Curved-Space Supersymmetry and the Partition Function

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Quantum Field Theory and the Path Integral

Consider the *path integral* of a QFT:

$$Z[\lambda; J_i] = \left\langle e^{-\int J_i \mathcal{O}^i} \right\rangle = \int [D\Phi] \, \exp\left(-S_0[\Phi; \lambda] - \int J_i \mathcal{O}^i \right)$$

In general, we can only evaluate Z *perturbatively* in λ. Moreover, the perturbative expansion has a vanishing radius of convergence.

Nevertheless, there exists examples of non-trivial, interacting quantum field theories where the partition function can be computed *exactly* (with some caveats).

Supersymmetry and the Path Integral

We all appreciate the beauty of supersymmetry as a simplifying assumption in our study of QFT.

We will consider:

- Field theories with supersymmetry
- Field theories defined on a curved compact manifold

The *intersection* of the two is especially beautiful —and powerful:

The path integral of supersymmetric field theories on compact spaces can (often) be computed exactly (for supersymmetric sources).

Supersymmetry and the Path Integral

Let us denote $Z_{\mathcal{M}}(J)$ the supersymmetric partition function on the manifold $\mathcal{M}$ as a function of sources J:

$$\delta_Q Z_{\mathcal{M}}(J) = 0$$

- $\mathcal{M}$ compact makes $Z_{\mathcal{M}}$ well-defined in the IR.
- $Z_{\mathcal{M}}$ has UV divergences, which are dealt with like in flat space. We consider diff invariant counterterms.
- The sources J are *supersymmetric sources*. They can be thought of as supersymmetric background fields. Consequently, only some special observables can be computed from $Z_{\mathcal{M}}$:

$$\partial_J \cdots \partial_J Z_{\mathcal{M}}(J) \sim \langle \mathcal{O}_J \cdots \mathcal{O}_J \rangle_{\mathcal{M}}$$

Some examples of supersymmetric $Z_{\mathcal{M}}$

- Topologically twisted $\mathcal{N} = 2$ SYM on four-manifolds. [Witten, 1988]
- 4d $\mathcal{N} = 2$ on the Ω-background. [Nekrasov, 2002]
- 4d $\mathcal{N} = 2$ on S^4. [Pestun, 2007]
- $\mathcal{N} = 1$ on $S^3 \times S^1$ (supersymmetric index). [Romelsberger, 2007]
- 3d $\mathcal{N} = 2$ theories on S^3. [Kapustin, Willet, Yakoov, 2010; Jafferis, 2010]
- 2d $\mathcal{N} = (2, 2)$ on S^2. [Benini, Cremonesi, 2012; Doroud, Gomis, Le Floch, Lee, 2012]
- 2d $\mathcal{N} = (0, 2)$ on T^2 (elliptic genus). [Benini, Eager, Hori, Tachikawa, 2013]

These results have led to many surprising developments.

(For instance: AGT; new dualities; F-theorem; exact Zamolochikov metric; ...)

See many of the talks at this workshop.

Outline

Defining Supersymmetric Theories on Curved Manifolds

Exact Results and Supersymmetric Localization (In Brief)

General Properties of $Z_{\mathcal{M}}$

Comments on Future Directions

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Sources and background fields

Sources in the path integral can be described as background fields. Useful because of spurion analysis.

For instance, consider a theory with a global symmetry current j^μ. The source for j^μ is a background gauge field A_μ:

$$\mathscr{L}' = \mathscr{L} + A_\mu j^\mu + \mathcal{O}(A^2) \ .$$

- The higher order terms in A_μ are fixed by gauge invariance.
- A_μ is non-dynamical (no e.o.m.).
- Variations of A_μ are captured by correlation functions of j^μ.

Coupling to background metric

Similarly, we always consider Poincaré invariant theories, which possess a (conserved, symmetric) stress tensor $T_{\mu\nu}$. The corresponding source is a background metric $g_{\mu\nu}$.

- Note: We work in Euclidean signature. $g_{\mu\nu}$ Riemannian.

Around flat space, $g_{\mu\nu} = \delta_{\mu\nu} + \Delta g_{\mu\nu}$:

$$\mathcal{L}' = \mathcal{L} - \frac{1}{2}\Delta g_{\mu\nu}\, T^{\mu\nu} + \mathcal{O}(\Delta g^2)$$

- Effect of $\Delta g_{\mu\nu}$ captured by correlation functions of $T_{\mu\nu}$.
- *Some* higher order terms fixed by diffeomorphism invariance.
- Ambiguities remain. Around flat space, this corresponds to improvement transformations of $T_{\mu\nu}$.

Supersymmetry on a curved manifold

Consider a supersymmetric field theory $\mathcal{T}$ in flat space. For definiteness, assume it is described by some UV Lagrangian $\mathscr{L}_0$:

$$\delta \mathscr{L}_0 = \partial_\mu(\cdots)\,, \qquad \tilde{\delta} \mathscr{L}_0 = \partial_\mu(\cdots)\,.$$

$$\{\delta, \tilde{\delta}\} \sim P\,.$$

Question: Given a Riemannian manifold $(\mathcal{M}, g_{\mu\nu})$, can we define a corresponding supersymmetric theory on that curved space ?

$$\begin{array}{ccc} (\mathcal{T}, \mathbb{R}^d, \delta_{\mu\nu}) & \to & (\mathcal{T}', \mathcal{M}, g_{\mu\nu}) \\ \delta\mathcal{T} & \to & \delta'\mathcal{T}' \end{array}$$

- Note: We will only consider deformations which do not modify the UV behavior. Compact geometry provides IR cut-off.

