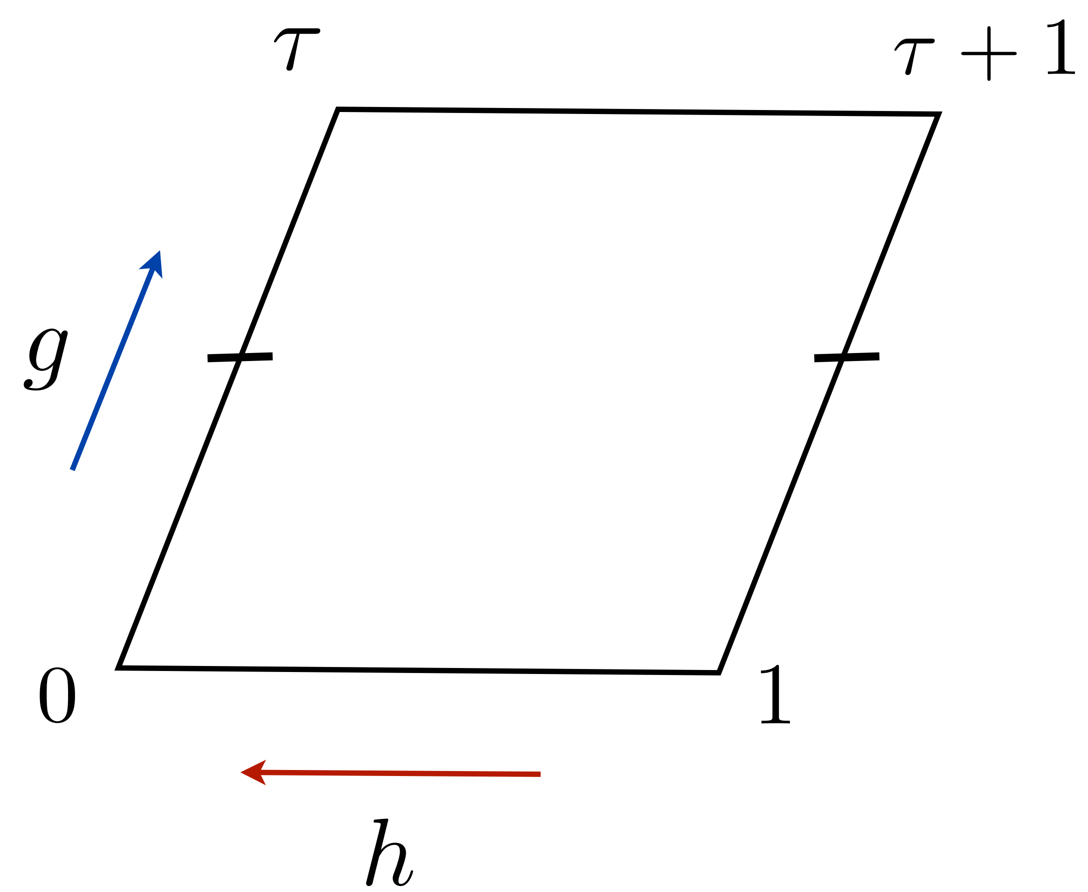
Can we also interpret generalized Mathieu moonshine in terms of orbifolds?

In the path-integral framework we should have

$$\phi_{g,h}(\tau, z) =$$



Modular transformations mixes the boundary conditions

$$SL(2,\mathbb{Z}) \quad : \quad \phi_{g,h} \longrightarrow \phi_{g^a h^c, g^b h^d}$$

