Localization and quantum AdS_4/CFT_3 holography

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Based on: `arXiv:1406.0505` - Atish Dabholkar, N.D. and João Gomes

Exact Results in SUSY Gauge Theories in Various Dimensions

CERN

Aug 21, 2014.

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- In 3d one can calculate an S^3 partition function and Wilson loop, $S^2 \times S^1$ index (for $\mathcal{N} \geq 2$) and again a few more.
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- So if localization works for non-renormalizable field theories, then why not supergravity?

Exact results and AdS/CFT

- Today I will present work on AdS_4/CFT_3, whose most famous example is ABJM theory.
- This is of course dual to M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$ (or IIA on $AdS_4 \times \mathbb{CP}^3$).
- The partition function of ABJM on S^3 in the large N and large $\lambda = N/k$ indeed agrees with the renormalized classical action of these SUGRA theories. [Drukker Mariño,Putrov]
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- We will attempt to go beyond that, to all orders in $1/N$.

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- But we don't know how to define M-theory, nor do we know how to extend SUSY transformations off-shell, which is necessary for SUSY localization.
- Workaround: Use an effective 4d SUGRA with an AdS_4 solution.

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Outline

- Introduction and motivation
- Localization of 3d theories on S^3
  - Solving the ABJM matrix model
  - Universal Airy function
- Localization in AdS_4 supergravity
  - The theory
  - The localization solution
  - The action
- Summary

Localization on S^3

[Kapustin Willett, Yaakov] [Hama Hosomichi, Lee] [Jafferis]

- Consider any $\mathcal{N} = 4$ super Chern-Simons matter theory on S^3.
- Add to the action a Q-exact term of the form $t\, Q(\Psi Q \Psi)$.
- VEV of Q-invariant observables is unmodified by this insertion.
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- Each $U(N)$ vector multiplet will reduce an N dimensional integral where a Chern-Simons at level $k^{(i)}$ gives

$$\int\prod_{a=1}^{N}\frac{d\nu_a^{(i)}}{2\pi}\,e^{\frac{ik^{(i)}}{4\pi}(\nu_a^{(i)})^2}\prod_{a<b}\left(2\sinh\frac{\nu_a^{(i)}-\nu_b^{(i)}}{2}\right)^2$$

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- bifundamentals contribute

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- In the case of ABJM theory we have a two node circular quiver with no fundamental matter, only bifundamentals, so one finds

$$Z = \frac{1}{N!^2} \int \prod_{a=1}^{N} \frac{d\nu_a}{2\pi} \frac{d\mu_a}{2\pi} \, e^{\frac{ik}{4\pi} \sum_a (\nu_a^2 - \mu_a^2)} \, \frac{\prod_{a<b} \left(2 \sinh \frac{\nu_a - \nu_b}{2}\right)^2 \left(2 \sinh \frac{\mu_a - \mu_b}{2}\right)^2}{\prod_{a,b} \left(2 \cosh \frac{\nu_a - \mu_b}{2}\right)^2}$$

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- For a more general circular quiver

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Solving the matrix model

- The ABJM matrix model is very similar to that of pure Chern-Simons on a $S^3/\mathbb{Z}_2$, a lens space and can be solved exactly. [Aganagic,Klemm Mariño,Vafa][Halmagyi Yasnov][Drukker Mariño,Putrov]
- Explicit expression exist for the free energy in the genus expansion (large N) to the first few orders. At each order one finds a principle contribution and a series of instanton corrections (both world-sheet and D-brane instantons).

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- The leading behavior of the free energy at large N and $\lambda = N/k$ can be written as

$$F = -\frac{\pi\sqrt{2}}{3}k^2\hat{\lambda}^{3/2} = -\frac{\pi\sqrt{2}}{3}\sqrt{k}\,\hat{N}^{3/2}$$

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- The full all-genus partition function can also be computed (recursively).
- This should be captured by the full quantum string theory partition function on $AdS_4 \times \mathbb{CP}^3$.