Comments on a standard lore

Two known approaches to this problem:

- Twisting.
- Rigid supersymmetry in curved space.

We will examine and compare the two. A few comments:

- It is often stated that the two approches are distinct. This can be misleading.
- A more accurate characterization is that the second approach is a slight extension of the first. Many similar properties.
- The curved-space theory is “quasi-topological” in both cases, as we will discuss in detail.

Twisting

A well-known way to define supersymmetric field theories on a curved manifold $\mathcal{M}$ is by twisting. [Witten, 1988]

Twisting: Identify an R-symmetry with (part of) the rotations of $\mathbb{R}^d$. Some of the supercharges become scalars, which can be defined on any manifold $\mathcal{M}_d$.

Example: For $\mathcal{N} = 2$ theories, identify $SU(2)_R$ with $SU(2)_+$ in $SU(2)_+ \times SU(2)_-$. Left-moving Q_α become scalars. [Witten, 1988]

After the twist, bosonic and fermionic fields become geometrical objects (forms).

In curved space, supersymmetry parameters are Killing spinors which solve an equation of the form

$$(\nabla - iA^{(R)})\zeta = 0 \ ,$$

for some connection ∇. Use the R-symmetry gauge field to cancel (part of) the holonomy of ∇.

Example: $\mathcal{N} = 1$ theories with $U(1)_R$ on Kähler manifolds, which have $U(2)$ holonomy. [Witten, 1994; Johanssen, 1995]

Background supergravity

A more systematic approach uses off-shell supergravity.
[Karlhede, Rocek, 1988; Johanseen, 1995; Festuccia, Seiberg, 2011]

Supersymmetric theories have a supermultiplet.
It can be minimally coupled to supergravity:

$$
\begin{array}{ccc}
T_{\mu\nu} & g_{\mu\nu} & \\
S_\mu & \Psi_\mu & \\
\mathcal{O} & X & \cdots
\end{array}
$$

Consider background off-shell supergravity. [Festuccia, Seiberg, 2011]

$$\Psi = 0 \,, \qquad \delta\Psi = 0 \,.$$

A few comments:

- It is important that the supergravity be off-shell. $g_{\mu\nu}$ arbitrary. We take a rigid limit. The local supersymmetry parameters solving $\delta\Psi = 0$ become (generalized) Killing spinors.
- The full knowledge of SUGRA is not always necessary. Linearized supergravity can be enough.
- There is a choice of supercurrent multiplet involved at the start.
- Otherwise, the procedure is completely model independent.

We will consider in some detail the following example:

- 4d $\mathcal{N} = 1$ theories with a $U(1)_R$.

One can similarly discuss:

- 3d $\mathcal{N} = 2$ theories with a $U(1)_R$.
- 2d $\mathcal{N} = (2,2)$ theories with a vector-like $U(1)_R$.

More cases have been worked out in the literature (or in drawers).

$\mathcal{N} = 1$ theories with an R-symmetry

Consider $\mathcal{N} = 1$ supersymmetric theories in 4d, with an R-symmetry.

We have the $\mathcal{R}$-multiplet

$$j_\mu^{(R)}\,, \qquad S_{\mu\alpha}\,, \qquad \tilde{S}_\mu^{\dot{\alpha}}\,, \qquad T_{\mu\nu}\,, \qquad \mathcal{F}_{\mu\nu}$$

which couples to the "new minimal" supergravity multiplet of [Sohnius, West, 1981] :

$$A_\mu^{(R)}\,, \qquad \Psi_{\mu\alpha}\,, \qquad \tilde{\Psi}_\mu^{\dot{\alpha}}\,, \qquad g_{\mu\nu}\,, \qquad B_{\mu\nu}$$

We often use the dual field strength of $B_{\mu\nu}$, denoted V_μ.

There is a complete classification of supersymmetric backgrounds in this case. We focus on one supercharge Q in this talk.
[Dumitrescu, Festuccia, Seiberg, 2012; Klare, Tomassielo, Zaffaroni, 2012]

Supersymmetry and complex geometry

4d Killing spinor equations $\delta\Psi = \delta\tilde{\Psi} = 0$:

$$\left(\nabla_\mu - iA_\mu^{(R)}\right)\zeta = \tfrac{i}{2}V_\mu\zeta - iV^\nu\sigma_{\mu\nu}\zeta$$
$$\left(\nabla_\mu + iA_\mu^{(R)}\right)\tilde{\zeta} = -\tfrac{i}{2}V_\mu\tilde{\zeta} + iV^\nu\tilde{\sigma}_{\mu\nu}\tilde{\zeta}$$

The spinors ζ_α, $\tilde{\zeta}^{\dot{\alpha}}$ are non-vanishing sections of $S_+ \otimes L$, $S_- \otimes L^{-1}$.

The most important result is

One supercharge on $\mathcal{M}_4$ $\Leftrightarrow$ $\mathcal{M}_4$ a complex manifold

The background metric $g_{\mu\nu}$ must be Hermitian.