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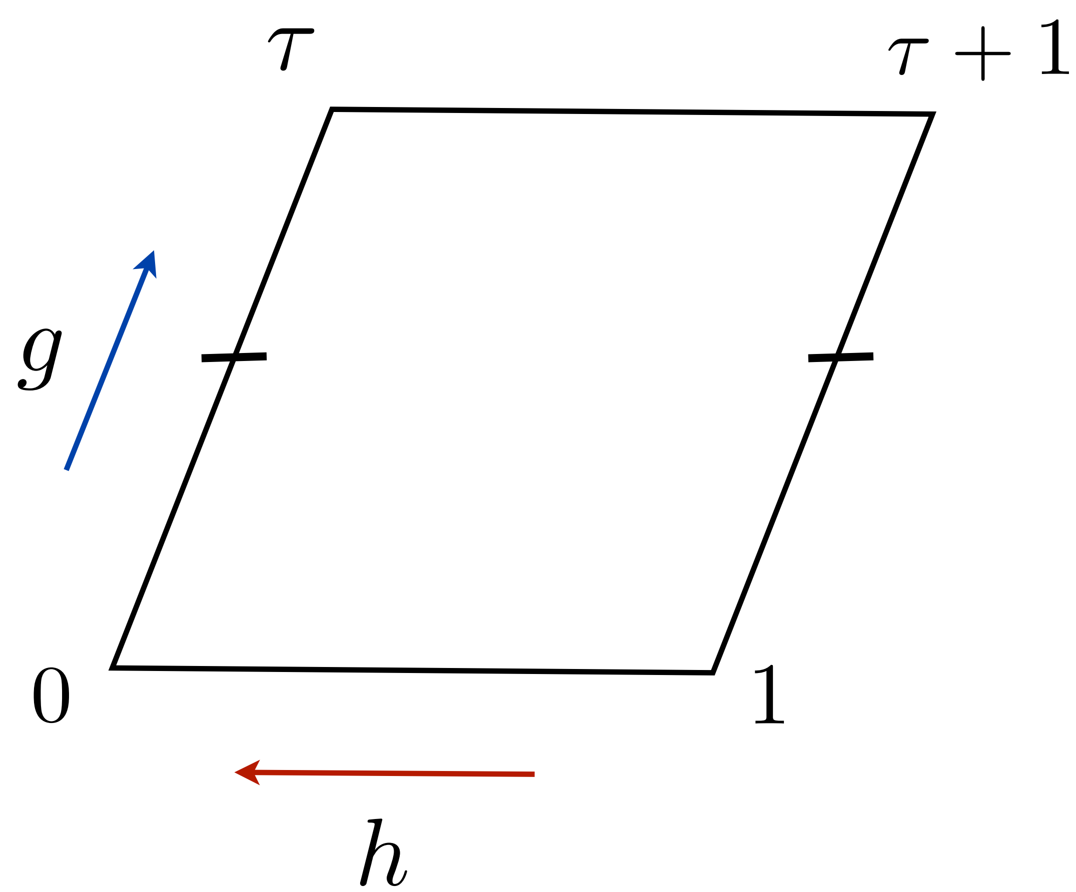

In the operatorial description this is equivalent to

$$\phi_{g,h}(\tau,z) = \mathrm{Tr}_{\mathcal{H}_g}\left(h\,(-1)^{J_0+\bar{J}_0} q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24} y^{J_0}\right)$$

$\mathcal{H}_g$ is the g -twisted sector in the orbifold theory

To make sense of this let us recall some facts about **orbifolds**

Holomorphic Orbifolds and Group Cohomology

Our main **assumption** is that *the twisted twining genera behave similarly as for characters of a holomorphic orbifold*

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Consider a ***holomorphic*** CFT $\mathcal{V}$ with $G = \mathrm{Aut}(\mathcal{V})$ a finite group

→ For each $g \in G$ there is a *unique twisted sector* $\mathcal{V}_g$ with character

$$Z_{g,h}(\tau) = \mathrm{Tr}_{\mathcal{V}_g}\left(\rho_g(h)\, q^{L_0-1}\right) \qquad \forall h \in C_G(g) = \mathrm{Aut}(\mathcal{V}_g)$$

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where $\rho_g(h)$ is a *projective representation* of $C_G(g) = \{h \in G | hg = gh\}$

$$\rho_g(h_1)\rho_g(h_2) = c_g(h_1, h_2)\rho_g(h_1 h_2)$$

with c_g a 2-cocycle

$$c_g \,:\, C_G(g) \times C_G(g) \to U(1)$$

[Dijkgraaf, Vafa, Verlinde, Verlinde][Dijkgraaf, Witten][Dong, Li, Mason]

Holomorphic Orbifolds and Group Cohomology

Fact: Consistent holomorphic orbifolds are classified by $H^3(G, U(1))$.

[Dijkgraaf, Witten][Dijkgraaf, Pasquier, Roche][Bantay][Coste, Gannon, Ruelle]

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The 2-cocycle c_g is determined by a class $[\alpha] \in H^3(G, U(1))$ via the formula

$$c_h(g_1, g_2) = \frac{\alpha(h, g_1, g_2)\alpha(g_1, g_2, (g_1 g_2)^{-1} h (g_1 g_2))}{\alpha(g_1, h, h^{-1} g_2 h)}$$

and this also implies that $[c_g] \in H^2(C_G(g), U(1))$

Then, under modular transformations the twisted twining characters of a holomorphic orbifold obey the relations

