wall-crossing, quiver invariants, and indices of d=1 GLSM

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S.J.Lee + Z.L.Wang + P.Y. 2012/2013
K.Hori + H.Kim + P.Y. 2014

Manschot + Pioline + Sen 2010-2013

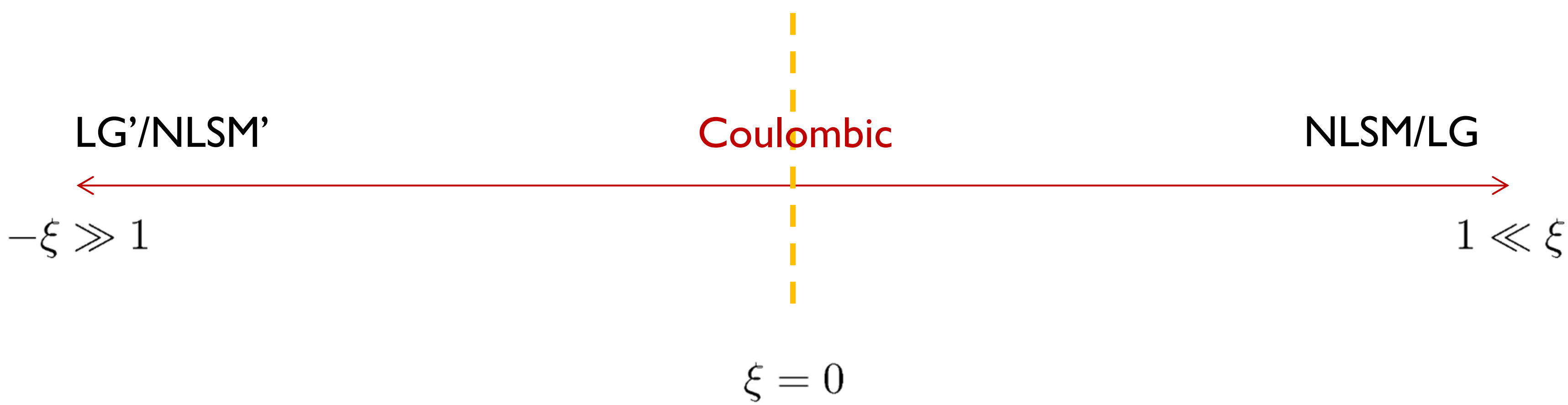
1d N=2 Gauged Linear Sigma Models

gauge fields $(A_\mu, \lambda_-, D)^a$

chirals $(X, \psi_+)^I$

fermi $(\psi_-, F)^I$

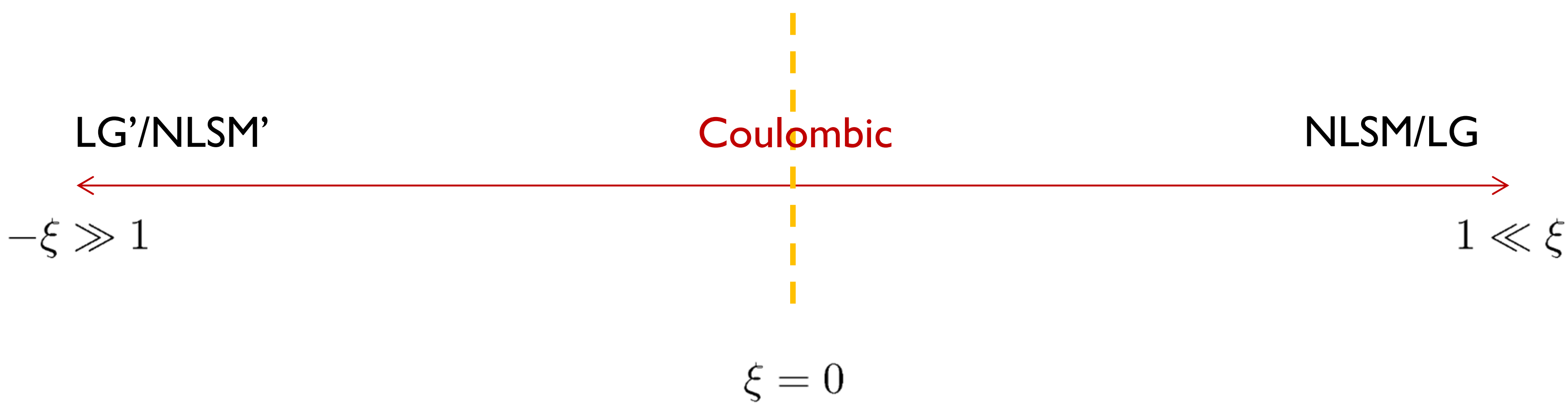
FI constants ξ^i for U(1)'s



1d N=4 Gauged Linear Sigma Models

gauge fields $(A_\mu, \lambda_\alpha, \sigma, D)^a$ FI constants ξ^i for U(1)'s

chiral matter $(X, \psi_\alpha, F)^I$



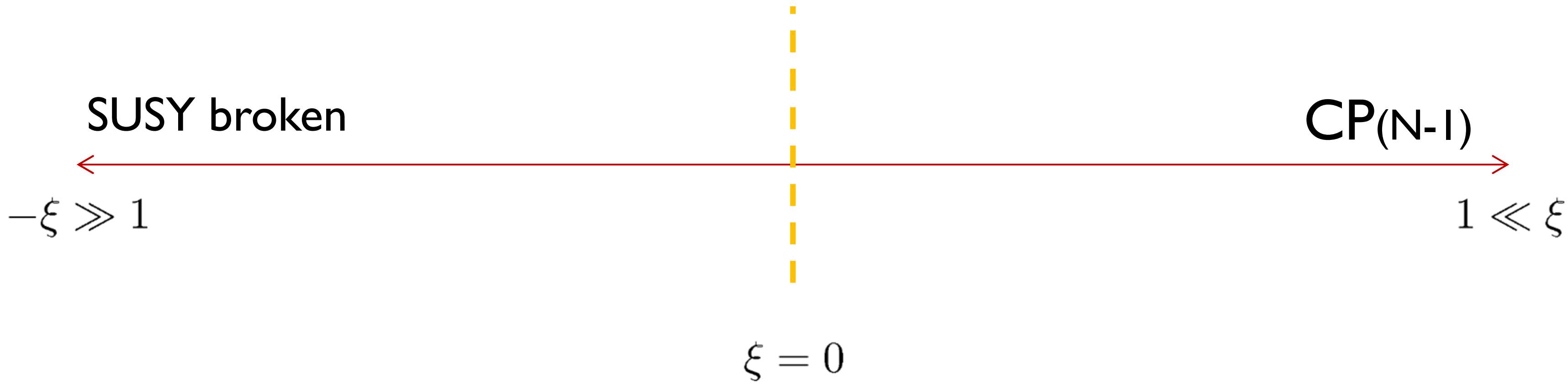
canonical example : CP(N-1)

$\{X^{1,\ldots,N}\}//U(1)$

$$(A_\mu, \lambda_\alpha, \sigma, D)$$

$$(X, \psi_\alpha, F)^{I=1,\cdots,N}_{Q=+1}$$

$$V \quad \sim \quad |\sigma|^2 \left(\sum_I |X_I|^2 \right) + \left(-\xi + \sum_I |X_I|^2 \right)^2$$



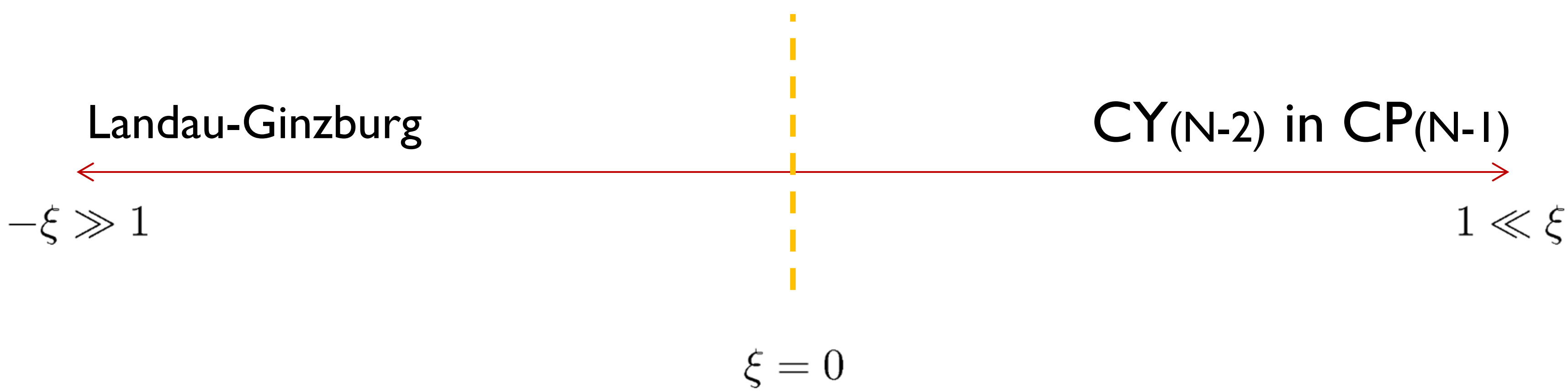
canonical example : CY(N-2) hypersurface in CP(N-1)