The complex structure is expressed in term of the Killing spinor :

$$J^\mu{}_\nu = -\frac{2i}{|\zeta|^2}\zeta^\dagger\sigma^\mu{}_\nu\zeta$$

[Dumitrescu, Festuccia, Seiberg, 2012; Klare, Tomassielo, Zaffaroni, 2012]

The background supergravity fields are determined in term of the metric and complex structure:

$$
\begin{aligned}
V_\mu &= \frac{1}{2}\nabla_\nu J^\nu{}_\mu + U_\mu \,, \\
A_\mu^{(R)} &= -\frac{1}{4}J_\mu{}^\nu \partial_\nu \log\sqrt{g} - \frac{1}{4}(2\delta_\mu{}^\nu - iJ_\mu{}^\nu)\nabla_\rho J^\rho{}_\nu \,,
\end{aligned}
$$

Given a complex Riemannian manifold $(\mathcal{M}, g_{\mu\nu}, J^\mu{}_\nu)$, one can always solve the Killing spinor equation for ζ.

If we have more supercharges the geometry is more constrained. [Dumitrescu, Festuccia, Seiberg, 2012]

Global symmetry and holomorphic G-bundle

When our theory has a global symmetry group G , we can couple it to a background vector multiplet.

$$A_\mu \,, \qquad D\,, \qquad \lambda = \delta\lambda = 0\,.$$

We ask that it preserve the Killing spinor ζ. This requires

$$F^{(0,2)} = 0\,, \qquad D = -\frac{1}{2} J^{\mu\nu} F_{\mu\nu}$$

The first equation implies that we have an holomorphic G-bundle.

We will focus on G Abelian: Holomorphic line bundles.

Supersymmetric Lagrangian (Example)

Consider a chiral multiplet. The supersymmetry variations are

$$\begin{aligned} \delta\phi &= \sqrt{2}\zeta\psi\ , & \delta\tilde{\phi} &= \sqrt{2}\tilde{\zeta}\tilde{\psi}\ , \\ \delta\psi &= \sqrt{2}\zeta F + \sqrt{2}i\sigma^\mu\tilde{\zeta}\, D_\mu\phi\ , & \delta\tilde{\psi} &= \sqrt{2}\tilde{\zeta}\tilde{F} + \sqrt{2}i\tilde{\sigma}^\mu\zeta\, D_\mu\tilde{\phi}\ , \\ \delta F &= \sqrt{2}iD_\mu\big(\tilde{\zeta}\,\tilde{\sigma}^\mu\psi\big)\ , & \delta\tilde{F} &= \sqrt{2}iD_\mu\big(\zeta\sigma^\mu\tilde{\psi}\,\big)\ . \end{aligned}$$

with

$$D_\mu\Phi^{(r)} = \Big(\nabla_\mu - ir\big(A_\mu^{(R)} + \frac{3}{2}V_\mu\big)\Big)\Phi^{(r)}\ .$$

The standard kinetic Lagrangian is given by

$$\begin{aligned} \mathscr{L}_{\tilde{\Phi}\Phi} = \quad & D^\mu\tilde{\phi}D_\mu\phi + i\tilde{\psi}\tilde{\sigma}^\mu D_\mu\psi - \tilde{F}F - \tfrac{r}{4}\left(R - 6V^\mu V_\mu\right)\tilde{\phi}\phi \\ & -iV^\mu\left(\tilde{\phi}D_\mu\phi - \phi D_\mu\tilde{\phi}\right) + \tfrac{1}{2}V_\mu\tilde{\psi}\tilde{\sigma}^\mu\psi\ . \end{aligned}$$

Here R is the Ricci scalar and A_μ, V_μ take their supersymmetric value, as given above.

The Holomorphic Twist

From the Killing spinor ζ, we can also construct the two-form

$$P_{\mu\nu} = \zeta\sigma_{\mu\nu}\zeta \ .$$

It is a $(2,0)$-form with respect to the complex structure $J^{\mu}{}_{\nu}$.
Consider the object $p = P_{12}$, as section of $\mathcal{K} \otimes L^2$:

$$P = p\, dz^1 \wedge dz^2 \ , \qquad\qquad p \in \Gamma(\mathcal{K} \otimes L^2) \ .$$

p is nowhere vanishing and therefore the line bundle $\mathcal{K} \otimes L^2$ is trivial. Consequently

$$L \cong \mathcal{K}^{-\frac{1}{2}}$$

This is what we will call the holomorphic twist.

A similar twist was considered by [Vyas, 2010].

We can also define

$$s = p g^{-\frac{1}{4}} \,, \qquad g = \det\left(g_{\mu\nu}\right) \ .$$

We have

$$p'(z') = p(z) \det\left(\frac{\partial z'^i}{\partial z^j}\right) \,, \quad s'(z') = s(z) \left(\det\left(\frac{\partial z'^i}{\partial z^j}\right)\right)^{\frac{1}{2}} \left(\det\left(\frac{\partial \bar{z}'^i}{\partial \bar{z}^j}\right)\right)^{-\frac{1}{2}} \ .$$

under holomorphic coordinate changes. After the twist, s is therefore an "holomorphic scalar". Because $\zeta \sim s$, it follows that the supercharge preserved on the complex manifold is a scalar under coordinate transformations.

This is a generalization the more familiar topological twist in 2d $\mathcal{N} = (2,2)$ theories or 4d $\mathcal{N} = 2$ theories.

Twisted variables

We can introduce variables which are more natural after the holomorphic twist.