[Bantay][Coste, Gannon, Ruelle]

$$Z_{g,h}(\tau+1) = c_g(g,h) Z_{g,gh}(\tau)$$

$$Z_{g,h}(-1/\tau) = \overline{c_h(g,g^{-1})} Z_{h,g^{-1}}(\tau)$$

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Moreover, under conjugation of g, h one has the general relation

$$Z_{g,h}(\tau) = \frac{c_g(h,k)}{c_g(k,k^{-1}hk)} Z_{k^{-1}gk,k^{-1}hk}(\tau) \qquad \forall k \in G$$

Cohomological Obstructions from $H^3(G)$

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Whenever k commutes with both g and h one finds

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So $Z_{g,h} = 0$ unless the 2-cocycle c_g is *regular*:

$$c_g(h,k) = c_g(k,h)$$

When this is not satisfied we have **obstructions**! [Gannon]

Conjecture (generalized Mathieu moonshine) [GHPV]:

For each $g \in M_{24}$ *there exists a graded unitary representation* $\mathcal{H}_g$ *of* $\mathcal{N} = 4$

with central charge $c = 6$ *carrying a projective representation*

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For each commuting pair $g, h \in M_{24}$ *there exists functions* $\phi_{g,h}(\tau, z)$ *satisfying:*

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- $\phi_{g,h}(\tau, z) = \xi(k)\phi_{k^{-1}gk,k^{-1}hk}(\tau, z), \qquad \forall k \in M_{24}$

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- $\phi_{g,h}\left(\dfrac{a\tau+b}{c\tau+d}, \dfrac{z}{c\tau+d}\right) = \chi_{g,h}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) e^{2\pi i \frac{cz^2}{c\tau+d}} \phi_{g^a h^c, g^b h^d}(\tau,z), \qquad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$

Conjecture (generalized Mathieu moonshine) [GHPV]:

For each $g \in M_{24}$ *there exists a graded unitary representation* $\mathcal{H}_g$ *of* $\mathcal{N}=4$ *with central charge* $c=6$ *carrying a projective representation*

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$R_{g,r}$ representation of a central extension of $C_{M_{24}}(g)$

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- *The phases* $\xi_{g,h}$, $\chi_{g,h}$ *and the central extension of* $C_{M_{24}}(g)$ *are determined by the same class* $[\alpha] \in H^3(M_{24}, U(1))$

Theorem [GHPV]:

- *For each commuting pair* $g, h \in M_{24}$ *there exists functions* $\phi_{g,h}(\tau, z)$ *satisfying all the expected modular properties with respect to subgroups* $\Gamma_{g,h} \subset SL(2,\mathbb{Z})$
- *There is a unique class* $[\alpha] \in H^3(M_{24}, U(1))$ *which determines all the modular phases.*
- *Many of the* $\phi_{g,h}$ *vanish due to cohomological obstructions controlled by* $H^3(M_{24}, U(1))$

(in deriving these results we use the fact that $H^3(M_{24}, U(1)) \cong \mathbb{Z}_{12}$ [Ellis, Dutour-Sikiric] *)*