Universal Airy function behavior

- The genus expansion of ABJM theory satisfies a holomorphic anomaly equation.
- Ignoring the instanton terms in the planar free energy F_0 the solution to this equation is remarkably simple [Fuji,Hirano Moriyama]

$$Z \approx \mathrm{Ai}\left[(c/k)^{-1/3}\hat{N}\right] \qquad c = \frac{2}{\pi^2}$$

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- By use of the Cauchy identity the matrix model can be written as

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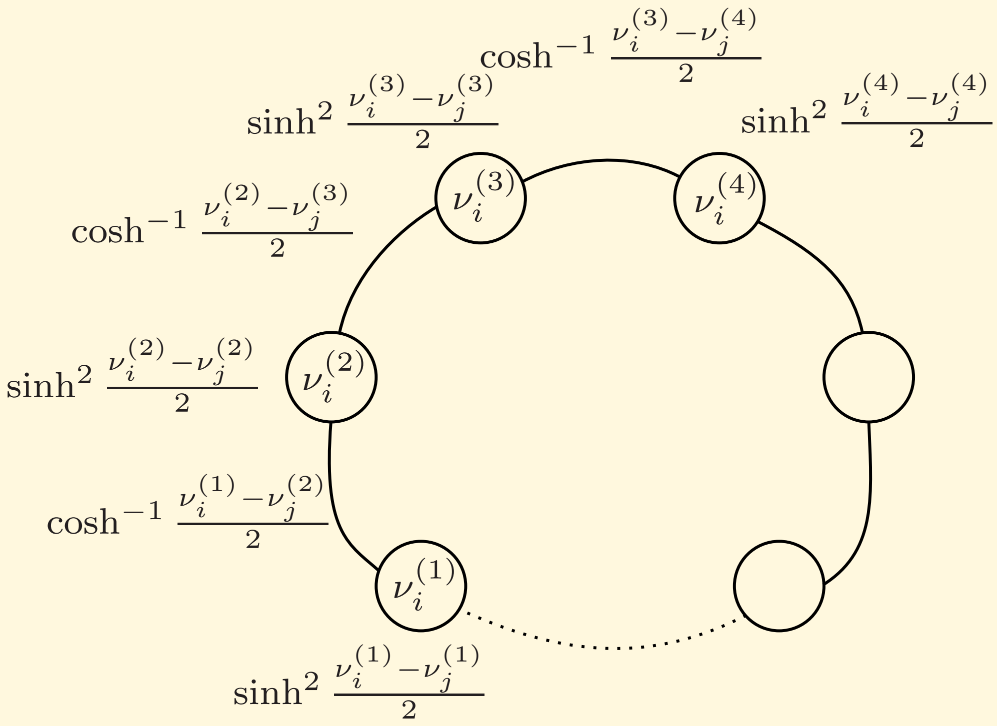
$$= \int \prod_{i=1}^{r} \prod_{a=1}^{N} \frac{d\nu_a^{(i)}}{2\pi} \, e^{\sum_{i=1}^{r} \frac{ik^{(i)}}{4\pi}(\nu_a^{(i)})^2} \sum_{\sigma^{(i)}\in S_N} \prod_{i=1}^{r} (-1)^{\sigma^{(i)}} \frac{1}{\prod_a 2\cosh\frac{\nu_a^{(i)}-\nu_{\sigma_{i+1}(a)}^{(i+1)}}{2}}$$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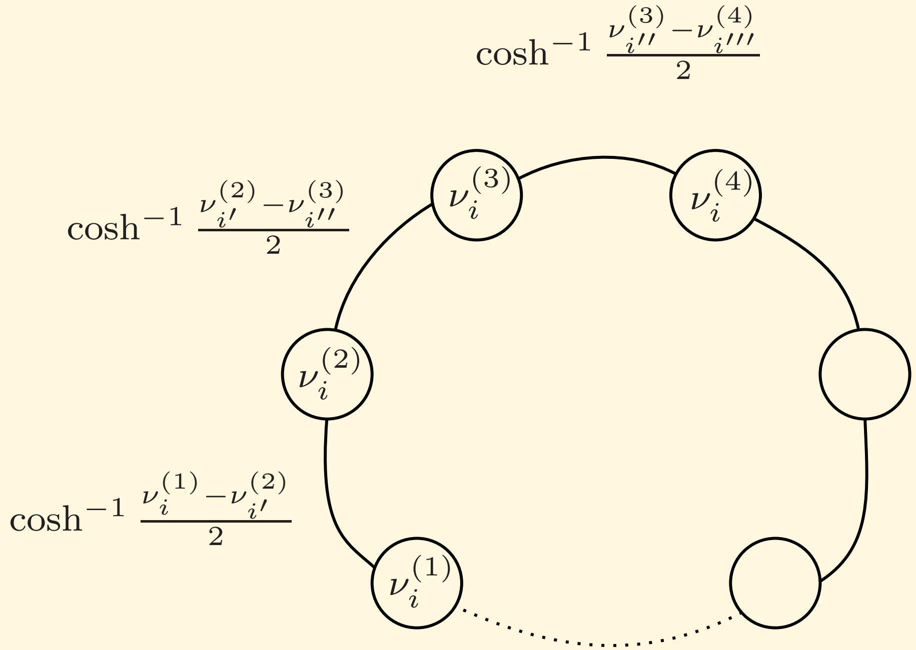
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and we find terms like $\mathrm{Tr}\,\rho^l$.