For instance, consider a chiral multiplet of R-charge r:

$$\phi \in \Gamma(L^r)\,, \quad \psi_\alpha \in \Gamma(L^{r-1} \otimes S_+)\,, \quad F \in \Gamma(L^{r-2})\,.$$

We can define the $\mathcal{K}^{-\frac{r}{2}}$-valued (p,q)-forms

$$\begin{aligned}
\mathcal{C} &= p^{-\frac{r}{2}}\phi && \in \Gamma(\mathcal{K}^{-\frac{r}{2}}) \\
\mathcal{X} &= \sqrt{2}p^{-\frac{r}{2}}\zeta\psi && \in \Gamma(\mathcal{K}^{-\frac{r}{2}}) \\
\mathcal{X}_{ij} &= -\sqrt{2}ip^{-\frac{r}{2}}\zeta\sigma_{ij}\zeta\frac{1}{|\zeta|^2}\zeta^\dagger\psi && \in \Gamma(\mathcal{K}^{-\frac{r}{2}} \otimes \Lambda^{2,0}\mathcal{M}) \\
\mathcal{M}_{ij} &= -2ip^{-\frac{r}{2}}\zeta\sigma_{ij}\zeta F && \in \Gamma(\mathcal{K}^{-\frac{r}{2}} \otimes \Lambda^{2,0}\mathcal{M})
\end{aligned}$$

with supersymmetry transformations

$$\delta\mathcal{C} = \mathcal{X}\,, \quad \delta\mathcal{X} = 0\,, \qquad \delta\mathcal{X}_{ij} = \mathcal{M}_{ij}\,, \quad \delta\mathcal{M}_{ij} = 0\,.$$

Similarly, we have the twisted anti-chiral multiplet with supersymmetry transformations

$$\delta\tilde{\mathcal{C}} = 0\ , \qquad \delta\tilde{\mathcal{X}}_{\bar{i}} = 2i\partial_{\bar{i}}\tilde{\mathcal{C}}\ , \qquad \delta\tilde{\mathcal{M}}_{\bar{i}\bar{j}} = -4\left(\partial_{\bar{i}}\tilde{\mathcal{X}}_{\bar{j}} - \partial_{\bar{j}}\tilde{\mathcal{X}}_{\bar{i}}\right)\ .$$

For the vector multiplet in WZ gauge:

$$\begin{aligned} \delta A_i &= \tilde{\mathcal{L}}_i\ , \qquad & \delta\tilde{\mathcal{L}}_i &= 0\ , \\ \delta\mathcal{L} &= \mathcal{D}\ , \qquad & \delta\mathcal{D} &= 0\ , \\ \delta\mathcal{L}_{\bar{i}\bar{j}} &= 4F_{\bar{i}\bar{j}}\ , \qquad & \delta A_{\bar{i}} &= 0\ . \end{aligned}$$

Note that, in those twisted variables:

- The susy transformations are scalar and metric independent.
- They depend on the complex structure.

Lagrangian in twisted variables (Example)

In twisted variables, the chiral multiplet reads

$$\begin{aligned}\mathscr{L}_{\tilde{\Phi}\Phi} = \quad & 2g^{i\bar{j}}\partial_{\bar{j}}\tilde{\mathcal{C}}\Big(\nabla_i^c - ir\big(V_i + \tfrac{1}{2}U_i\big)\Big)\mathcal{C} - \tfrac{1}{8}\tilde{\mathcal{M}}_{\bar{i}\bar{j}}\mathcal{M}^{\bar{i}\bar{j}} + \partial_{\bar{i}}\tilde{\mathcal{X}}_{\bar{j}}\mathcal{X}^{\bar{i}\bar{j}} \\ & + i\tilde{\mathcal{X}}^i\Big(\nabla_i^c - ir\big(V_i + \tfrac{1}{2}U_i\big)\Big)\mathcal{X} + U^{\bar{i}}\Big(2i\partial_{\bar{i}}\tilde{\mathcal{C}}\mathcal{C} - \tilde{\mathcal{X}}_{\bar{i}}\mathcal{X}\Big)\end{aligned}$$

up to total derivatives. Here ∇^c is the Chern connection.

It is easy to check that this Lagrangian is Q-exact:

$$\mathscr{L}_{\tilde{\Phi}\Phi} = \delta\left(-i\tilde{\mathcal{X}}^i\Big(\nabla_i^c - ir\big(V_i + \frac{1}{2}U_i\big)\Big)\mathcal{C} - \frac{1}{8}\tilde{\mathcal{M}}_{\bar{i}\bar{j}}\mathcal{X}^{\bar{i}\bar{j}} + U^{\bar{i}}\tilde{\mathcal{X}}_{\bar{i}}\mathcal{C}\right) .$$

Comparing Twisting and Rigid Curved Supersymmetry

- The two formulations are *equivalent*. Invertible map. (Assuming no anomalies; trivial Jacobian.)
- We can write the Killing spinor equation as $(\nabla^c - \hat{A})\zeta = 0$. [Dumitrescu, Festuccia, Seiberg, 2012]
- The Rigid Supersymmetric approach cleanly separates supersymmetric background from Lagrangians and supersymmetry transformations.
- The Twisting approach is more convenient to study some geometric properties of the field theory.
- The reason for the term “holomorphic twist” will become clear by the end of the talk.