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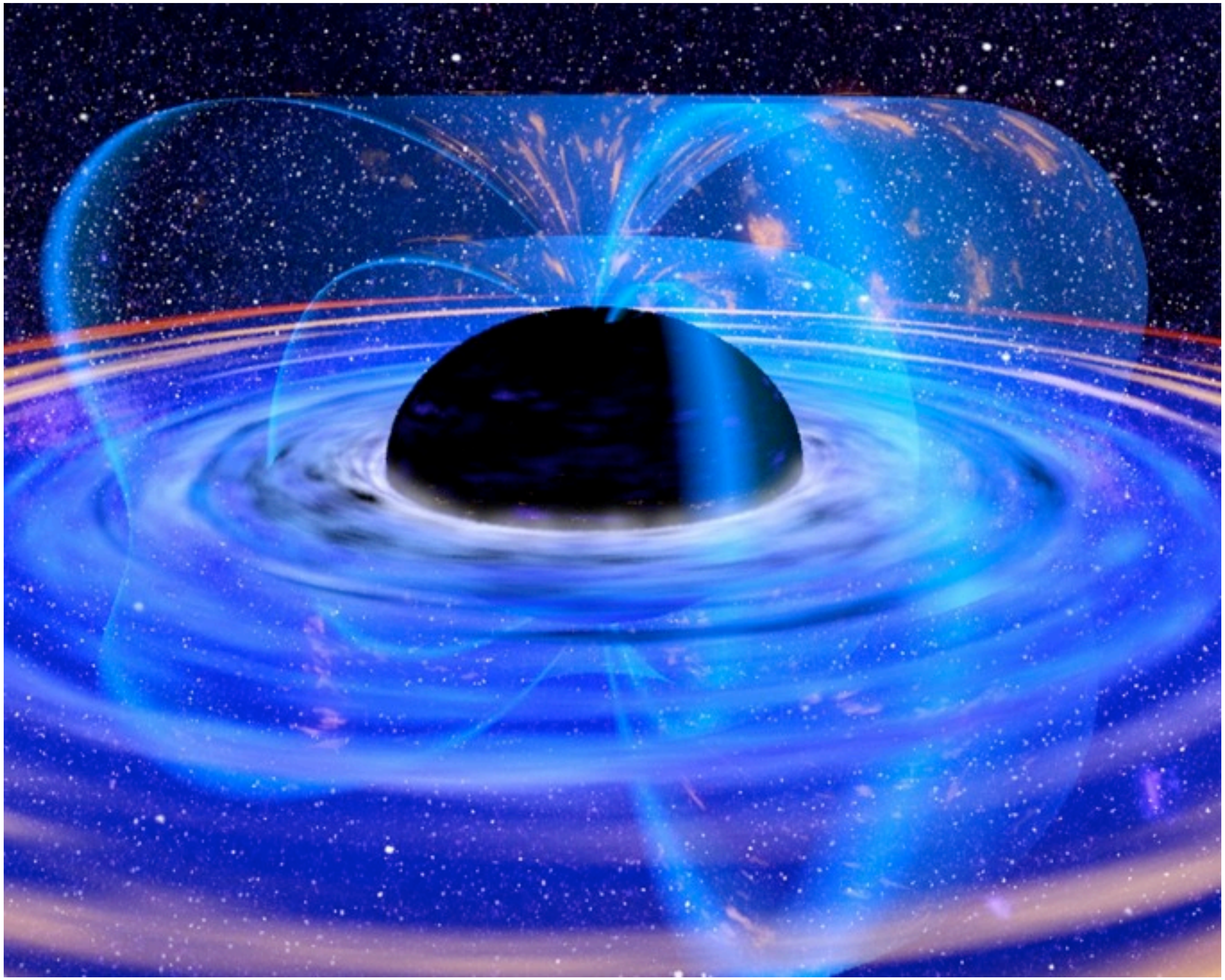
This is very strong evidence that generalized Mathieu Moonshine holds!

But what is the physical interpretation?

2. Second quantization & black hole counting



Second quantized elliptic genus

Let X be a Calabi-Yau manifold and $\chi(X;\tau,z)$ its elliptic genus.

$\longrightarrow \chi(X;\tau,z)$ is a weak Jacobi form of weight zero and index $(\dim_{\mathbb{C}} X)/2$ [Gritsenko]

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This is the generating function of elliptic genera of symmetric products of X

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DMVV proved the following remarkable formula:

$$\Psi_X(\sigma,\tau,z) = \exp\left[\sum_{L=1}^{\infty} p^L T_L \chi(X;\tau,z)\right] = \prod_{\substack{n>0,\, m\geq 0\\ \ell\in\mathbb{Z}}} (1-p^n q^m y^\ell)^{-c_X(mn,\ell)}$$

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Hecke operator

$$T_L : J_{0,m} \to J_{0,mL}$$

Fourier coefficients of

$$\chi(X;\tau,z) = \sum_{k\geq 0,\ell\in\mathbb{Z}} c_X(k,\ell) q^k y^\ell$$

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Gritsenko later showed that

$$\Phi_X(\sigma,\tau,z) := \frac{A_X(\sigma,\tau,z)}{\Psi_X(\sigma,\tau,z)}$$

Siegel modular form of weight $c_X(0,0)/2$

A_X is called the "Hodge anomaly"

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This is an example of a **Borcherds lift:**

$$\Phi : \boxed{\begin{matrix}\text{Jacobi}\\ SL(2,\mathbb{Z})\end{matrix}} \longrightarrow \boxed{\begin{matrix}\text{automorphic}\\ SO(3,2;\mathbb{Z})\end{matrix}}$$

Counting dyons in $\mathcal{N}=4$ string theory

For X a K3-manifold we have that

Igusa cusp form of weight 10 for

$Sp(4;\mathbb{Z})$

$$\Phi_X = \Phi_{10} = pqy \prod_{m,n,\ell)>0} (1-p^m q^n y^\ell)^{c(mn,\ell)}$$

$$\chi(K3;\tau,z) = 2\phi_{0,1}(\tau,z) = \sum_{n\geq 0,\ell\in\mathbb{Z}} c(n,\ell) q^n y^\ell$$