$G_N(X) = 0$ $\{X^{1,\dots,N}\}//U(1)$

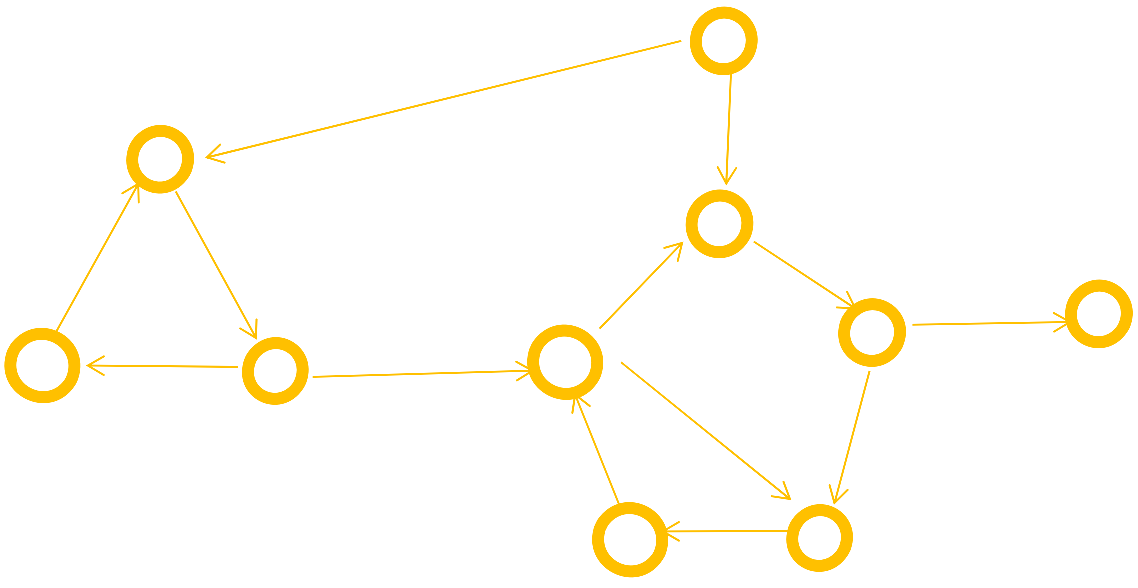
$(A_\mu, \lambda_\alpha, \sigma, D)$

$(X, \psi_\alpha, F)^{I=1,\cdots,N}_{Q=+1}$

$(P, \chi_\alpha, G)_{Q=-N}$ $W(X;P) = P \cdot G_N(X)$



canonical example : N=4 quivers



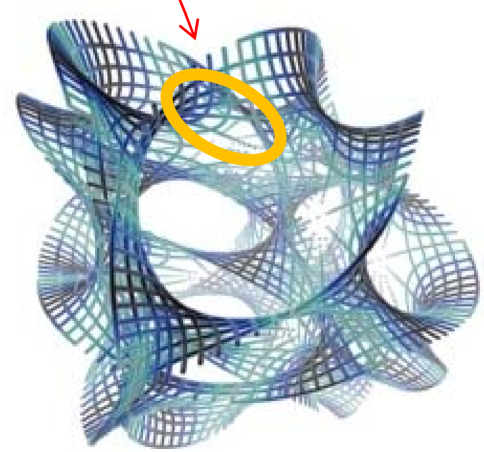
wall-crossing for $d=4$ BPS states

and $d=1$ quiver GLSM

D3 wrapped on a SL cycle in CY3 → 4d BPS particle

charged particle-like
BPS state in 4d

$$R^{1+3} \quad \times$$



wall-crossing

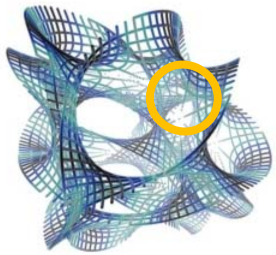
marginal stability wall $\xi = 0$

γ_1

γ_2

γ_3

$\vdots$



γ_1 $\quad$ $\gamma_1 + \gamma_2$

γ_2 $\quad$ $\gamma_2 + \gamma_3$

γ_3 $\quad$ $\gamma_1 + \gamma_3$

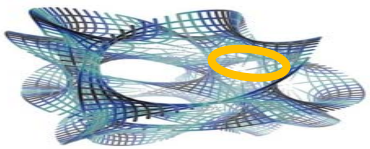
$\vdots$ $\quad$ $\vdots$

$\gamma_1 + \gamma_2 + \gamma_3$

$\vdots$

$\sum k_i \gamma_i$

$\vdots$



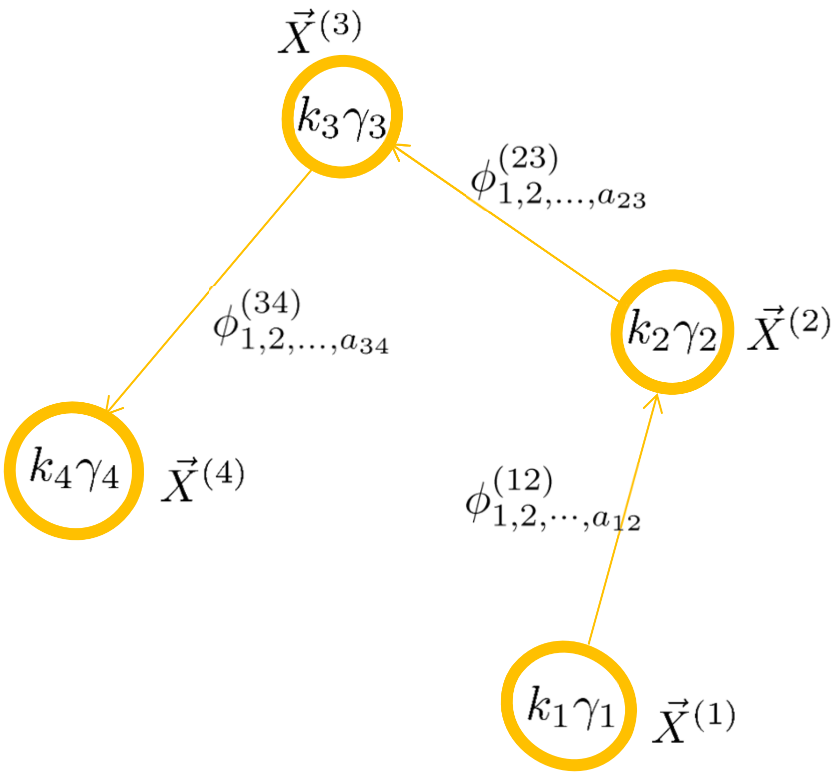
Denef 2002

→ d=1 quiver GLSM for particle-like BPS states in 4d

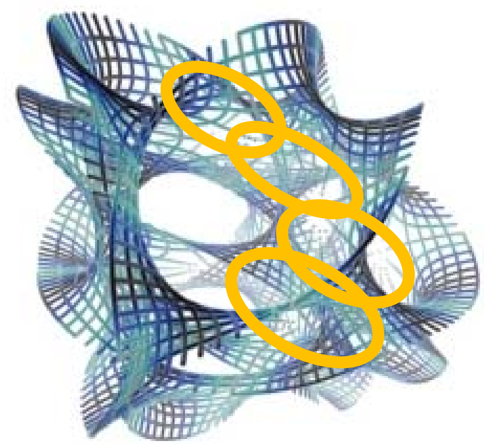
$$\vec{X}^{(1)} \quad \vec{X}^{(2)} \quad \vec{X}^{(3)} \quad \vec{X}^{(4)}$$

$$U(k_1) \times U(k_2) \times U(k_3) \times U(k_4)$$

$$\phi^{(12)}_{1,2,\cdots,a_{12}} \quad \phi^{(23)}_{1,2,\ldots,a_{23}} \quad \phi^{(34)}_{1,2,\ldots,a_{34}}$$



$$a_{ik} = \langle \gamma_i, \gamma_k \rangle$$



$$V_{\mathrm{Higgs}}^{tree} \sim \sum_j \left(\sum_{i \neq j} \phi^{(ij)} \phi^{(ij)\dagger} - \xi_j \right)^2$$

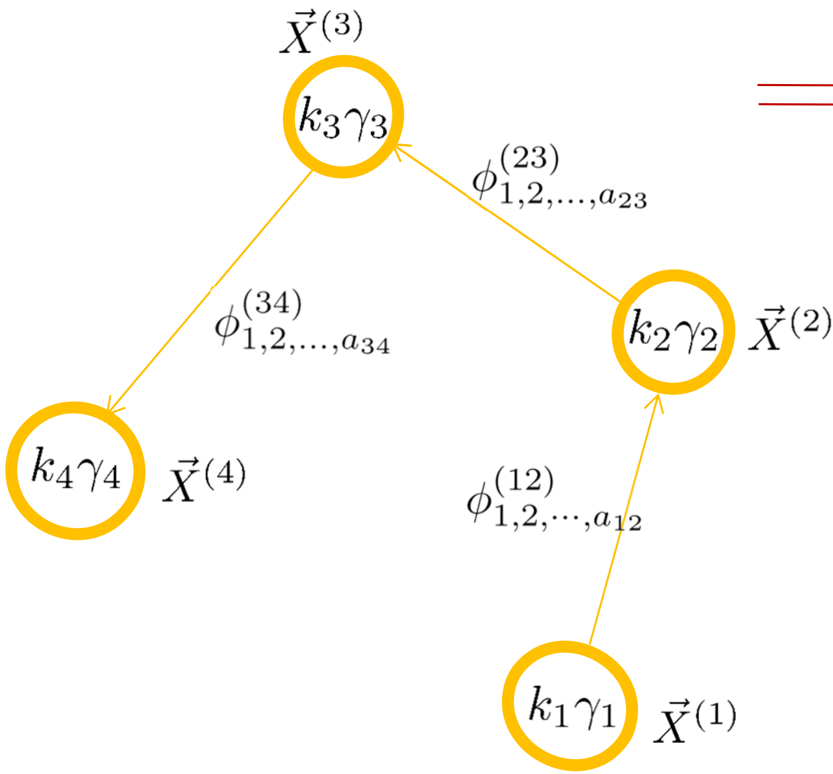
$$+ \sum_{ij} |\partial W / \partial \phi^{(ij)}|^2$$

versus

$$V_{\mathrm{Coulomb}}^{one-loop} \sim \sum_j \left(\sum_i \frac{a_{ij}/2}{|X^{(i)} - X^{(j)}|} - \xi_j \right)^2$$

Coulomb : wrapped D-branes are separated along real space

small & “positive”
FI constants

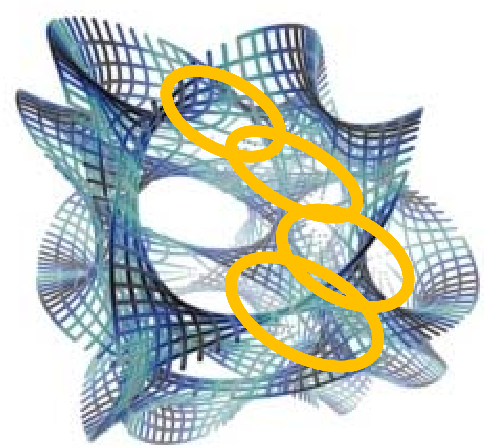


$$\vec{X}^{(1)} \quad \vec{X}^{(2)} \quad \vec{X}^{(3)} \quad \vec{X}^{(4)}$$

$$U(k_1) \times U(k_2) \times U(k_3) \times U(k_4)$$

$$\phi^{(12)}_{1,2,\cdots,a_{12}} \quad \phi^{(23)}_{1,2,\ldots,a_{23}} \quad \phi^{(34)}_{1,2,\ldots,a_{34}}$$

$$a_{ik} = \langle \gamma_i, \gamma_k \rangle$$

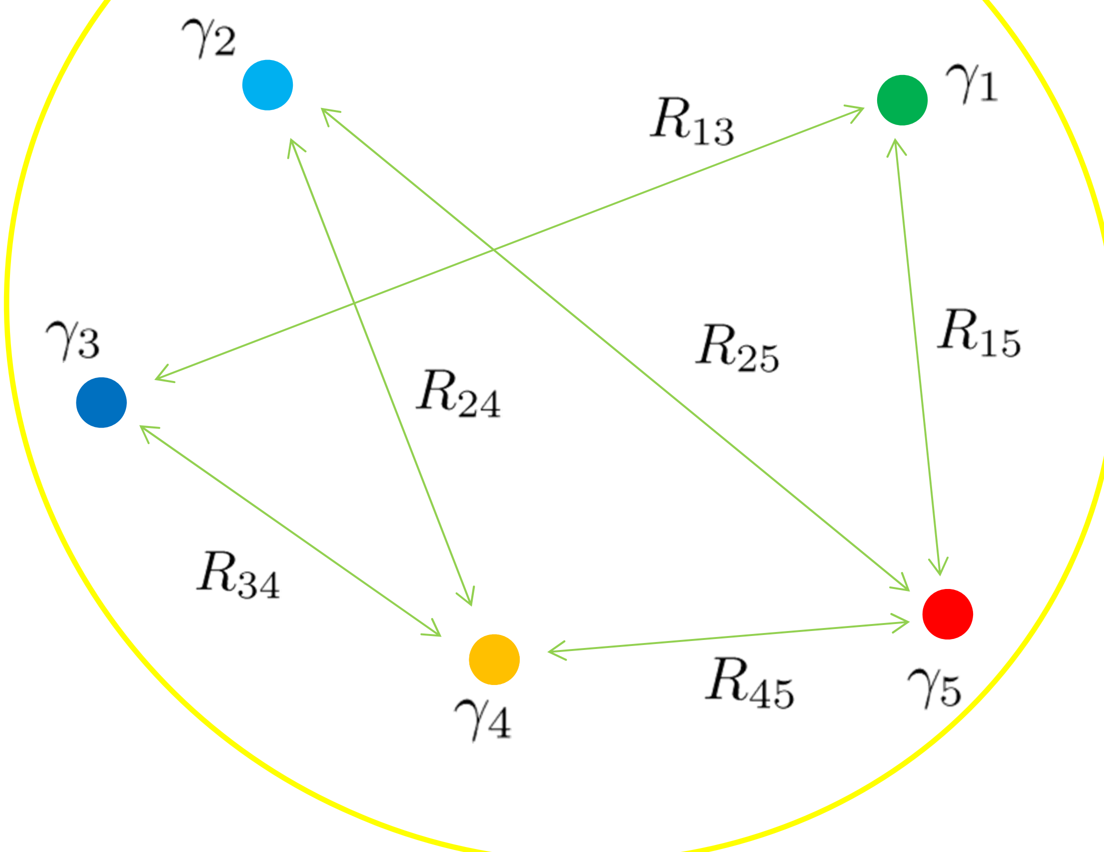


generic BPS particle is a multi-center bound state

$$\mathrm{Im}[\zeta^{-1} Z_{\gamma_A}] = \sum_{B \neq A} \frac{\langle \gamma_A, \gamma_B \rangle / 2}{R_{AB}}$$

$$\zeta \equiv \frac{\sum_A Z_{\gamma_A}}{|\sum_A Z_{\gamma_A}|}$$

$$\gamma_{\text{total}} = \sum_A \gamma_A$$



1998 Lee + P.Y.

N=4 SU(n) ¼ BPS states via semiclassical multi-center dyon solitons

1999 Bak + Lee + Lee + P.Y.

N=4 SU(n) ¼ BPS states via semi-classical multi-center monopole dynamics

1999-2000 Gauntlett + Kim + Park + P.Y. / Gauntlett + Kim + Lee + P.Y. / Stern + P.Y.

N=2 SU(n) BPS state counting via semi-classical multi-center monopole dynamics

2000 Denef

N=2 supergravity via classical multi-center black holes attractor solutions

$$\mathrm{Im}[\zeta^{-1} Z_{\gamma_A}] = \sum_{B\neq A} \frac{\langle \gamma_A, \gamma_B \rangle/2}{R_{AB}} \qquad \zeta \equiv \frac{\sum_A Z_{\gamma_A}}{|\sum_A Z_{\gamma_A}|}$$

2001 Argyres + Narayan / Ritz + Shifman + Vainshtein + Voloshin

UV-incomplete string-web picture for N=2 BPS dyons

2000 Stern + P.Y.

wall-crossing formula for simple magnetic charges; weak coupling regime

2002 Denef

quiver dynamics representation of N=2 supergravity BH's

⋮

2008 Kontsevich + Soibelman

⋮

2010/2011 Manschot + Pioline + Sen

general n-particle conjecture for Quantum Mechanics Counting

2011 Lee+P.Y. / Kim+Park+Wang+P.Y.

general n-particle solution to Quantum Mechanics Counting

2011 Sen

Coulomb counting = Higgs counting = Kontsevich-Soibelman counting

for nonscaling examples

universal wall-crossing formulae from quantum mechanics of BPS particles

Manchot+Pioline-Sen 2010/2011
Kim+Park+P.Y.+Wang 2011

$$\bar{\Omega}^-\left(\sum \gamma_A\right) - \bar{\Omega}^+\left(\sum \gamma_A\right) = (-1)^{\sum_{A>B}\langle\gamma_A,\gamma_B\}+n-1}\frac{\prod_A \bar{\Omega}^+(\gamma_A)}{|\Gamma|}\int_{\mathcal{M}} ch(\mathcal{F})$$

$$\vdots$$

$$+(-1)^{\sum_{A'>B'}\langle\gamma'_{A'},\gamma'_{B'}\}+n'-1}\frac{\prod_{A'} \bar{\Omega}^+(\gamma'_{A'})}{|\Gamma'|}\int_{\mathcal{M}'} ch(\mathcal{F}')$$

$$\vdots$$

$$+(-1)^{\sum_{A''>B''}\langle\gamma''_{A''},\gamma''_{B''}\}+n''-1}\frac{\prod_{A''} \bar{\Omega}^+(\gamma''_{A''})}{|\Gamma''|}\int_{\mathcal{M}''} ch(\mathcal{F}'')$$

$$\vdots$$

$$\bar{\Omega}(\gamma) = \sum_{p|\gamma} \Omega(\gamma/p)/p^2$$

$$\sum_{A=1}^{n}\gamma_A = \cdots = \sum_{A'=1}^{n'}\gamma'_{A'} = \cdots = \sum_{A''=1}^{n''}\gamma''_{A''} = \cdots$$