Supersymmetric Localization

Main attraction of this whole suject: Supersymmetry allows to compute exactly the supersymmetric partition function.

$$Z_{\mathcal{M}} = \langle 1 \rangle_{\mathcal{M}} \; , \qquad \delta_Q Z_{\mathcal{M}} = 0 \; .$$

Similarly, various Q-closed observables are available:

$$\langle \mathcal{O} \rangle_{\mathcal{M}} \; , \qquad [Q, \mathcal{O}] = 0 \; .$$

We will focus on $Z_{\mathcal{M}}$.

The main tool is the identity

$$\langle \{Q, \Psi\} \rangle_{\mathcal{M}} = 0 \; .$$

This lead to the supersymmetric localization argument...
Some semi-classical (one-loop) result becomes exact.

The S^3 partition function

Let us focus on one example, the squashed S^3 partition function for $\mathcal{N} = 2$ theory.

It has been computed for many choices of metrics and other supersymmetric background fields on S^3. [Hama, Hosomishi, Lee, 2010; Imamura, Yokoyama, 2011; Alday, Martelli, Richmond, Sparks, 2013; ...]

It was found that:

The S^3 partition function only depends on a single parameter $b \in \mathbb{C}$.

It was then understood that this b is the moduli of a one-parameter family of THF's on manifolds of S^3 topology.
[C.C., Dumitrescu, Festucia, Komargodski, 2013]

Why?

This example and few others suggest that $Z_{\mathcal{M}}$ is “quasi-topological”.

Can we make this more precise?... *Yes.*

General Properties of $Z_{\mathcal{M}}$

Let's go back to the 4d $\mathcal{N}=1$ case, for definiteness. (3d is similar.)

We saw that we can take any R-symmetric $\mathcal{N}=1$ supersymmetric quantum field theory and couple it to a given compact complex four-manifold $\mathcal{M}_4$.

When the theory has a global symmetry G, we also couple it to an holomorphic G-bundle over $\mathcal{M}_4$.

Consider the partition function $Z_{\mathcal{M}_4}$:

$$Z_{\mathcal{M}_4}(J^{\mu}{}_{\nu}, g_{\mu\nu}, A_{\mu}, \lambda)$$

What does it depend on ?

$$Z_{\mathcal{M}_4}(J^\mu{}_\nu, g_{\mu\nu}, A_\mu, \lambda)$$

$Z_{\mathcal{M}_4}$ could *a priori* depend on :

- The choice of complex structure $J^\mu{}_\nu$
- The choice of Hermitian metric $g_{\mu\nu}$
- The background gauge field A_μ (choice of holomorphic G-bundle)
- The couplings λ of the original flat-space theory

All this data can be varied continuously. There might also be discrete choices, such as for the topology of G-bundles.

Dependence of $Z_{\mathcal{M}_4}$ on coupling constants

Using the twisted variables, it is straightforward to show that:

- Any D-term Lagrangian is Q-exact. (For well-defined D-term.)
- Any F-term Lagrangian is Q-exact
- The $\tilde{F}$-term Lagrangian is not Q-exact unless there is a second supercharge $\tilde{Q}$ of opposite chirality.

Therefore, the partition function $Z_{\mathcal{M}_4}$ is independent of D-term and of holomorphic F-term couplings, while it might depend on anti-holomorphic $\tilde{F}$-term couplings.

If there are at least two supercharges, $Z_{\mathcal{M}_4}$ is independent of all λ's.

Dependence of $Z_{\mathcal{M}_4}$ on the geometry

Choose a single supercharge Q (corresponding to ζ).
The strategy we follow is:

- Consider small variations of any of the continuous geometric data above, $\Delta g_{\mu\nu}$, $\Delta J^{\mu}{}_{\nu}$, ΔA_{μ}.
- These variations induce a deformation $\Delta\mathscr{L}$ of the theory.
- Study which of these $\Delta\mathscr{L}$ are Q-exact. The partition function cannot depend on Q-exact deformations.

For simplicity, I will present an analysis around flat space. A similar analysis can be done around any background using twisted variables. See [C.C., Dumitrescu, Festucia, Komargodski, 2013].

Basics of deformation theory

Consider a complex structure $J^\mu{}_\nu$ on $\mathcal{M}_4$. Introduce adapted coordinates z^i, $\bar{z}^{\bar{i}}$. We have $J^i{}_j = i\,\delta^i{}_j\,, \qquad J^{\bar{i}}{}_{\bar{j}} = -i\,\delta^{\bar{i}}{}_{\bar{j}}$.

Non-zero elements of the general variation $\Delta J^\mu{}_\nu$ are

$$\Delta J^i{}_{\bar{j}}\,, \qquad \partial_{\bar{j}}\left(\Delta J^i{}_{\bar{k}}\right) - \partial_{\bar{k}}\left(\Delta J^i{}_{\bar{j}}\right) = 0$$

and its complex conjugates $\Delta J^{\bar{i}}{}_j$. We quotient by diffeomorphisms,

$$\Delta J^i{}_{\bar{j}} = 2i\,\partial_{\bar{j}}\,\varepsilon^i$$

Thus, first order deformations of the complex structure correspond to

$$\Theta^i = \Delta J^i{}_{\bar{j}}\,d\bar{z}^{\bar{j}}\,, \qquad \left[\Theta^i\right] \in H^{0,1}\left(\mathcal{M}_4, T^{1,0}\mathcal{M}_4\right)$$

This cohomology contains the complex structure moduli.