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$$\Xi(z) = 1 + \sum_{N=1}^{\infty} z^N Z(N) = \det(1+z\rho) = \prod_n (1 + e^{\mu - E_n})$$

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$$H(p,q) = \sum_{i=1}^{r} \log\left[2\cosh\frac{p - \sum_{j=1}^{i}\frac{k^{(j)}}{k}q}{2}\right]$$

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$$H(p,q) \approx \frac{1}{2}\sum_{i=1}^{r}\left|p - \sum_{j=1}^{i}\frac{k^{(j)}}{k}q\right|$$

- and the density of states is given by the derivative of the area of the polygon enclosed by $H = E$

$$\rho(E) = \frac{dn}{dE} \qquad n(E) = \frac{c}{k}E^2 + n_0(1 - e^{-E})$$

- The grand potential is then

$$J(\mu) = \int_0^\infty dE\, \rho(E) \log(1 + e^{\mu - E}) = -\frac{2c}{k} \operatorname{Li}_3(-e^\mu) + n_0 \mu (1 + e^{-\mu}) \log(1 + e^{-\mu}) - n_0$$

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- Now the canonical partition function can be derived from the canonical potential by

$$Z(N) = \frac{1}{2\pi i}\int d\mu\, e^{J(\mu)-N\mu} = \left(\frac{c}{k}\right)^{-1/3} e^{A}\,\mathrm{Ai}\left[\left(\frac{c}{k}\right)^{-1/3}\left(N - \frac{\pi^2 c}{3k} - n_0\right)\right],$$

c and n_0 can be evaluated for any particular model. A depends more intimately on the instanton corrections and can be evaluated perturbatively.

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- We again use localization for that.
  - Choose an off-shell formulation of supergravity.
  - Find all solutions to the BPS equations.
  - Make sure they arise as the saddle point of a Q exact action.
  - Evaluate the one-loop determinant about the saddle points.
  - Integrate over them.

Localization in supergravity

- To reproduce all the $1/N$ corrections we need to do the full quantum gravity path integral.
- We again use localization for that.
  - Choose an off-shell formulation of supergravity.
  - Find all solutions to the BPS equations.
  - Make sure they arise as the saddle point of a Q exact action.
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- I will describe only some of these steps for asymptotically AdS_4 - those which we have succeeded in doing.

Superconformal gravity

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- There is one vector multiplet whose scalar captures the volume of the SE space.
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- We work in a frame where there is no tensor, and a the prepotential has sqare-root form.

Gravity multiplet

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- We did show that AdS_4 is one such solution (and allowed the extra conformal factor).
- Writing a localizing action without rigid SUSY is rather perplexing.
- How does one calculate the 1-loop determinant?

Vector multiplets

- The bosonic action for the vector multiplet is

$$S_{\text{vec}} = \int d^4x\sqrt{g}\Big[N_{IJ}\bar{X}^I X^J\left(\frac{R}{6}+D\right)+N_{IJ}\partial\bar{X}^I\partial X^J-\frac{1}{8}N_{IJ}Y^{ijI}Y^J_{ij}+\Big]$$

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$$\delta\Omega^i_+ = -i\partial\!\!\!/ X\xi^i_- - \frac{1}{2}Y^i_j\xi^j_+ + X\eta^i_+ = 0$$
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- Writing $X = iJ + H$ we find that to preserve *all* SUSYs we need

$$J = \text{const}\,, \qquad H = 0\,, \qquad Y^1{}_1 = -y^2{}_2 = -2i\frac{J}{L}\,, \qquad Y^1{}_2 = Y^2{}_1 = 0$$

where L is the AdS radius and the constant J gets related to each-other by the identification of the coefficient of R with the Newton constant (together with the hypers).