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⋮

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general n-particle conjecture for Quantum Mechanics Counting

2011 Lee+P.Y. / Kim+Park+Wang+P.Y.

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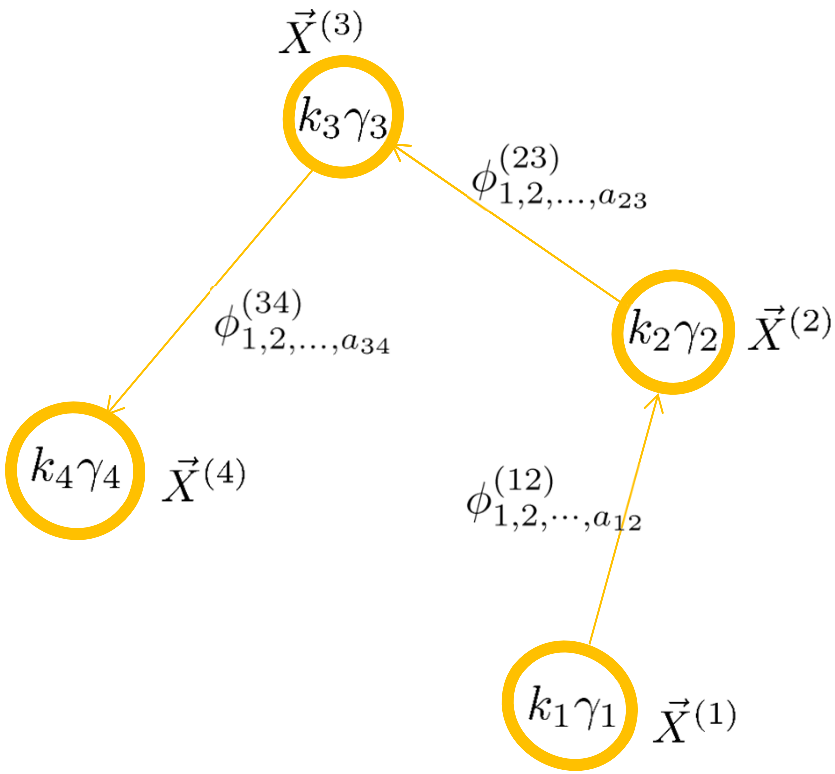
Coulomb counting = Higgs counting = Kontsevich-Soibelman counting

for nonscaling examples

Higgs : wrapped D-branes are fused into a single object

large & "positive"
FI constants



$$U(k_1) \times U(k_2) \times U(k_3) \times U(k_4)$$

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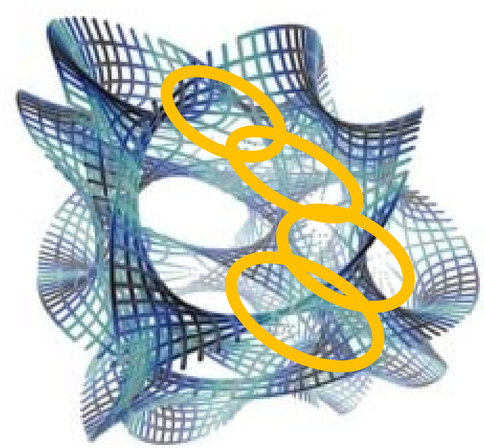
$$a_{ik} = \langle \gamma_i, \gamma_k \rangle$$



Higgs regime have different geometries in different chambers

marginal stability wall

$$\xi^{(1)} > \xi^{(2)} > \xi^{(3)} > \xi^{(4)}$$

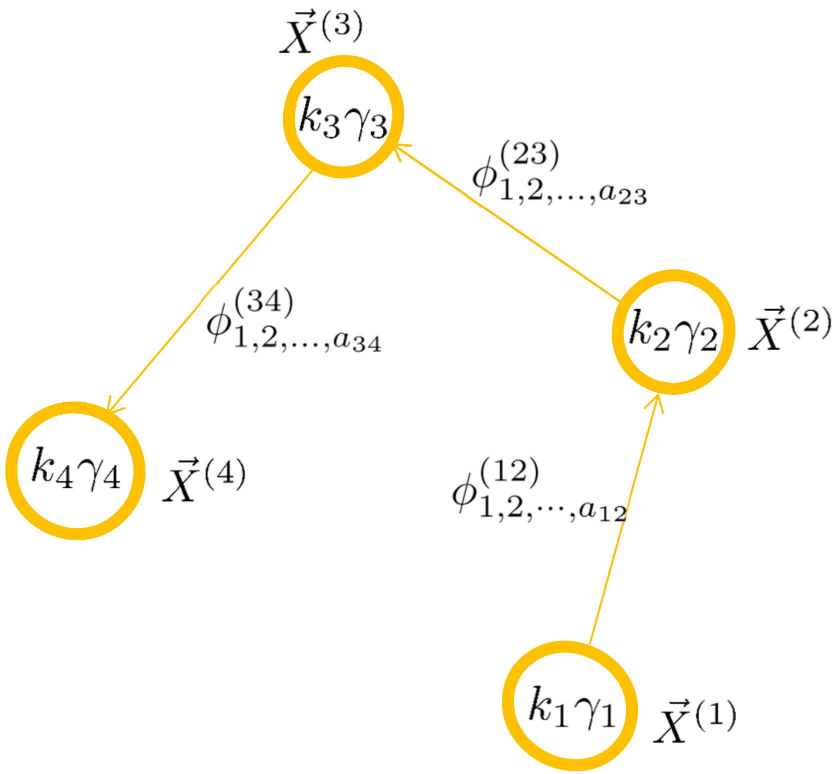
$$a_{ik} = \langle \gamma_i, \gamma_k \rangle$$

$$\mathcal{M}_{\mathrm{H}} \left(\sum_i \gamma_i ; \xi^{(i)} \right) = 0$$

$$\mathcal{M}_{\mathrm{H}} \left(\sum_i \gamma_i ; \xi^{(i)} \right)$$

$$= CP^{a_{12}-1} \times CP^{a_{32}-1} \times CP^{a_{34}-1}$$

BPS states → cohomology → the Hirzebruch character



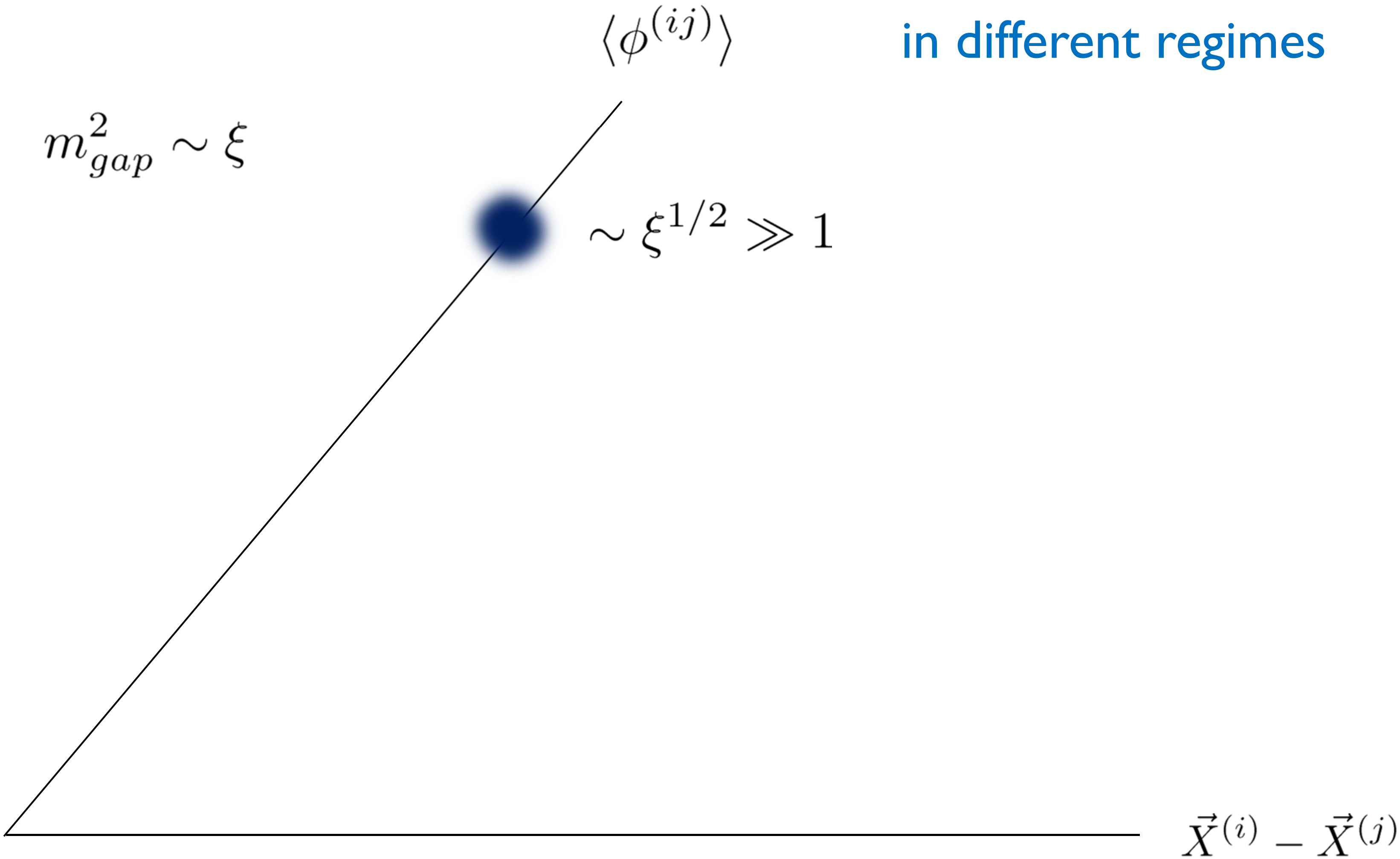
$$\Omega_{\text{Higgs}}[y]\left(\sum_i k_i\gamma_i;\xi^{(i)}\right)$$