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1/4 BPS states are counted by the sixth helicity supertrace [Kiritsis]

$$B_6(P,Q) := \frac{1}{6!}\mathrm{Tr}_{\mathcal{H}_{P,Q}}\left((-1)^J(2J)^6\right)$$

J = helicity

electric-magnetic charges $(P,Q)\in\Gamma^{6,22}\oplus\Gamma^{6,22}$

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invariant under $SL(2,\mathbb{Z}) \times SO(6,22;\mathbb{Z})$ $\longrightarrow$ can only depend on the combinations

$$P^2, Q^2, Q \cdot P$$

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The formula is

$$\frac{1}{\Phi_{10}} = \sum_{m,n,\ell} d(m,n,\ell) p^m q^n y^\ell$$

with the identification

$$B_6(P,Q) = d\left(\frac{Q^2}{2}, \frac{P^2}{2}, P \cdot Q\right)$$

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The index is locally constant but *jumps at codimension one walls* in moduli space:

"wall-crossing formula"

$$\lim_{z \to 0} \frac{\Phi_{10}(\sigma, \tau, z)}{(2\pi i z)^2} = \eta(\sigma)^{24} \eta(\tau)^{24}$$

1/2-BPS

1/4-BPS

1/2-BPS

Second quantized twisted twining genera

Inspired by this we seek a similar **spacetime interpretation** for the twisted twining genera $\phi_{g,h}(\tau, z)$ of generalized Mathieu moonshine.

This generalizes earlier results by Cheng for the case $(g,h)=(e,h)$

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$$\mathcal{T}_L^{\alpha} \phi_{g,h}(\tau, z) := \frac{1}{L} \sum_{\substack{a,d>0 \\ ad=L}} \sum_{b=0}^{d-1} \chi_{g,h} \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \phi_{g^d, g^{-b}, h^a} \left(\frac{a\tau + b}{d}, az\right)$$

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multiplier phase determined by

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moduli space of principal M_{24} -bundles on the elliptic curve E_τ

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This implies that for L sufficiently large $\mathcal{T}_L^{\alpha}\phi_{g,h}$ has trivial multiplier phase

→ Even if $\phi_{g,h}$ vanishes by cohomological obstructions, all the second quantized twisted twining genera $\Psi_{g,h}$ are unobstructed!

Results (D.P., Volpato):

The second quantized twisted twining genera satisfy the following properties

- *Infinite product formula*

$$\frac{1}{\Psi_{g,h}(\sigma,\tau,z)} = \prod_{d=1}^{\infty}\prod_{m=0}^{\infty}\prod_{\ell\in\mathbb{Z}}\prod_{t=0}^{M-1}(1-e^{\frac{2\pi it}{M}}q^{\frac{m}{N\lambda}}y^{\ell}p^{d})^{\hat{c}_{g,h}(d,m,\ell,t)}$$

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$$M=\mathcal{O}(h) \qquad N=\mathcal{O}(g)$$

λ *length of the shortest cycle of* g *in its 24-dim permutation reps*

$$\hat{c}_{g,h}(d,m,\ell,t) := \sum_{k=0}^{M-1}\sum_{b=0}^{\lambda N-1}\frac{e^{-\frac{2\pi i t k}{M}}}{M}\frac{e^{\frac{2\pi i b m}{\lambda N}}}{\lambda N}\chi_{g,h}\left(\begin{smallmatrix} k & b \\ 0 & d\end{smallmatrix}\right)c_{g^d,g^{-b}h^k}(\frac{md}{N\lambda},\ell)$$

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- *The ratio*

$$\Phi_{g,h}(\sigma,\tau,z) := \frac{A_{g,h}(\sigma,\tau,z)}{\Psi_{g,h}(\sigma,\tau,z)}$$

is a Siegel modular form for a subgroup $\Gamma^{(2)}_{g,h} \subset Sp(4;\mathbb{R})$

For $g=e$ *this was conjectured by Cheng and partially proven by Raum.*

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is a Siegel modular form for a subgroup $\Gamma^{(2)}_{g,h}\subset Sp(4;\mathbb{R})$

For $g=e$ *this was conjectured by Cheng and partially proven by Raum.*

"Hodge anomaly"

$$\boxed{A_{g,h} = -p\frac{\vartheta_1(\tau,z)^2}{\eta(\tau)^6}\eta_{g,h}(\tau)}$$

Mason's generalized eta-products

Results (D.P., Volpato):