Hypermultiplets

- They hypermultiplet Lagrangian is

$$\mathcal{L}_{\rm hyp} = \Big[- D_\mu A^i{}_\beta D^\mu A_i{}^\alpha - \frac{1}{6} R A^i{}_\beta A_i{}^\alpha + \frac{1}{2} D A^i{}_\beta A_i{}^\alpha + F^i{}_\beta F_i{}^\alpha$$
$$+ 4g^2 A^i{}_\beta \bar{X}^\alpha_\gamma X^\gamma_\delta A_i{}^\delta + g A^i{}_\beta Y^{jk\,\alpha}_\gamma A_k{}^\gamma \epsilon_{ij} \Big] d^\beta_\alpha + \text{fermionic terms}$$

- The fields are fixed (assymptotically) to carry a charge under the vector X_I

$$t_I A_i{}^\alpha = P_I (i\sigma^3)^\alpha{}_\beta A_i{}^\beta$$

- The SUSY variation of the the hypers leads to

$$F_i^\alpha = -i2g A_j^\alpha \sigma^j_{3i} (H \cdot P) = 0, \qquad 2g(J \cdot P) = -\frac{1}{L}$$

- With the prepotential for our model

$$F = \sqrt{X^0 (X^1)^3}$$

  the last relation is refined to

$$2gJ^0 P_0 = -\frac{1}{4L}, \qquad 2gJ^1 P_1 = -\frac{3}{4L}$$

Going off-shell

- We take the AdS metric to be $ds^2 = L^2(d\eta^2 + \sinh^2(\eta)d\Omega_3^2)$.
- With a specific choice of a single SUSY generator (ϵ are S^3 Killing spinors)

$$\begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} = \begin{pmatrix} \chi_+ \times \epsilon_-^1 \\ (\sigma_3 \chi_+) \times \epsilon_-^2 \end{pmatrix} \qquad \chi_+ = \begin{pmatrix} \sinh(\eta/2) \\ -i\cosh(\eta/2) \end{pmatrix}$$

we find a more general solution

$$X = iJ + H = iJ + \frac{Jh}{\cosh(\eta)}, \qquad Y^1{}_1 = -Y^2{}_2 = -2i\frac{J}{L} + 2\frac{Jh}{L\cosh^2(\eta)}$$

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- The hypers A are constant for both the vacuum solution and the BPS instanton, but the auxiliary field is proportional to H

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- These field configurations arise as the saddle points of a localizing action.

Action on localization locus

- The action for the vector multiplet is

$$S_{\text{vec}} = \int d^4x\sqrt{g}\Big[N_{IJ}\bar{X}^I X^J\frac{R}{6} + N_{IJ}\partial\bar{X}^I\partial X^J - \frac{1}{8}N_{IJ}Y^{ijI}Y^J_{ij}\Big]_{\text{Loc. locus}}$$

$$= \Omega_3 L^2\frac{\sqrt{J^0(J^1)^3}}{2i}\int dr(r^2-1)\Big[-\left(1+\frac{(h^0)^2}{r^2}\right)(t+\bar{t})^3 + \frac{3}{4}h^2\left(\frac{1}{t}+\frac{1}{\bar{t}}\right)\frac{r^2-1}{r^4}$$

$$+\frac{3}{2}h^1h^0\,(t+\bar{t})\,\frac{r^2-1}{r^4} - \frac{1}{4}(h^0)^2\left(t^3+\bar{t}^3\right)\frac{r^2-1}{r^4} - \frac{3}{4}\left(\frac{1}{t}+\frac{1}{\bar{t}}\right)\left(1+i\frac{h^1}{r^2}\right)^2$$

$$-\frac{3}{2}(t+\bar{t})\left(1+i\frac{h^1}{r^2}\right)\left(1+i\frac{h^0}{r^2}\right) + \frac{1}{4}\left(t^3+\bar{t}^3\right)\left(1+i\frac{h^0}{r^2}\right)^2\Big]$$

Here $N_{IJ} = \text{Im}(F_{IJ})$, $F_{IJ} = \partial_I\partial_J F$, $t = \sqrt{\frac{X^1}{J^1}/\frac{X^0}{J^0}} = \sqrt{\frac{i+h^1/r}{i+h^0/r}}$ and $r = \cosh(\eta)$.