$$= \text{tr}(-1)^{p+q-d}\mathbf{y}^{2p-d}$$

$$= \sum_{p,q}(-1)^{p+q-d}\mathbf{y}^{2p-d}h^{(p,q)}(\mathcal{M}_{\text{H}})$$

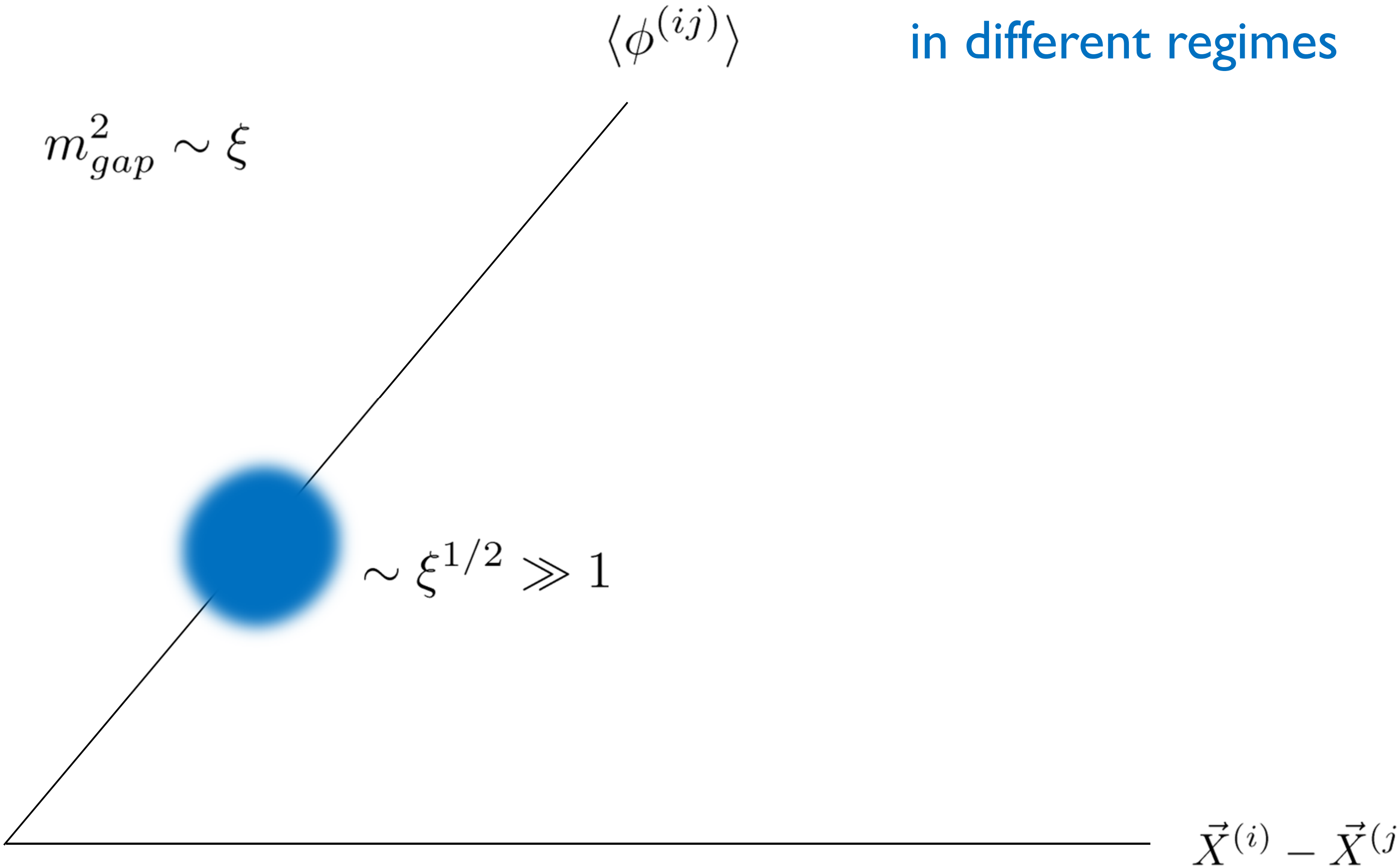
$\text{if } \xi > 0$

how wavefunctions look in different regimes



if $\xi > 0$

how wavefunctions look
in different regimes



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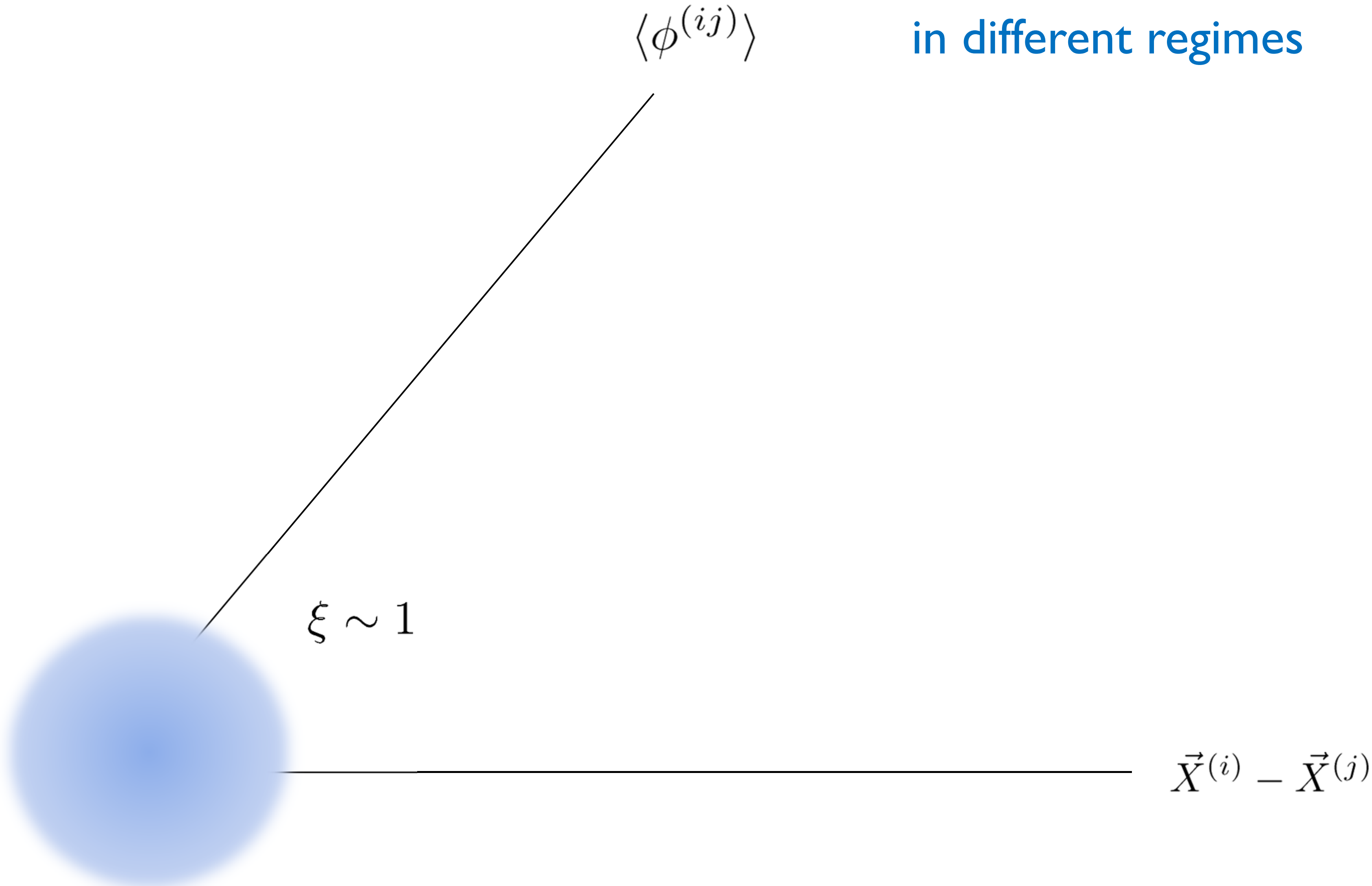

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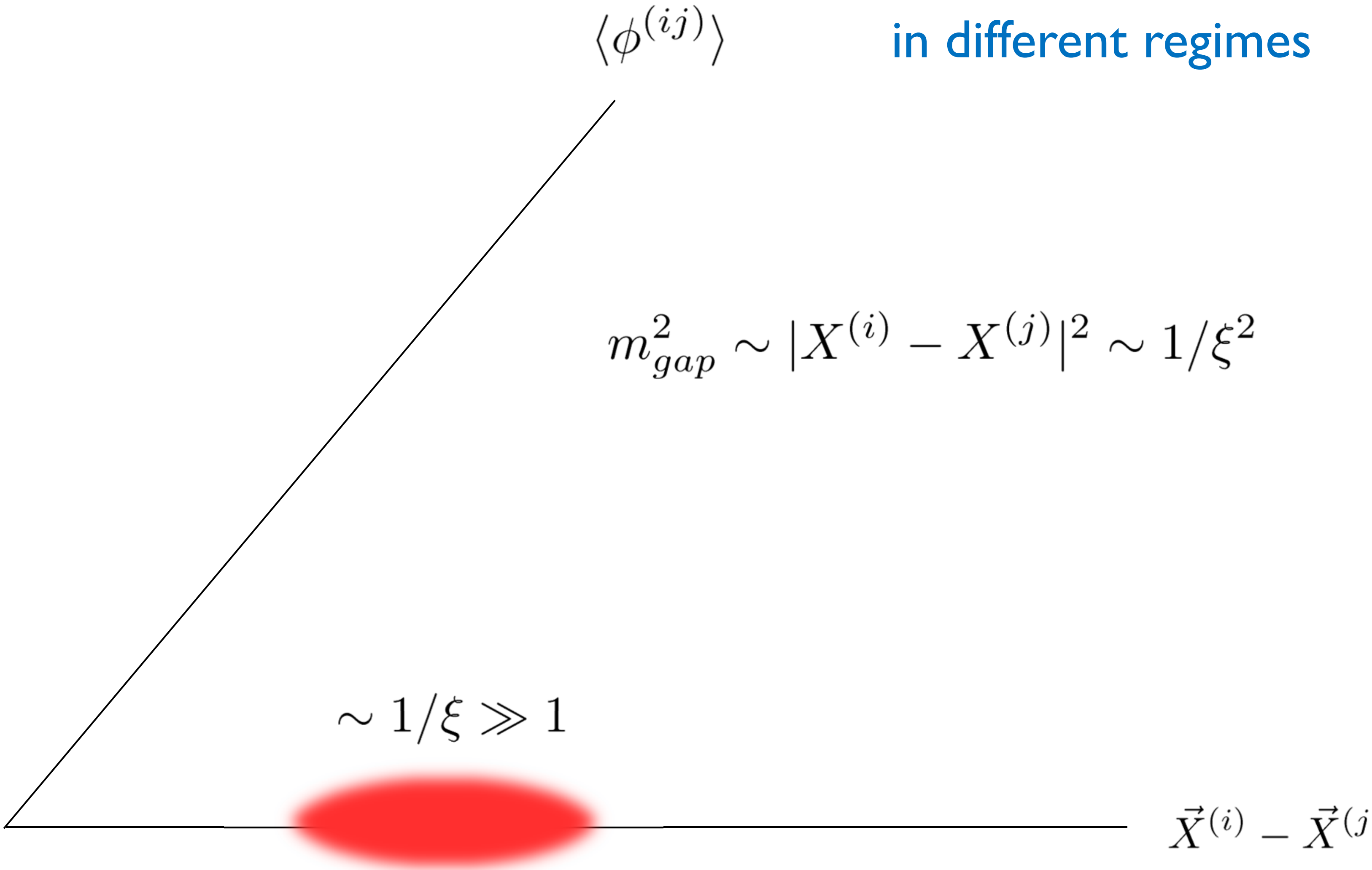
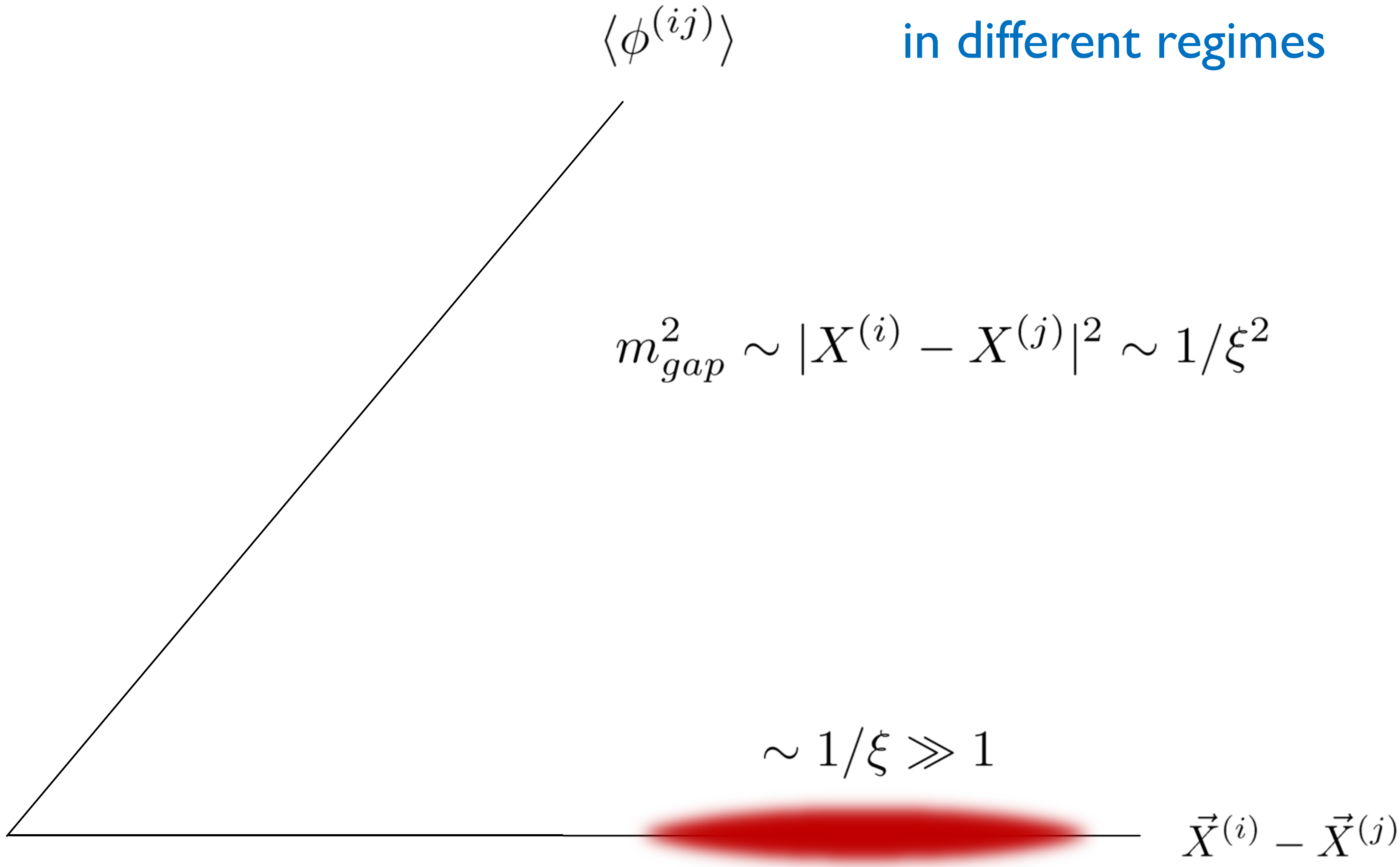