The second quantized twisted twining genera satisfy the following properties

- *Infinite product formula*

$$\frac{1}{\Psi_{g,h}(\sigma,\tau,z)} = \prod_{d=1}^{\infty}\prod_{m=0}^{\infty}\prod_{\ell\in\mathbb{Z}}\prod_{t=0}^{M-1}(1-e^{\frac{2\pi i t}{M}}q^{\frac{m}{N\lambda}}y^{\ell}p^{d})^{\hat{c}_{g,h}(d,m,\ell,t)}$$

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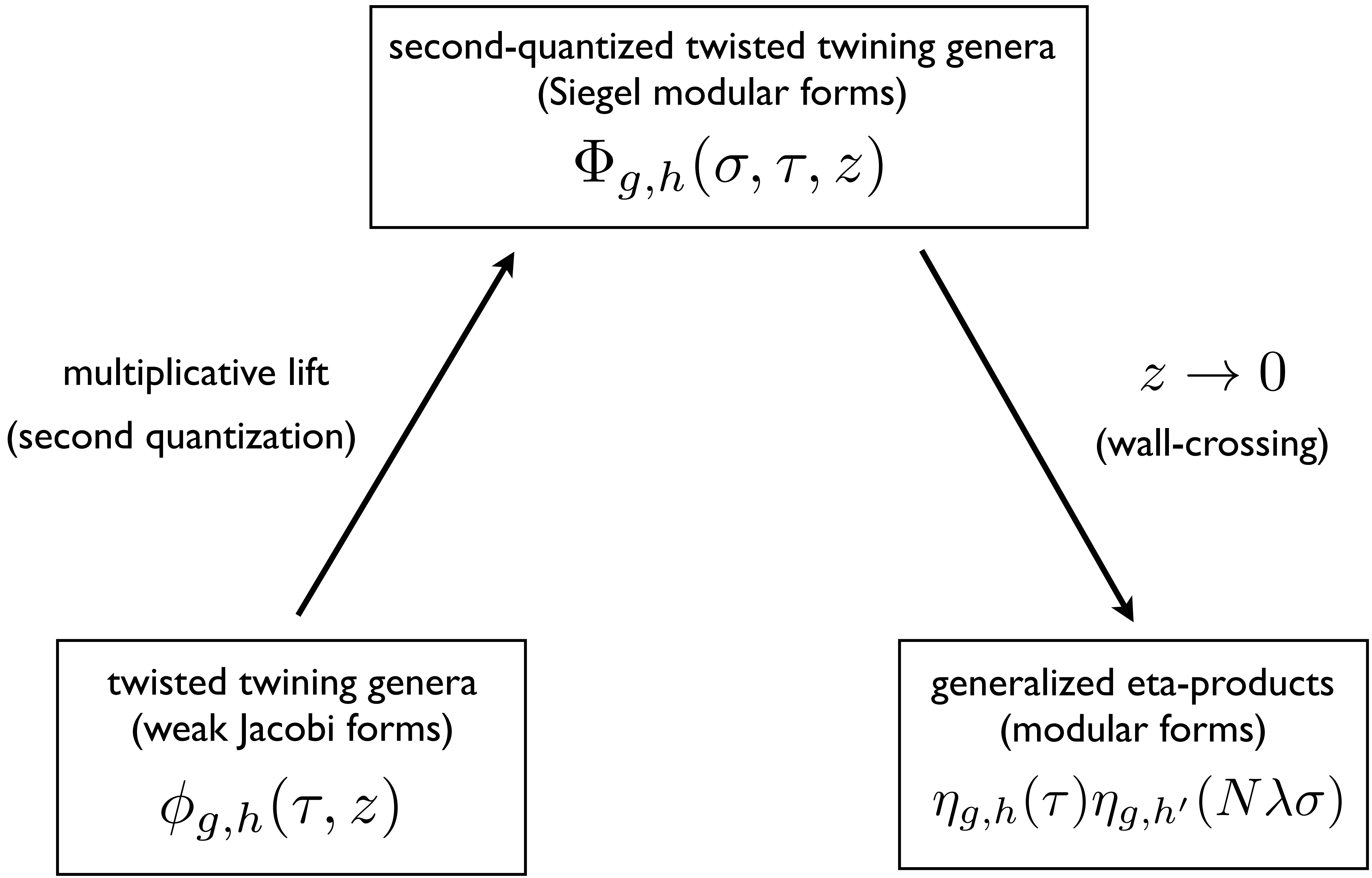
- *"Wall-crossing formula"*

$$\lim_{z\to 0}\frac{\Phi_{g,h}(\sigma,\tau,z)}{(2\pi i z)^2} = \eta_{g,h}(\tau)\eta_{g,h}(N\lambda\sigma)$$

This resolves a puzzle about the connection with Mason's old version of generalized M_{24} -moonshine for eta-products

(For $g = e$ this was observed previously by Cheng and Govindarajan.)