Basics of deformation theory

We also vary the metric $\Delta g_{\mu\nu}$. To preserve the compatibility with the complex structure, we have

- $\Delta g_{\bar{i}\bar{j}}$ unconstrained
- $\Delta g_{ij} = \frac{i}{2}\left(g_{i\bar{k}}\Delta J^{\bar{k}}{}_j + g_{j\bar{k}}\Delta J^{\bar{k}}{}_i\right)$ and its complex conjugate

Similarly, for Abelian background gauge fields, we have

$$\partial_{\bar{i}}\Delta A_{\bar{j}} - \partial_{\bar{j}}\Delta A_{\bar{i}} = 0$$

modulo (complexified) gauge transformations. The holomorphic line bundle moduli sit in $H^{0,1}(\mathcal{M}_4)$.

The deformation Lagrangians

At first order around flat space, the supergravity background fields couple to the $\mathcal{R}$-multiplet of the flat space theory. We have the deformation Lagrangian

$$\Delta\mathscr{L} = -\frac{1}{2}\Delta g^{\mu\nu} T_{\mu\nu} + A^{(R)\mu} j^{(R)}_{\mu} + \frac{i}{4}\varepsilon^{\mu\nu\rho\lambda} B_{\mu\nu}\mathcal{F}_{\rho\lambda}$$

Similar for the background gauge fields coupling to a linear multiplet:

$$\Delta\mathscr{L} = \Delta A_{\mu} j^{\mu} + DJ$$

One plugs the supersymmetric values of $A^{(R)}_{\mu}, B_{\mu\nu}, D$.

Q-exactness of $\mathcal{T}_{\mu\nu}$

One can study the Q-cohomology of the $\mathcal{R}$-multiplet. In particular,

$$\{Q, \tilde{S}^{\dot{\alpha}}{}_{\mu}\} = 2i\,(\tilde{\sigma}^{\nu}\zeta)^{\dot{\alpha}}\,\mathcal{T}_{\mu\nu}$$

with

$$\mathcal{T}_{\mu\nu} = T_{\mu\nu} + \frac{i}{4}\varepsilon_{\mu\nu\rho\lambda}\mathcal{F}^{\rho\lambda} - \frac{i}{4}\varepsilon_{\mu\nu\rho\lambda}\partial^{\rho} j^{(R)\lambda} - \frac{i}{2}\partial_{\nu} j^{(R)}_{\mu}$$

We can see that the eight operators $\mathcal{T}_{\mu\bar{i}}$ are Q-exact. They are also the only Q-closed operators in the $\mathcal{R}$-multiplet.

Note: Coupling the metric to a shifted stress tensor is a standard description of twisting.

Geometric deformations

We plug the supersymmetric values of the (linearized) background supergravity fields in $\Delta\mathcal{L}$, and find :

$$
\begin{aligned}
\Delta\mathscr{L} \;=\;& -\Delta g^{\bar{i}\bar{j}}\mathcal{T}_{\bar{i}\bar{j}} - i\sum_{j=\bar{j}} \Delta J^{\bar{i}}{}_{j}\mathcal{T}_{\bar{j}\bar{i}} \\
& + i\Delta J^{w}{}_{\bar{w}}\left(T_{ww} + \frac{i}{2}\partial_w j_w^{(R)}\right) + i\Delta J^{z}{}_{\bar{z}}\left(T_{zz} + \frac{i}{2}\partial_z j_z^{(R)}\right) \\
& + i\Delta J^{w}{}_{\bar{z}}\left(T_{wz} + \frac{i}{2}\mathcal{F}_{wz} - \frac{i}{4}\partial_w j_z^{(R)} + \frac{3i}{4}\partial_z j_w^{(R)}\right) \\
& + i\Delta J^{z}{}_{\bar{w}}\left(T_{wz} - \frac{i}{2}\mathcal{F}_{wz} - \frac{i}{4}\partial_z j_w^{(R)} + \frac{3i}{4}\partial_w j_z^{(R)}\right)
\end{aligned}
$$

Note that $\Delta g^{\bar{i}\bar{j}}$ and $\Delta J^{\bar{i}}{}_{j}$ couple to operators which are Q-exact. Moreover, the whole Lagrangian is Q-exact for a trivial complex structure deformation, as follows from diff invariance.

Holomorphicity of the partition function

We conclude :

- $Z_{\mathcal{M}_4}$ is independent of $\Delta g_{i\bar{j}}$.
- $Z_{\mathcal{M}_4}$ is independent on the anti-holomorphic complex structure deformations $\Delta J^{\bar{i}}{}_j$, and only depends on the holomorphic ones through its cohomology class $H^{0,1}\left(\mathcal{M}_4, T^{1,0}\mathcal{M}_4\right)$. Thus it only depends on holomorphic complex structure moduli.

$Z_{\mathcal{M}_4}$ is independent of the Hermitian metric.
It is a locally holomorphic function of the complex structure moduli.

Similar analysis for the G-bundle moduli.
We conclude :

$Z_{\mathcal{M}_4}$ is locally holomorphic in the G-bundle moduli.

Example: The $S^3 \times S^1$ partition function

Consider the following quotient of $\mathbb{C}^2 - \{(0,0)\}$:

$$(z_1, z_2) \sim (p\, z_1, q\, z_2)\,, \qquad 0 < |p| \leq |q| < 1$$

It is a complex manifold known as a primary Hopf surface, that we denote $\mathcal{M}_4^{p,q}$.

$$\mathcal{M}_4^{p,q} \cong S^3 \times S^1$$

We can write it in term of real angles:

$$z_1 = p^x \cos\frac{\theta}{2}\, e^{i\varphi}\;, \qquad z_2 = q^x \sin\frac{\theta}{2}\, e^{i\chi}$$

The quotient corresponds to the identification $x \sim x + 1$.