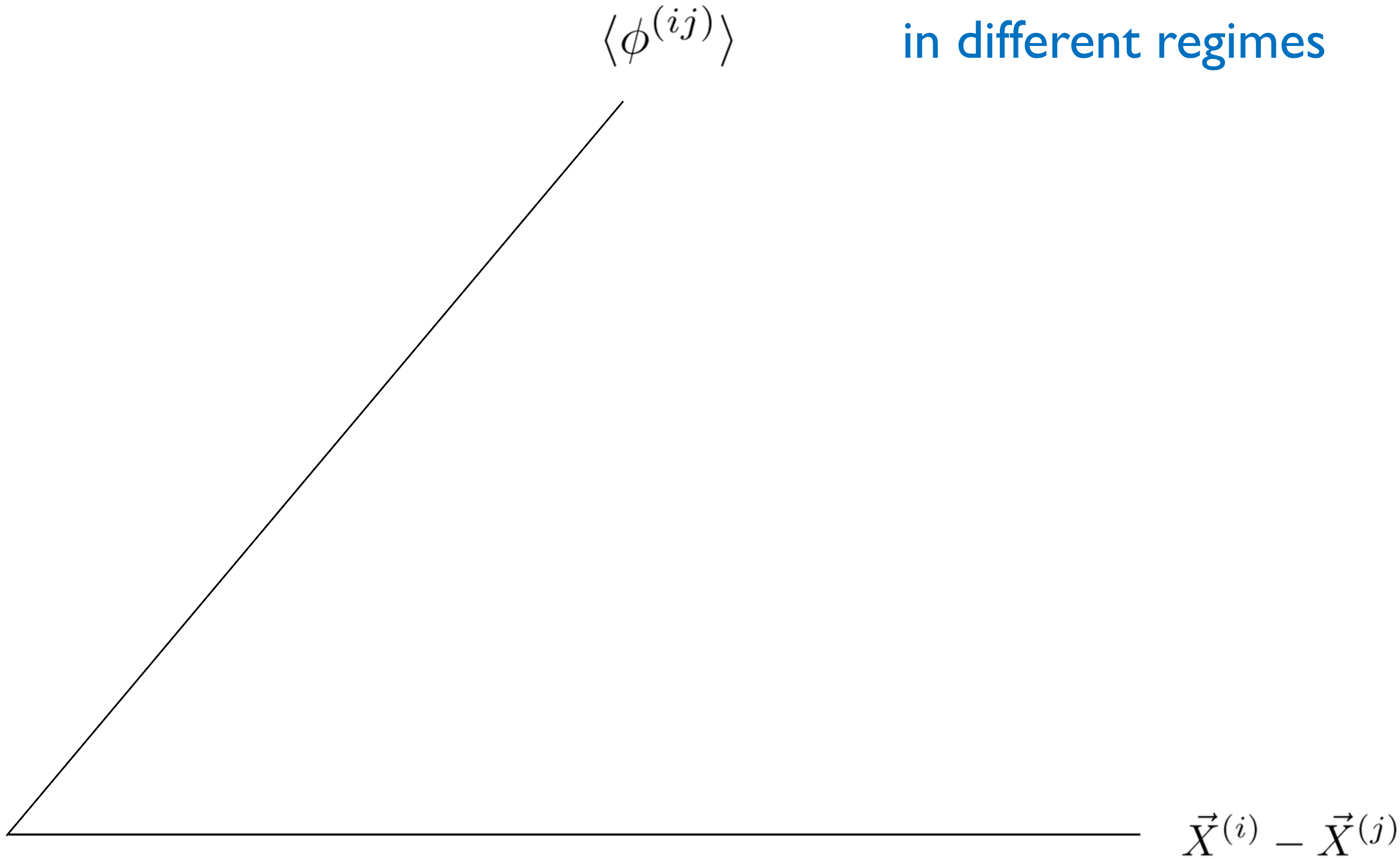
if $\xi > 0$

how wavefunctions look
in different regimes



$\text{if } \xi \to 0^+$

how wavefunctions look
in different regimes



for d ≤ 2, what we mean by regimes is

truncation processes where we integrate out partially:

if both regimes are reliable, the two answers must agree

large FI constants

small FI constants

$$\Omega_{\text{Higgs}} = \Omega_{\text{Coulomb}}$$

F. Denef 2002 + A. Sen 2011

for d ≤ 2, what we mean by regimes is
truncation processes where we integrate out partially:
if both regimes are reliable, the two answers must agree

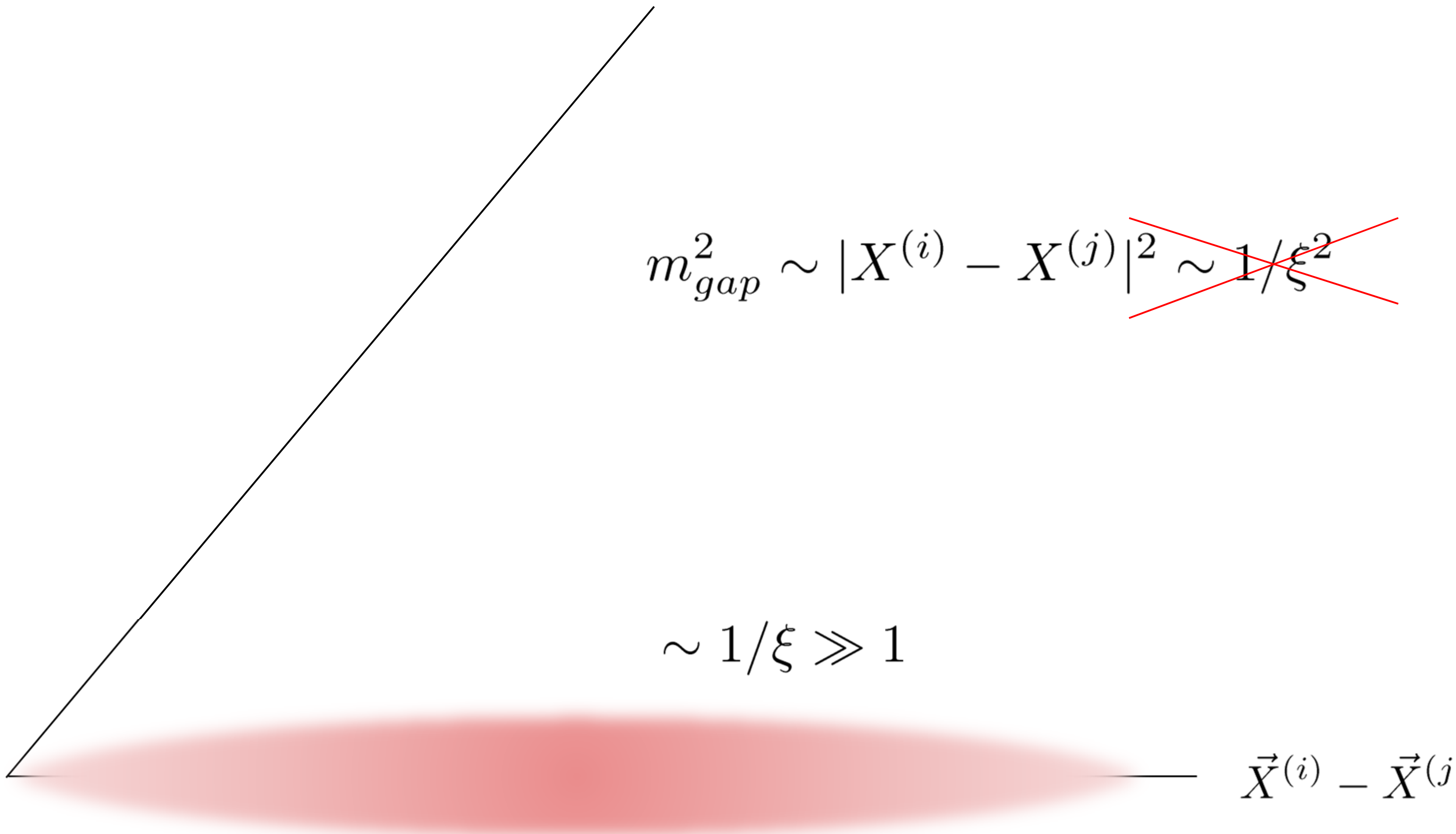
however, such processes can sometimes fail,
if the "heavy" multiplet in question become light
somewhere in the vacuum moduli space

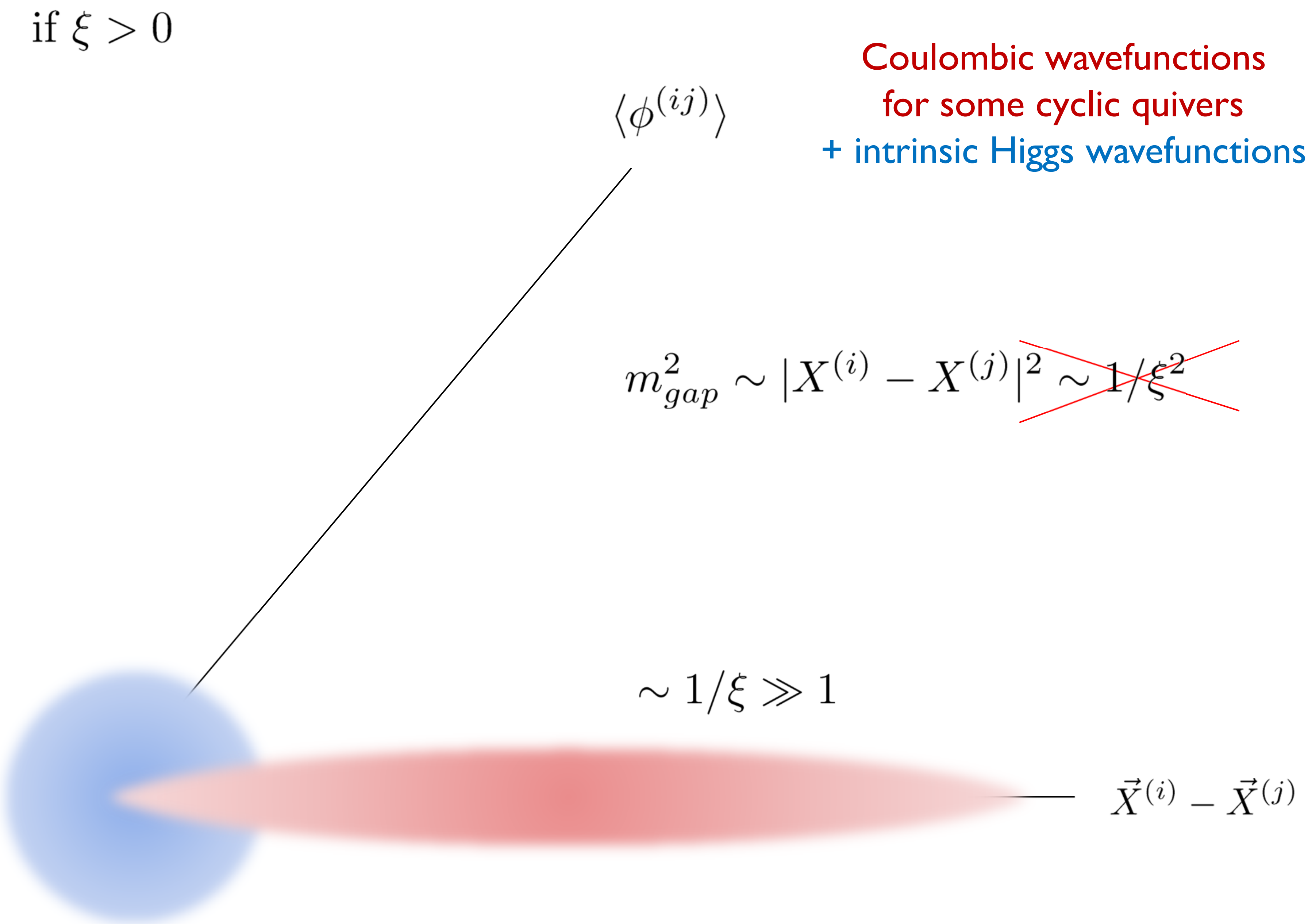
$$\Omega_{\text{Higgs}} \neq \Omega_{\text{Coulomb}}$$

$\text{if } \xi > 0$

Coulombic wavefunctions for some cyclic quivers

$\langle \phi^{(ij)} \rangle$




if $\xi > 0$
Coulombic wavefunctions
for some cyclic quivers
+ intrinsic Higgs wavefunctions
$\langle \phi^{(ij)} \rangle$
$m_{gap}^2 \sim |X^{(i)} - X^{(j)}|^2 \sim 1/\xi^2$
$\sim 1/\xi \gg 1$
$\vec{X}^{(i)} - \vec{X}^{(j)}$

quiver invariant

large FI constants

small FI constants

$$\Omega_{\text{Higgs}} \neq \Omega_{\text{Coulomb}}$$

$$\Omega^{(k)}_{\text{Higgs}} = \boxed{\Omega_{\text{Invariant}}} + \boxed{\Omega^{(k)}_{\text{Coulomb}}}$$

provided that

1) superpotentials generic & consistent with gauge symmetry

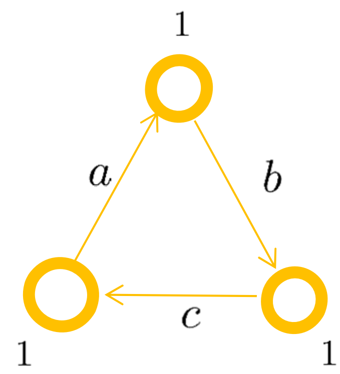
2) in all chambers, Higgs "phase" is nontrivial