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$\boxed{\text{new } \mathcal{N} = 4 \text{ theory}}$

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[Chaudhuri, Hockney, Lykken]

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In this orbifold theory we have "twisted" dyon states counted by the twisted BPS-index

$$B_{6;g,h}(P,Q):=\frac{1}{6!}\mathrm{Tr}_{\mathcal{H}^g_{Q,P}}\left(h(-1)^{2J}(2J)^6\right)$$

[Sen]

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Computed for some pairs of symmetries [Dabholkar, Gaiotto][Dabholkar, Nampuri][Jatkar, Sen][David][Dabholkar, Cheng][Govindarajan][Sen]...

$$\longrightarrow \quad B_{6;g,h}(P,Q) = d_{g,h}\left(\frac{Q^2}{2},\frac{P^2}{2},Q\cdot P\right)$$

Coincides with Fourier coefficients of

$$\Phi_{g,h}$$

for some pairs (g,h) *!*

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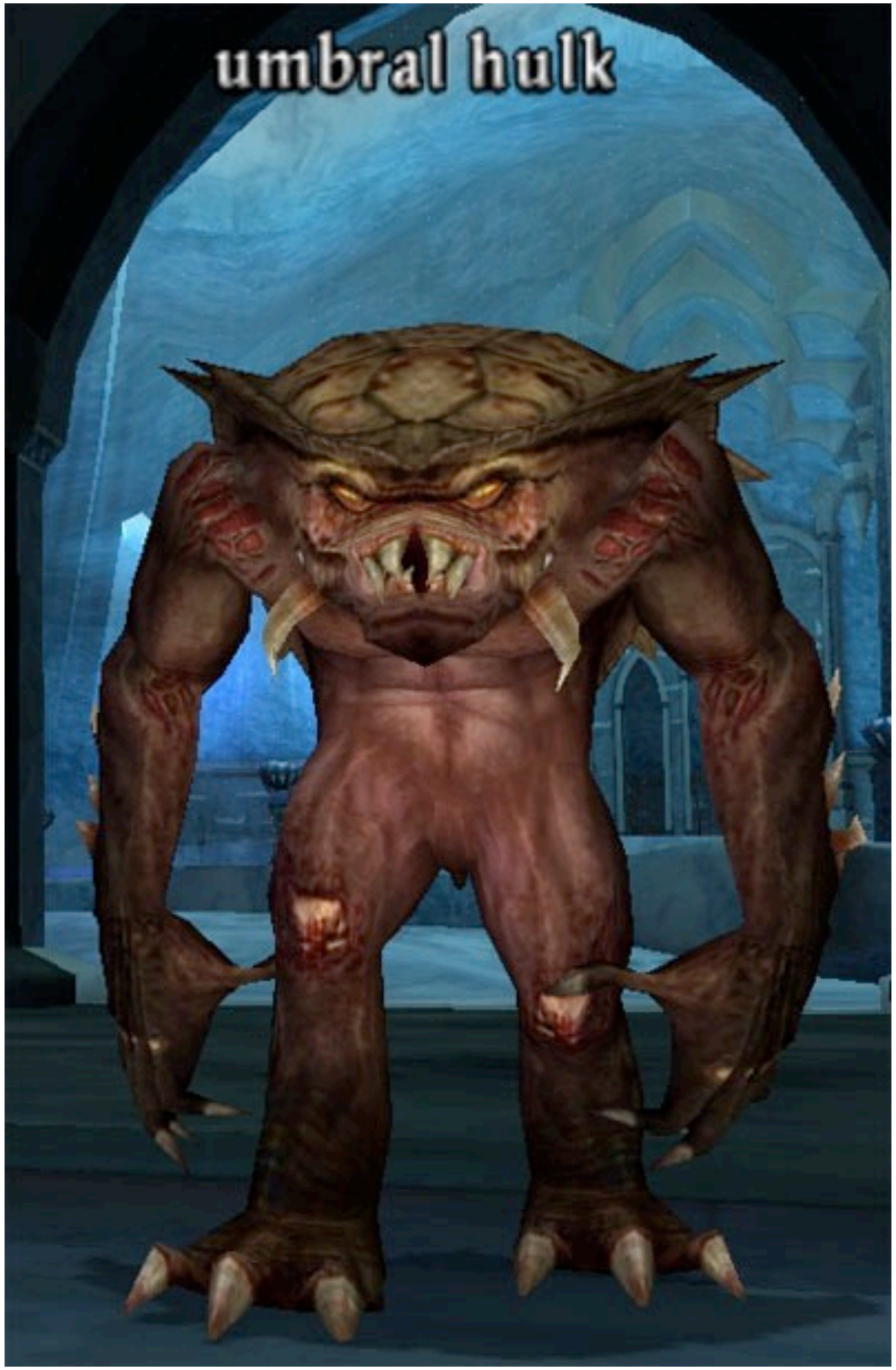
Could it be that all of the $\Phi_{g,h}$ have interpretations as partition functions for BPS-dyons?