For generic p, q, $\mathcal{M}_4^{p,q}$ is compatible with two supercharges.
For the particular choice $p = \bar{q}$, we have four supercharges.

The $S^3 \times S^1$ partition function on this space is known as the supersymmetric index

$$\mathcal{I}(p, q, u) = \mathrm{Tr}_{S^3}\left((-1)^F p^{J_3 + J_3' - \frac{R}{2}} q^{J_3 - J_3' - \frac{R}{2}} u^{Q_f}\right)$$

Explicit formula known for general gauge theories. [Romelsberger, 2007]

- The fugacities p, q above are naturally complex parameters, corresponding to complex structure moduli of $\mathcal{M}_4^{p,q}$.
- The fugacity u corresponds to holomorphic line bundle moduli.
- $\mathcal{I}$ is a meromorphic function on the $\mathcal{M}_4^{p,q}$ moduli space.
- We can compute the index for any Hermitian metric. Same answer. See also [C.C., I. Shamir, 2013; Assel, Cassani, Martelli, 2014].

R-charge dependence

In $\mathcal{N} = 1$ theories with an R-symmetry, it is often the case that there is a continuum of possible (non-anomalous) R-charges, corresponding to the shift

$$j^\mu_{(R)} \to j^\mu_{(R)} + \Delta r\, j^\mu$$

by a $U(1)$ current. Since we have the minimal coupling to background gauge fields

$$A^{(R)}_\mu j^\mu_{(R)} + A_\mu j^\mu \ ,$$

the change of R-charge corresponds to a shift in the background gauge field A_μ for a $U(1)$ symmetry. It has a supersymmetric completion

$$\Delta\mathscr{L} = \Delta r \left(j^\mu \left(A^{(R)}_\mu + \frac{3}{2} V_\mu \right) - \frac{1}{4} J (R - 6 V_\mu V^\mu) \right) + \dots$$

This corresponds to a shift of the $U(1)$ background gauge field

$$A_\mu \to A_\mu + \Delta r \left(A_\mu^{(R)} + \frac{3}{2} V_\mu \right)$$

- Since $L \cong \mathcal{K}^{-\frac{1}{2}}$, this makes sense only for $c_1(\mathcal{K}) = 0$.
- However, $\mathcal{K}$ need not be holomorphically trivial.
  Let $\nu_{\mathcal{K}} \in H^{0,1}(\mathcal{M}_4)$ be value of the moduli of $\mathcal{K}$ seen as a complex line bundle.

We have

$$(A_\mu)^{0,1} = \nu + \bar{\partial}(\cdots) , \qquad (\Delta A_\mu)^{0,1} = -\frac{\Delta r}{2} \nu_{\mathcal{K}} + \bar{\partial}(\cdots)$$

In other words, we have the following shift in the moduli space of flat line bundles over $\mathcal{M}_4$:

$$L_\nu \to L_{\nu - \frac{\Delta r}{2} \nu_{\mathcal{K}}} \cong L_\nu \otimes \mathcal{K}^{-\frac{\Delta r}{2}} .$$

It follows that the partition function only depends on the R-charges through the holomorphic combination $\nu - \frac{\Delta r}{2}\nu_{\mathcal{K}}$.

A simple consequence is that whenever $\mathcal{K}$ is holomorphically trivial, the partition function is independent of the R-charges.

Example: for the $S^3 \times S^1$ partition function (Hopf surface $\mathcal{M}_{p,q}$), we have $\nu_{\mathcal{K}} = \frac{1}{2\pi i}\log(pq)$.

Similarly, in $3d$, we can prove that the S^3 partition function depends holomorphically on $m + i\Delta r$, which give a proof of an important "experimental" fact. [Jafferis, 2010]

Conclusions and generalizations

To summarize:

- In recent years, large progress in our understanding of curved space supersymmetry with four supercharges in $d \leq 4$. Especially in the presence of an R-symmetry.
- In 4d, $\mathcal{M}_4$ is a Hermitian manifold. $Z_{\mathcal{M}}$ independent of Hermitian metric, and locally holomorphic in complex structure moduli. Also holomorphic dependence on bundle moduli for global symmetry.
- R-charge dependence of $Z_{\mathcal{M}}$ is constrained by holomorphy as well.

This generalizes to lower dimensions:

- In 3d, $\mathcal{M}_3$ is a transversely holomorphic foliation (THF). $Z_{\mathcal{M}}$ independent of adapted metric, and locally holomorphic in THF moduli.
- Similar story in 2d, with interesting subtleties.

Some interesting open problems

A few interesting problems, among many:

- Detailed analysis of Q-closed observables? Holomorphic sector.
- In examples, $Z_{\mathcal{M}}$ is meromorphic in the geometric moduli. Understand the physics of the poles.
- Explicit computation of the partition function in examples with only one supercharge? Would give access to marginal deformations.
- Understand manifolds with boundaries, and gluing rules.
- Analysis with 8 supercharges. (See [Klare, Zaffaroni, 2013])
- ...

$\mathcal{N} = 2$ supersymmetric field theories on $\mathcal{M}_3$

One can similarly define 3d $\mathcal{N} = 2$ supersymmetric field theories on three-manifolds.