S.J. Lee + Z.L. Wang + P.Y., 2012

Bena + Berkooz + de Boer + El-Showk + d. Bleeken, 2012

a simple 3-body problem

Denef + Moore 2007



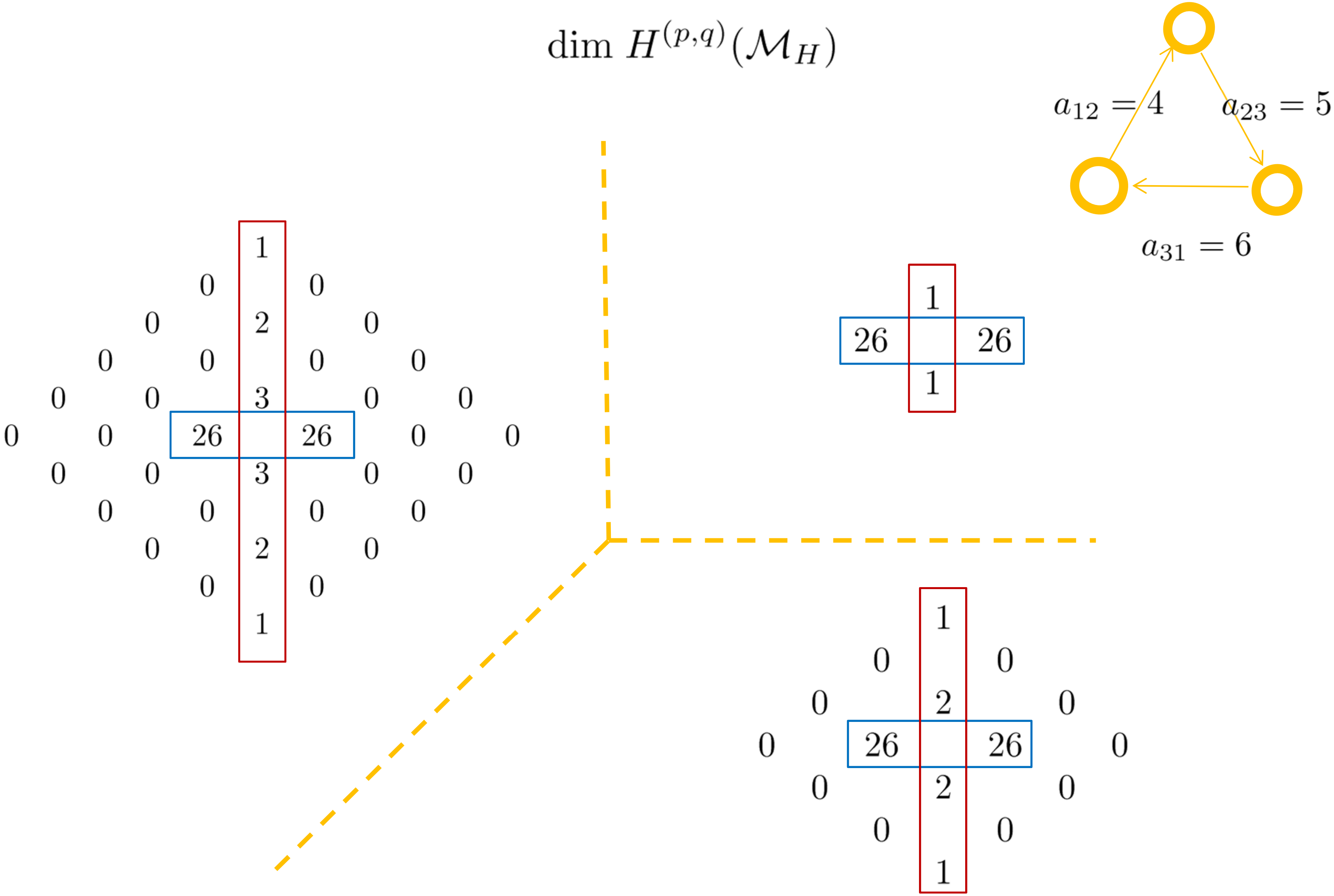
$$\Omega_{\text{Higgs}} = \begin{pmatrix} a \cdot (c-b) \\ b \cdot (a-c) \\ c \cdot (b-a) \end{pmatrix} + \# \cdot 2^{(a+b+c)/2} + \cdots$$

$$= \Omega_{\text{Coulomb}}$$

S.J. Lee + Z.L. Wang + P.Y., 2012

a simple 3-body example

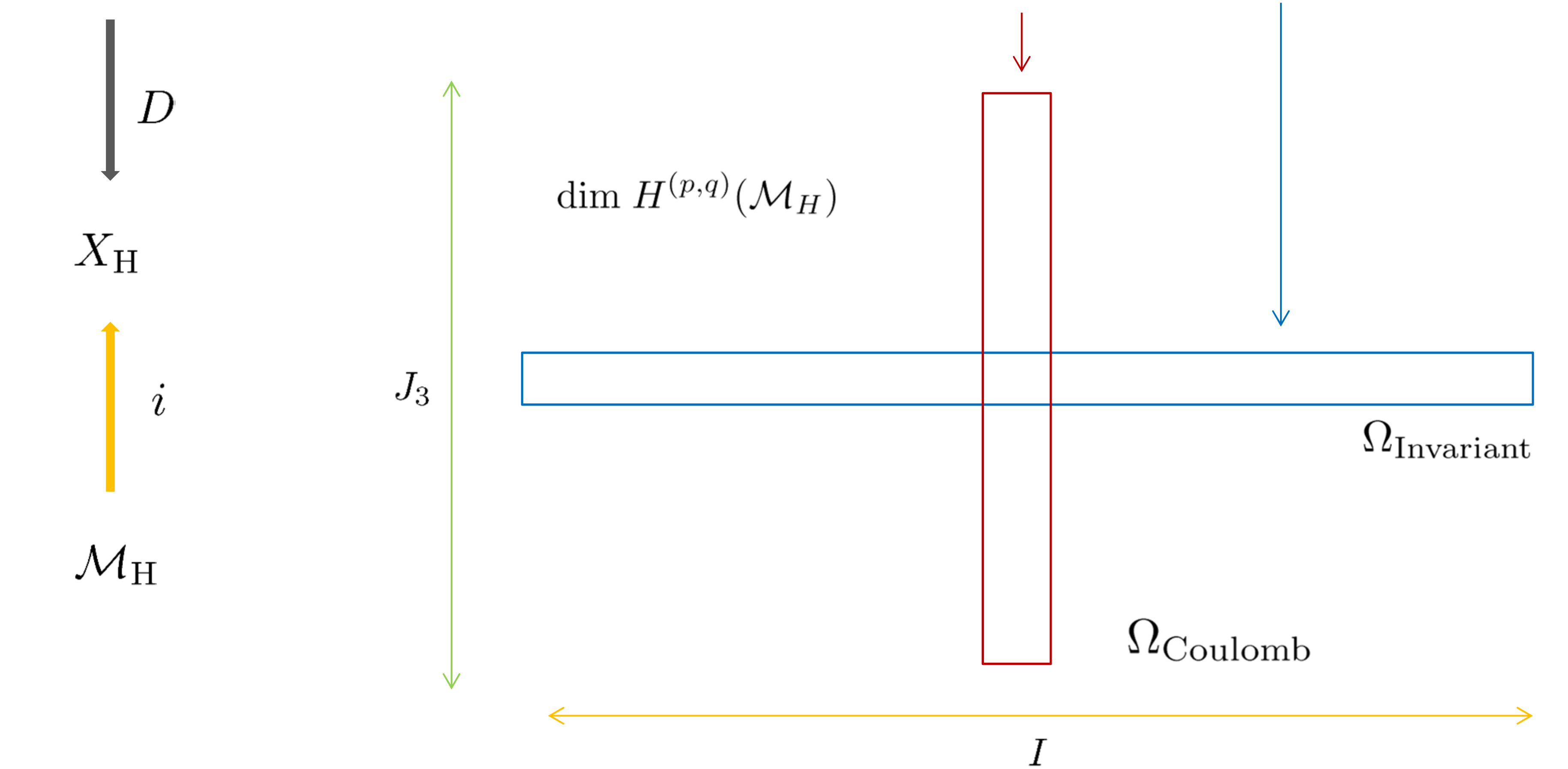
$$\dim H^{(p,q)}(\mathcal{M}_H)$$



wall-crossing states vs. wall-crossing-safe states

$\{\phi^{(12)}, \dots\}$

$$H^*(\mathcal{M}_H) = i^*\left[H^*(X_H)\right] \oplus H^*(\mathcal{M}_H)_{\mathrm{Intrinsic}}$$



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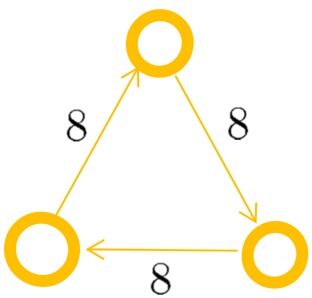

$\dim H^{(p,q)}(\mathcal{M}_H)$

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 0 & & 0 & & & & & \\
 & & & & 0 & & 2 & & 0 & & & & \\
 & & & 0 & & 0 & & 0 & & 0 & & & \\
 & & 0 & & 0 & & 3 & & 0 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
1 & & 322 & & 6803 & & 4+18212 & & 6803 & & 322 & & 1 \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 0 & & 3 & & 0 & & 0 & & \\
 & & & 0 & & 0 & & 0 & & 0 & & & \\
 & & & & 0 & & 2 & & 0 & & & & \\
 & & & & & 0 & & 0 & & & & & \\
 & & & & & & 1 & & & & & &
\end{array}$$

wall-crossing states vs. wall-crossing-safe states

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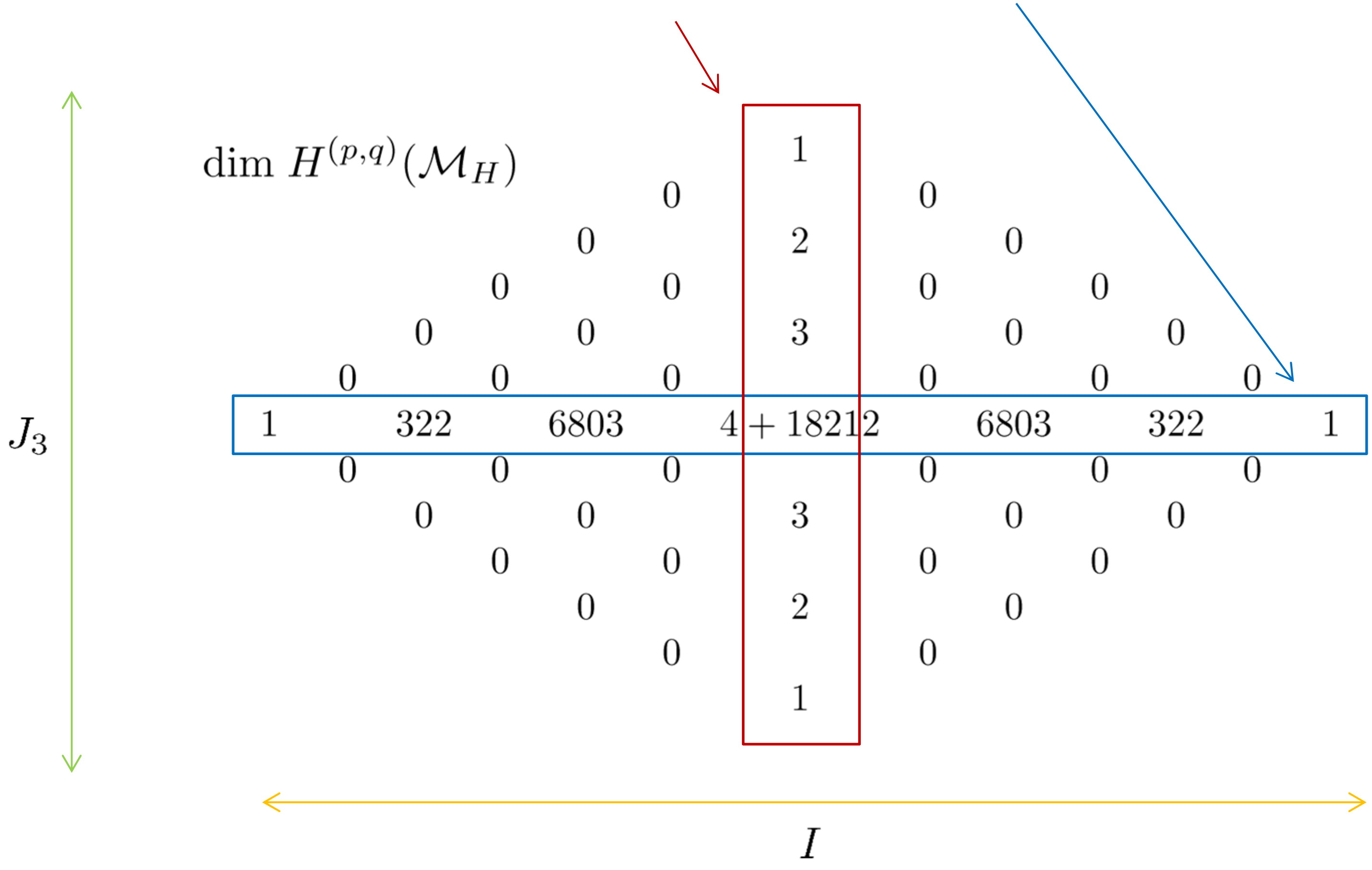


wall-crossing states vs. wall-crossing-safe states

$$H^*(\mathcal{M}_H) = i^*\left[H^*(X_H)\right] \oplus H^*(\mathcal{M}_H)_{\text{Intrinsic}}$$

$$\Updownarrow \qquad \mathrm{tr}(-1)^{p+q-d}\mathbf{y}^{2p-d} \qquad \Updownarrow$$

$$\Omega_{\text{Coulomb}} \qquad\qquad \Omega_{\text{Invariant}} = \Omega_{\text{Higgs}} - \Omega_{\text{Coulomb}}$$

angular momentum multiplets with polynomial degeneracy:

field theory BPS states typically belong here

angular momentum singlets with exponential degeneracy:

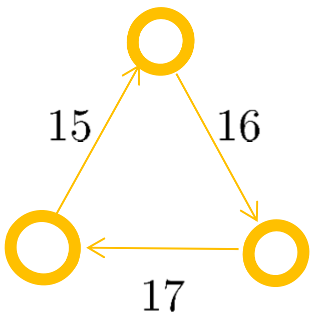
microstates of single-center BH's belong here

more examples of quiver invariants

$$H^*(\mathcal{M}_H) = i^*\left[H^*(X_H)\right] \oplus H^*(\mathcal{M}_H)_{\text{Intrinsic}}$$

$$\Omega(\mathbf{y})\Big|_{\text{Intrinsic}}^{\{a_{i,i+1}\}=(15,16,17)} = \text{tr}_{\text{Intrinsic}}(-1)^{2J_3}\mathbf{y}^{2J_3+2I} = \sum(-1)^{p+q-d}\mathbf{y}^{2p-d}h^{(p.q)}_{\text{Intrinsic}}$$

$$\begin{aligned}
&= 1665\mathbf{y}^{-12} \\
&+ 724674\mathbf{y}^{-10} \\
&+ 60686563\mathbf{y}^{-8} \\
&+ 1523273844\mathbf{y}^{-6} \\
&+ 13886938949\mathbf{y}^{-4} \\
&+ 50685934038\mathbf{y}^{-2} \\
&+ 77668453887 \\
&+ 50685934038\mathbf{y}^{2} \\
&+ 13886938949\mathbf{y}^{4} \\
&+ 1523273844\mathbf{y}^{6} \\
&+ 60686563\mathbf{y}^{8} \\
&+ 724674\mathbf{y}^{10} \\
&+ 1665\mathbf{y}^{12}
\end{aligned}$$

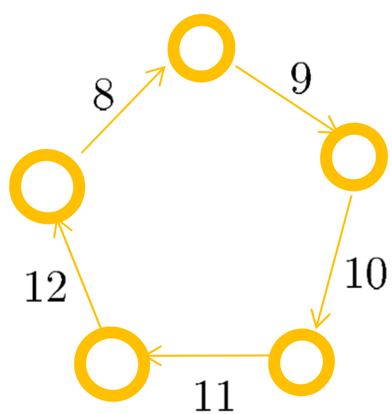


more examples of quiver invariants

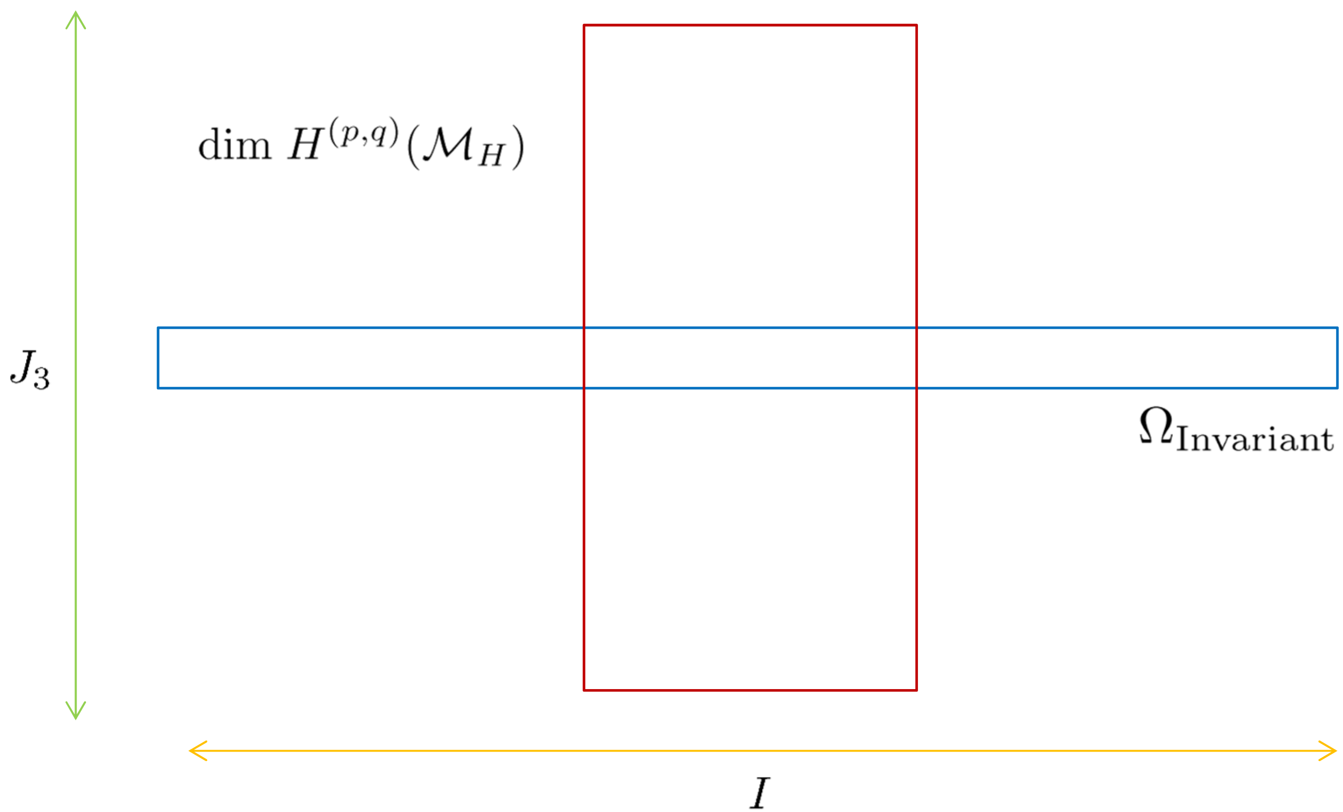
$$H^*(\mathcal{M}_H) = i^*\left[H^*(X_H)\right] \oplus H^*(\mathcal{M}_H)_{\text{Intrinsic}}$$

$$\Omega(\mathbf{y})\Big|_{\text{Intrinsic}}^{\{a_{i,i+1}\}=(8,9,10,11,12)} = \text{tr}_{\text{Intrinsic}}(-1)^{2J_3}\mathbf{y}^{2J_3+2I} = \sum(-1)^{p+q-d}\mathbf{y}^{2p-d}h^{(p,q)}_{\text{Intrinsic}}$$

$$\begin{aligned}
= \; & 32294250/\mathbf{y}^{22} + 58872952926/\mathbf{y}^{20} + 23086762587054/\mathbf{y}^{18} \\
& +3146301650299568/\mathbf{y}^{16} + 186529800766285403/\mathbf{y}^{14} \\
& +5480846262397291070/\mathbf{y}^{12} + 86780383421802203555/\mathbf{y}^{10} \\
& +783408269154731872224/\mathbf{y}^{8} + 4192271239441338802849/\mathbf{y}^{6} \\
& +13657486692285216220742/\mathbf{y}^{4} + 27560691162972524163666/\mathbf{y}^{2} \\
& +34791235315880411958041 + 27560691162972524163666y^{2} \\
& +13657486692285216220742\mathbf{y}^{4} + 4192271239441338802849\mathbf{y}^{6} \\
& +783408269154731872224\mathbf{y}^{8} + 86780383421802203555\mathbf{y}^{10} \\
& +5480846262397291070\mathbf{y}^{12} + 186529800766285403\mathbf{y}^{14} \\
& +3146301650299568\mathbf{y}^{16} + 23086762587054\mathbf{y}^{18} \\
& +58872952926\mathbf{y}^{20} + 32294250\mathbf{y}^{22}
\end{aligned}$$



this simple dichotomy is literally true only for cyclic Abelian quivers: for more general quivers, the cohomology is far more intricate



a more complete index computation ?

what if there are chambers without geometric limit ?

can one define/compute wall-crossing invariants for all gauged QM ?

large rank limit ?

index, wall-crossing, and quiver invariant
via direct path integral computation ?

$$\mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2I}\right] \quad = \quad \int_{\text{periodic}}[dX\cdots d\phi\cdots]\, e^{-S_E^{\mathbf{y}}+\cdots}$$

K.Hori + H.Kim + P.Y. 2014

index of $d=1$ GLSM

localization

$$e^2, g^2 \to 0$$

$$\mathcal{L}_{\rm vector} = \frac{1}{e^2}\ {\rm Re}\left(\int d\theta^2\ W_\alpha W^\alpha\right)\Big|_{\rm time\ only}$$

$$\mathcal{L}_{\rm chiral} = \frac{1}{g^2}\ \int d\theta^2 d\bar\theta^2\ \bar\Phi e^V \Phi\Big|_{\rm time\ only}$$

$$\mathcal{L}_{\rm usperpotential} = \int d\theta^2 W(\Phi) + c.c.$$

$$\mathcal{L}_{\rm FI} = \xi\ \int d\theta^2 d\bar\theta^2 V\Big|_{\rm time\ only}$$

Benini + Eager + Hori + Tachikawa 2013
Hori + Kim + P.Y. 2014

localization : but all four pieces are individually Q-exact

$$\mathcal{L}_{\text{vector}} = \frac{1}{e^2}\,\text{Re}\left(\int d\theta^2\; W_\alpha W^\alpha\right)\Bigg|_{\text{time only}}$$

$$\mathcal{L}_{\text{chiral}} = \frac{1}{g^2}\;\int d\theta^2 d\bar\theta^2\;\bar\Phi e^V\,\Phi\Big|_{\text{time only}}$$

$$\mathcal{L}_{\text{superpotential}} = \int d\theta^2 W(\Phi) + c.c.$$

$$\mathcal{L}_{\text{FI}} = \xi\;\int d\theta^2 d\bar\theta^2 V\Big|_{\text{time only}}$$

Benini + Eager + Hori + Tachikawa 2013
Hori + Kim + P.Y. 2014

to what extent will the result be independent of the superpotential or FI constants ?

$$\mathcal{L}_{\text{vector}} = \frac{1}{e^2}\,\text{Re}\left(\int d\theta^2\; W_\alpha W^\alpha\right)\Big|_{\text{time only}}$$

$$\mathcal{L}_{\text{chiral}} = \frac{1}{g^2}\;\int d\theta^2 d\bar{\theta}^2\;\bar{\Phi} e^V \Phi\Big|_{\text{time only}}$$

$$\mathcal{L}_{\text{superpotential}} = \int d\theta^2 W(\Phi) + c.c.$$

$$\mathcal{L}_{\text{FI}} = \xi\;\int d\theta^2 d\bar{\theta}^2 V\Big|_{\text{time only}}$$

Benini + Eager + Hori + Tachikawa 2013
Hori + Kim + P.Y. 2014

localization

$$\mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2I}e^{-\beta Q^2}\right] \qquad\qquad [Q, J_3 + I] = 0$$

$$= \quad \lim_{e^2\to 0}\mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2I}e^{-\beta Q^2}\right]$$

$$= \quad \lim_{e^2\to 0}\int_{\text{periodic}}[dX\cdots d\phi\cdots]\; e^{-\int_0^\beta d\tau\mathcal{L}_E}\Big|_{\partial_\tau\to\partial_\tau+(2J_3+2I)\log(\mathbf{y})/\beta}$$

Benini + Eager + Hori + Tachikawa 2013
Hori + Kim + P.Y. 2014

localization

$$\mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2I}e^{-\beta Q^2}\right]$$

$$u = A_3 + iA_\tau \Big|^{\mathrm{Cartan}}_{\mathrm{zeromode}}$$

$$= \int_{M_u} du\, d\bar{u} \int_{\mathbf{R}+i\delta} dD \left[h(u,\bar{u};D)\cdot g(u,\bar{u};D)\cdot e^{-\frac{D^2}{e^2}+i\xi D}\right]$$

↑ integral over Cartan gaugino zero mode

↑ one-loop determinants from chirals + off-diagonal vectors

Benini + Eager + Hori + Tachikawa 2013
Hori + Kim + P.Y. 2014

localization

$$\mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2I}e^{-\beta Q^2}\right] \qquad u = A_3 + iA_\tau\Big|_{\mathrm{zeromode}}^{\mathrm{Cartan}}$$

$$= \int_{M_u} du\, d\bar{u} \int_{\mathbf{R}+i\delta} dD \left[h(u,\bar{u};D)\cdot g(u,\bar{u};D)\cdot e^{-\frac{D^2}{e^2}+i\xi D}\right]$$

$$h(u,\bar{u};D)\cdot g(u,\bar{u};D)$$

$$\sim \frac{1}{D}\partial_{\bar{u}} g(u,\bar{u};D) + O(e^2)$$

Benini + Eager + Hori + Tachikawa 2013
Hori + Kim + P.Y. 2014

localization

$$\mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2I}e^{-\beta Q^2}\right] \qquad\qquad u = A_3 + iA_\tau\Big|_{\text{zeromode}}^{\text{Cartan}}$$

$$= \int_{M_u} du\, d\bar{u} \int_{\mathbf{R}+i\delta} dD\, \left[h(u,\bar{u};D)\cdot g(u,\bar{u};D)\cdot e^{-\frac{D^2}{e^2}+i\xi D}\right]$$

$$= \int_{\partial M_u} du \int_{\mathbf{R}+i\delta} \frac{dD}{D}\, g(u,\bar{u};D)\cdot e^{-\frac{D^2}{e^2}+i\xi D}$$

↑

???