3. Connection with umbral moonshine

Umbral moonshine

Cheng, Duncan, Harvey proposed a generalization of Mathieu moonshine including 5 additional examples of pairs:

$$(G^{(\ell)}, Z^{(\ell)}) \qquad\qquad \ell \in \{2, 3, 4, 5, 7, 13\}$$

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None of the other umbral Jacobi forms appear to have interpretations as elliptic genera.

Most of the umbral groups are not sporadic.

Starting from the umbral Jacobi forms Cheng-Duncan-Harvey constructed a class of Siegel modular forms using a standard **Borcherds lift:**

$$\Phi^{(\ell)} = p^{A(\ell)} q^{B(\ell)} y^{C(\ell)} \prod_{(m,n,r)>0} (1 - p^m q^n y^r)^{c^{(\ell)}(mn,r)}$$

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For $\ell \in \{2,3,4,5\}$ one has $\Phi^{(\ell)} = (\Delta_k)^2$ $\qquad k = \frac{7-\ell}{\ell-1}$

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We observe that these Siegel modular forms coincide with some of the functions in generalized Mathieu moonshine

$$(2A, 2A) \,:\, \Phi_{g,h} = (\Delta_2)^2 = \Phi^{(3)}$$

$$(3A, 3A) \,:\, \Phi_{g,h} = (\Delta_1)^2 = \Phi^{(4)}$$

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Overlap between umbral moonshine and generalized Mathieu moonshine!

A modular coincidence or an indication of some deeper relation?

4. Summary and outlook

Summary

- We have established that generalised Mathieu moonshine holds by computing all twisted twining genera $\phi_{g,h}$.

- Twisted twining genera can be expanded in projective characters of $C_{M_{24}}(g)$.

- A key role is played by the third cohomology group $H^3(M_{24}, U(1))$.

- All the second quantized twisted twining genera found and verified to be Siegel modular forms

- Some of these correspond to partition functions of twisted dyons in CHL-models

Outlook

- Can one construct a generalised Kac-Moody algebra for each conjugacy class $[g] \in M_{24}$? (c.f. [Borcherds][Carnahan])

- Relation with BPS-algebras à la Harvey Moore...?

- Generalised Umbral Moonshine...? [Cheng, Duncan, Harvey]

- Recent interesting results indicate that there is an N=2 version of Mathieu Moonshine in heterotic string theory. [Cheng, Dong, Duncan, Harvey, Kachru, Wrase]

- Umbral Moonshine from multiple NS5-branes? [Harvey, Murthy]

- Can one construct an action of M_{24} on the topological (half-twisted) sigma model on K3? (for attempts in this direction, see [Creutzig, Höhn])

What does M_{24} act on?

Our results strongly suggests that there is something like a **holomorphic vertex operator algebra** underlying Mathieu Moonshine

...but which one remains a mystery...

Example: $8A$ -**twist** and $2B$ -**twine**:

$$\phi_{8A,2B}(\tau,z)=\frac{\eta\left(\frac{\tau}{2}\right)^6}{\eta(\tau)^6}\frac{\vartheta_1(\tau,z)^2}{\vartheta_4(\tau,0)^2}$$

$8A$ = M_{24} -conjugacy class of order 8 elements.

$\phi_{8A,2B}(\tau,z)$ is a Jacobi form of weight 0 index 1 for the group

$$\Gamma_{8A,2B}:=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})\mid b\equiv 0 \mod 4\right\}=\Gamma^0(4)$$

Multiplier given by:

$$\phi_{8A,2B}(\tau+4,z)=\frac{\prod_{i=0}^{3}c_g(g,g^ih)}{c_{g^4h}(g,g^{-1})c_{g^{-1}}(g^4h,g^4h)}\frac{c_{g^{-1}}(g^4h,k)}{c_{g^{-1}}(k,h)}\phi_{8A,2B}(\tau)=-\phi_{8A,2B}(\tau)$$

using our result for $c_{g_1}(g_2,g_3)$ in terms of $\alpha\in H^3(M_{24},U(1))$