- This integrates to

$$\Omega_3 L^2 \frac{\sqrt{J^0(J^1)^3}}{2i}\Bigg[\frac{(r-1)^2}{r}\sqrt{\frac{1+ih^1/r}{1+ih^0/r}}(-ih^1(1-2ih^0-r)-2r(2+r)-ih^0(3+r))+$$

$$+\frac{(r+1)^2}{r}\sqrt{\frac{1-ih^1/r}{1-ih^0/r}}(-ih^1(1-2ih^0+r)+2r(2-r)-ih^0(3-r))\Bigg]$$

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- This diverges at large r, which requires careful regularization. With a cutoff r_0 the divergent and finite pieces are

$$S_{\rm vec}=-\frac{\Omega_3 L^2}{32\pi G}\Big[4r_0^3+\frac{r_0}{2}(3(h^1)^2+6h^1h^0-(h^0)^2-24)+2ir_0(3h^1+h^0)$$
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- One needs to add a Gibbons-Hawking term and a term assocciated to the intrinsic curvature of the boundary S^3.

- This leaves

$$S_{\text{vec}} + S_{\text{vec}}^{\text{bndry}} = -\frac{\Omega_3 L^2}{32\pi G}\Big[2ir_0(3h^1 + h^0) + 8(1-ih^1)^{3/2}\sqrt{1-ih^0}\Big]$$

- The hypermultiplet action is

$$S_{\text{hyp}} = \int d^4x\sqrt{g}\left[-\frac{1}{6}RA^2 + \left[4g^2 A^i{}_\beta \bar{X}^\alpha{}_\gamma X^\gamma{}_\delta A_i{}^\delta + gA^i{}_\beta Y^{jk\alpha}_\gamma A_k{}^\gamma \epsilon_{ij} + F_i{}^\alpha F^i{}_\beta\right] d_\alpha{}^\beta\right]$$

$$= -i\Omega_3 L^4 \int_1^{r_0} dr(r^2-1)\frac{1}{r^2}\frac{g}{2\pi GL}(h^0J^0P_0 + h^1J^1P_1)$$

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- P^I are the charges carried by the hypermultiplets and exactly these combinations are fixed to the values

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$$= -\frac{\Omega_3L^2}{4\pi G}\left[4(1-ih^1)^{3/2}\sqrt{1-ih^0}-\frac{3}{2}(1-ih^1)-\frac{1}{2}(1-ih^0)+2\right]$$

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- But the hyper action also has R, so we need to supplement it with the boundary Gibbons-Hawking term. leading to an extra divergence...

- The remaining divergence is independent of h_0 and h_1.
- It can be canceled by a term which arises in string theory from the integration of F_4 over AdS_4.
- This is proportional to the volume of AdS_4, which is $\frac{1}{3}r_0^3 - r_0 + \frac{2}{3}$. This cancels the remaining divergence and adds a finite piece.

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$$S_{ren} = -\frac{\pi\sqrt{2}}{3}k^{1/2}N^{3/2}\left[4(1-ih^1)^{3/2}\sqrt{1-ih^0} - \frac{3}{2}(1-ih^1) - \frac{1}{2}(1-ih^0)\right]$$

which was written in terms of the ABJM parameters

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- Recall that the scalars were $X^I = iJ(1 - \frac{ih^I}{r})$.
- It is convenient to define new variables proportional to $X^I(r=1)$

$$\phi^0 := \frac{\pi}{3\sqrt{2}}\frac{N^{3/2}}{k^{1/2}}(1-ih^0), \qquad \phi^1 := \frac{\pi}{\sqrt{2}}k^{1/2}N^{1/2}(1-ih^1).$$

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$$-S_{ren} = -k\left(\sqrt{\phi^0} - \frac{1}{\pi k}\sqrt{\frac{2}{3}}(\phi^1)^{3/2}\right)^2 + \frac{2}{3\pi^2 k}(\phi^1)^3 - N\phi^1$$

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If we can justify this measure, then this would lead to an agreement with the matrix model to all orders in N.

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- Gaps:
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- Open questions:
  - Can we get the instanton corrections?
  - Generalize:
    * Other AdS_4 solutions?
    * Other geometries altogether?
  - What can we learn more generally about quantum gravity from localization.

The end