Consider $\mathcal{N} = 2$ theories with an R-symmetry. The 3d $\mathcal{R}$-multiplet couples to a supergravity multiplet:

$$A_\mu^{(R)}\,, \qquad \Psi_{\mu\alpha}, \qquad \tilde{\Psi}_\mu^{\dot{\alpha}}\,, \qquad g_{\mu\nu}\,. \qquad V_\mu\,, \qquad H$$

The generalized Killing spinor equations are:

$$\begin{aligned}(\nabla_\mu - iA_\mu)\zeta_\alpha &= -\frac{1}{2}H(\gamma_\mu\zeta)_\alpha - \frac{1}{2}\epsilon_{\mu\nu\rho}V^\nu(\gamma^\rho\zeta)_\alpha - iV_\mu\zeta_\alpha\,,\\ (\nabla_\mu + iA_\mu)\tilde{\zeta}_\alpha &= -\frac{1}{2}H(\gamma_\mu\tilde{\zeta})_\alpha + \frac{1}{2}\epsilon_{\mu\nu\rho}V^\nu(\gamma^\rho\tilde{\zeta})_\alpha + iV_\mu\tilde{\zeta}_\alpha\,.\end{aligned}$$

Supersymmetric three-manifolds

Given one Killing spinor ζ on $\mathcal{M}_3$, we can construct a vector

$$\xi^\mu = \frac{\zeta^\dagger \gamma^\mu \zeta}{\zeta^\dagger \zeta}$$

It is nowhere vanishing and therefore defines a foliation of $\mathcal{M}_3$. (Orbits of ξ are leaves.) Let us also define the auxilliary objects

$$\eta_\mu = g_{\mu\nu}\xi^\nu \ , \qquad \Phi^\mu{}_\nu = -\epsilon^\mu{}_{\nu\rho}\xi^\rho \ .$$

Let $\mathcal{D}$ be the two-dimensional normal bundle of the foliation. We have a complex structure on $\mathcal{D}$:

$$\Phi\Big|_{\mathcal{D}} = J \ .$$

The KSE makes it integrable.

Supersymmetric three-manifolds

Conversely, on any manifold with such a structure we can define one supercharge.

One supercharge on $\mathcal{M}_3$

$\Leftrightarrow$

$\mathcal{M}_3$ admits a transversely holomorphic foliation (THF)

We can describe $\mathcal{M}_3$ in term of coordinates $\tau, z, \bar{z}$ such that

$$\xi = \partial_\tau$$

with z a complex coordinates, and that the transition functions between patches are holomorphic:

$$\tau' = \tau + t(z, \bar{z}) \,, \qquad z' = f(z) \,.$$

Supersymmetric three-manifolds

The background metric $g_{\mu\nu}$ must be adapted to that structure :

$$ds^2 = \eta^2 + c(\tau, z, \bar{z})^2 dz d\bar{z}\,, \qquad \eta = d\tau + h(\tau, z, \bar{z}) dz + \bar{h}(\tau, z, \bar{z})\ ,$$

This is known as a transversely Hermitian metric.

Transversally holomorphic foliations are classified. [Brunella, 1996]
It includes Seifert manifolds, which admit two supercharges (the ξ is Killing).

Moreover, there exists a notion of holomorphic line bundle, corresponding to background vector multiplets for $U(1)$ global symmetries.

$\mathcal{N} = (2,2)$ theories with R_V-symmetry.

We can again do a similar analysis of supersymmetric backgrounds in two dimensions. [C.C., Cremonesi, 2014].

Consider a theory with a R_V-symmetry acting trivially on twisted-chiral multiplets. It possesses a R-mutiplet, coupling to the supergravity fields:

$$A_\mu^{(R)}\,, \qquad \Psi_{\mu\alpha}, \qquad \tilde\Psi_\mu^{\dot\alpha}\,, \qquad g_{\mu\nu}\,. \qquad \mathcal{H}\,, \qquad \tilde{\mathcal{H}}$$

Any orientable two-manifold is complex, with metric

$$ds^2 = 2g_{z\bar z}(z,\bar z)\, dz d\bar z = e^1 e^{\bar 1}\ .$$

The Killing spinor equations are

$$(\nabla_z - iA_z)\zeta_- = 0\,, \qquad (\nabla_{\bar z} - iA_{\bar z})\zeta_- = \frac{1}{2}\mathcal{H}\, e_{\bar z}^{\bar 1}\,\zeta_+\ ,$$

$$(\nabla_z - iA_z)\zeta_+ = \tfrac{1}{2}\tilde{\mathcal{H}}\, e_z^1\,\zeta_-\ , \qquad (\nabla_{\bar z} - iA_{\bar z})\zeta_+ = 0\ .$$

Supersymmetric Riemann surfaces

We can run a similar analysis in this case. There are some interesting subtleties. Basic spinors have zeros.

In summary, for compact orientable two-manifolds:

- For genus $g > 1$, only possibility is the A-twist.

$$\frac{1}{2\pi}\int dA^{(R)} = \pm(g-1)$$

- For genus $g = 0$, we have:
  - Equivariant deformation of the A-twist (spherical Ω-background).
  - Sphere with vanishing flux. Arbitrary squashings of spheres, like in [Doroud, Gomis, Le Floch, Lee, 2012; Benini, Cremonesi, 2012; Gomis, Lee, 2012]