Benini + Eager + Hori + Tachikawa 2013
Hori + Kim + P.Y. 2014

localization

$$\mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2I}e^{-\beta Q^2}\right] \qquad u = A_3 + iA_\tau\Big|_{\text{zeromode}}^{\text{Cartan}}$$

$$= \int_{M_u} du\, d\bar{u} \int_{\mathbf{R}+i\delta} dD\, \left[h(u,\bar{u};D)\cdot g(u,\bar{u};D)\cdot e^{-\frac{D^2}{e^2}+i\xi D}\right]$$

$$= \int_{\partial M_u} du \int_{\mathbf{R}+i\delta} \frac{dD}{D}\, g(u,\bar{u};D)\cdot e^{-\frac{D^2}{e^2}+i\xi D}$$

Benini + Eager + Hori + Tachikawa 2013
Hori + Kim + P.Y. 2014

localization

$$\mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2I}e^{-\beta Q^2}\right] \qquad u = A_3 + iA_\tau\Big|_{\text{zeromode}}^{\text{Cartan}}$$

$$= \int_{M_u} du\, d\bar{u} \int_{\mathbf{R}+i\delta} dD \left[h(u,\bar{u};D)\cdot g(u,\bar{u};D)\cdot e^{-\frac{D^2}{e^2}+i\xi D}\right]$$

$$= \int_{\partial M_u} du \int_{\mathbf{R}+i\delta} \frac{dD}{D}\, g(u,\bar{u};D)\cdot e^{-\frac{D^2}{e^2}+i\xi D}$$

$$g(u,\bar{u};D) \sim \prod_Q \prod_n \frac{(2\pi ni + Qu - (R-2)\log(\mathbf{y}))\cdot(-2\pi ni + Qu - R\log(\mathbf{y}))}{|2\pi ni + Qu - R\log(\mathbf{y})|^2 - iQD}$$

localization

$$\mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2I}e^{-\beta Q^2}\right] = \int_{\partial M_u} du \int_{\mathbf{R}+i\delta} \frac{idD}{D}\, g(u,\bar{u};D)\cdot e^{-\frac{D^2}{e^2}+i\xi D}$$

$$u = A_3 + iA_\tau \Big|^{\mathrm{Cartan}}_{\mathrm{zeromode}}$$

d=2 $\quad M_u = \left(T^2\right)^{\mathrm{rank}} \setminus \cup H_*^Q$

d=1 $\quad M_u = (C^*)^{\mathrm{rank}} \setminus \cup H_*^Q$

N=4 CP(N-1)

chirals	$U(1)$	$U(N)_F$
X	1	N

$$g_{CP^{N-1}}(u;0) = \frac{1}{2\sinh[z/2]} \cdot \left[-\frac{\sinh[(u-z)/2]}{\sinh[u/2]}\right]^N$$

$\Omega = 0$

$(-1)^{N-1}\dfrac{\mathbf{y}^N - \mathbf{y}^{-N}}{\mathbf{y} - \mathbf{y}^{-1}}$

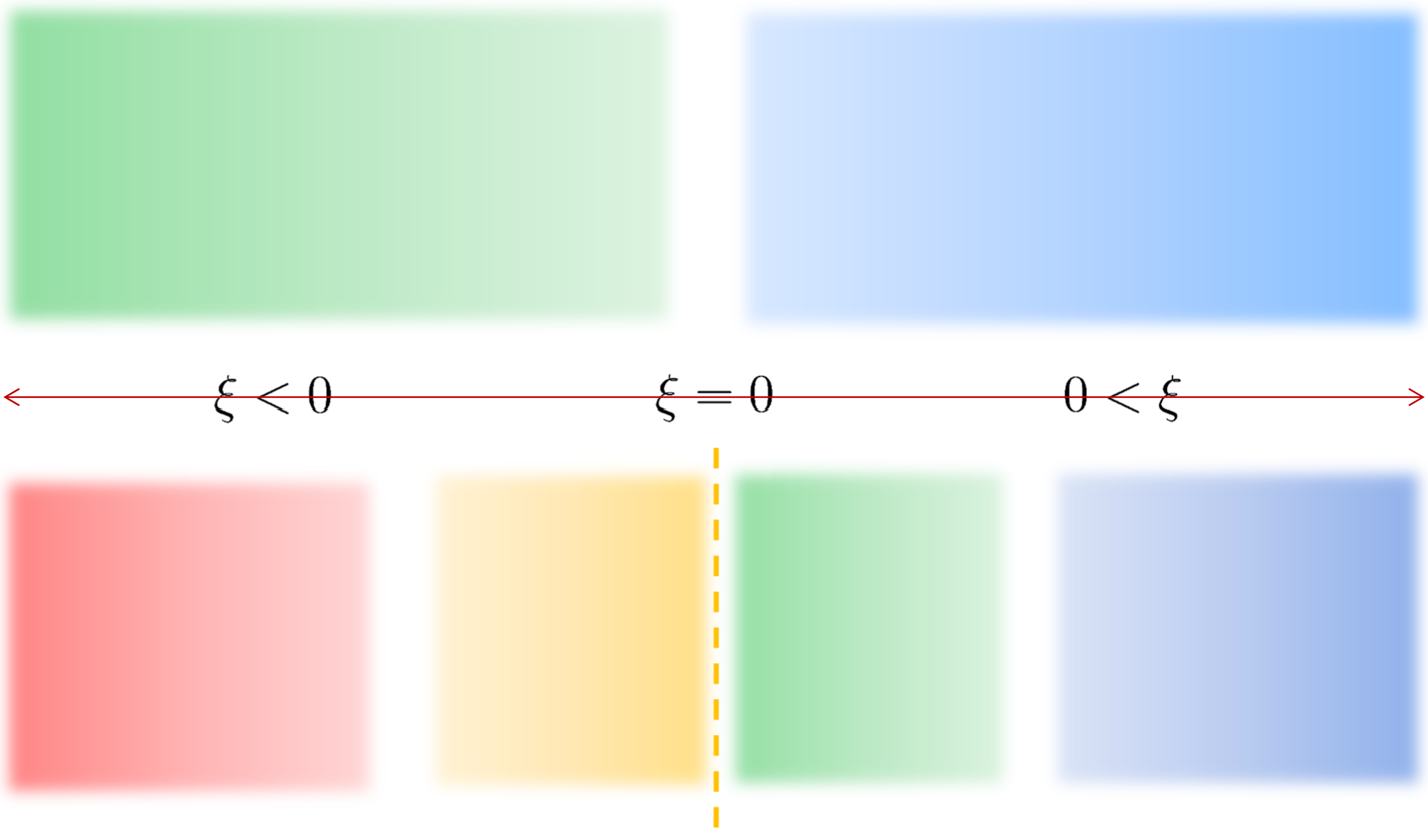
$-\xi \gg 1$

$1 \ll \xi$

$\xi = 0$

Benini + Eager + Hori + Tachikawa 2013

2d GLSM



1d GLSM

K.Hori + H.Kim + P.Y. 2014

so, whatever happened to Q-exactness of FI constant ?

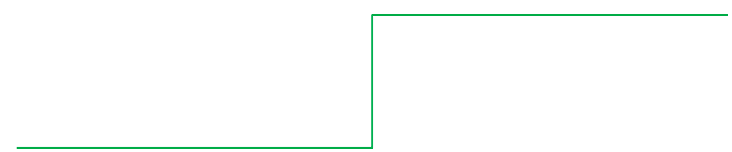
such a naïve invariance argument always assumes
“small” deformation of the parameters,
meaning, nothing drastic should happen asymptotically

however, vanishing FI constants always implies new
asymptotic runaway direction along vector multiplets,
invalidating Q-exactness across $\xi = 0$

Q-exactness → piece-wise constant behavior…

$$\int \frac{dD}{D} e^{-D^2/e^2 + i\xi D} \quad = \int \frac{dD'}{D'} e^{-\cancel{e^2 D'^2} + i(e^2\xi)D'} \quad \to \quad \int \frac{dD'}{D'} e^{i(e^2\xi)D'}$$

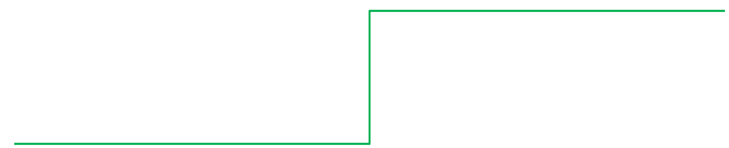
$$\to \quad \Theta(e^2\xi)$$



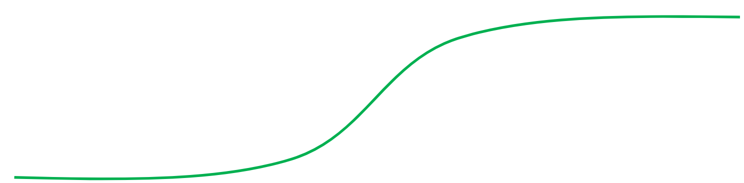
consistent with Q-exactness of the FI constant term

Q-exactness → piece-wise constant behavior…

$$\int \frac{dD}{D} e^{-D^2/e^2+i\xi D} = \int \frac{dD'}{D'} e^{-\cancel{e^2 D'}^2+i(e^2\xi)D'} \quad \rightarrow \quad \Theta(e^2\xi)$$



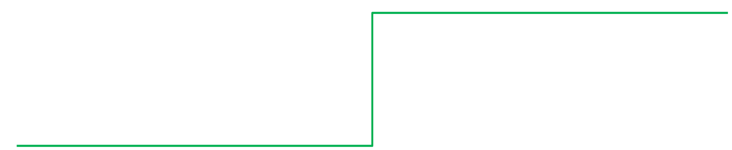
$$\int \frac{dD}{D} e^{-D^2/e^2+i\xi D} = \int \frac{dD''}{D''} e^{-D''^2+i(e\xi)D''} \quad \rightarrow \quad \Theta_{\mathrm{smooth}}(e\xi)$$



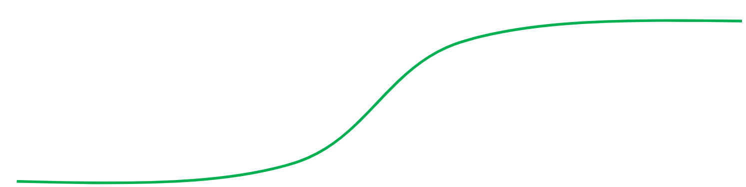
inconsistent with Q-exactness of the FI constant term

→ the localization **fails** unless $e^2\xi$ is kept nonzero

$$\int \frac{dD}{D} e^{-D^2/e^2 + i\xi D} = \int \frac{dD'}{D'} e^{-e^2 D'^2 + i(e^2\xi)D'} \quad \rightarrow \quad \Theta(e^2\xi)$$



$$\int \frac{dD}{D} e^{-D^2/e^2 + i\xi D} = \int \frac{dD''}{D''} e^{-D''^2 + i(e\xi)D''} \quad \rightarrow \quad \Theta_{\mathrm{smooth}}(e\xi)$$



inconsistent with Q-exactness of the FI constant term rescale to keep $e^2\xi$ finite,

then, after a long, long, long song and dance, ……

(Kentaro’s talk in the morning)

the Jeffrey-Kirwan residue tagged by FI constant

$$\mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2I}e^{-\beta Q^2}\right] = (2\pi i)^{-r}\int_{\partial\Delta^+_{e^2\xi}} d^r u\; g(u;0)$$

$$g^{(N=4)}(u;0) = \prod_A\left[\frac{1}{|W(G_A)|}\left(\frac{1}{2\sinh[z/2]}\right)^{r_A}\cdot\prod_{\alpha\in\Delta_A}\frac{\sinh[\alpha(u)/2]}{\sinh[(\alpha(u)-z)/2]}\right]$$

$$\times\prod_I\left[-\frac{\sinh[(Q_I(u)+(R_I/2-1)z)/2]}{\sinh[(Q_I(u)+R_I z/2)/2]}\right]$$

K.Hori + H.Kim + P.Y. 2014

cf) Cordova + Shao;
Hwang + Kim + Kim + Park

the Jeffrey-Kirwan residue tagged by FI constant

$$\mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2I}e^{-\beta Q^2}\right] = (2\pi i)^{-r}\int_{\partial\Delta^+_{e^2\xi}} d^r u\; g(u;0)$$

$$\partial M_u = \cancel{\partial M_\infty} + \cup_Q \partial\Delta^Q$$

$$\partial\Delta^+_{e^2\xi} = \cap_{Q^+}\partial\Delta^{Q^+}(\xi)$$

$$e^2\xi = \sum_{i=1}^{\mathrm{rank}} a_i Q_i^+ \qquad a_i > 0$$

K.Hori + H.Kim + P.Y. 2014

the Jeffrey-Kirwan residue tagged by FI constant

$$\mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2I}e^{-\beta Q^2}\right] \quad = \quad \text{JK-Res}_{e^2\xi:\{Q^+\}}\, g(u;0)$$

$$\text{JK-Res}_{\eta:\{Q_i\}} \frac{d^r u}{(Q_1\cdot u)(Q_2\cdot u)\cdots(Q_r\cdot u)}$$

$$= \left\{ \begin{array}{cc} \frac{1}{|\mathrm{Det}Q|} & \eta = \sum a_i^{>0} Q_i \\ \\ 0 & \text{otherwise} \end{array} \right\}$$

more examples

N=2 Grassmannian

chirals	$U(K)$	$U(N)_F$
X	K	N

$+\ q$ Wilson line

$\xi < 0$	$0 < \xi$
$\Omega = 0$	$\Omega = \begin{cases} \prod_{a=0}^{K-1} \dfrac{{}_{q+N/2-1-a}C_{N-K}}{{}_{N-K+a}C_a}\ , & q \geq \frac{N}{2} \\ 0\ , & -\frac{N}{2} < q < \frac{N}{2} \\ (-1)^{K(N-1)} \prod_{a=0}^{K-1} \dfrac{{}_{-q+N/2-1-a}C_{N-K}}{{}_{N-K+a}C_a} & q \leq -\frac{N}{2} \end{cases}$

N=4 Grassmannian

chirals	$U(K)$	$U(N)_F$
X	K	N

$\xi < 0$

$$\Omega = 0$$

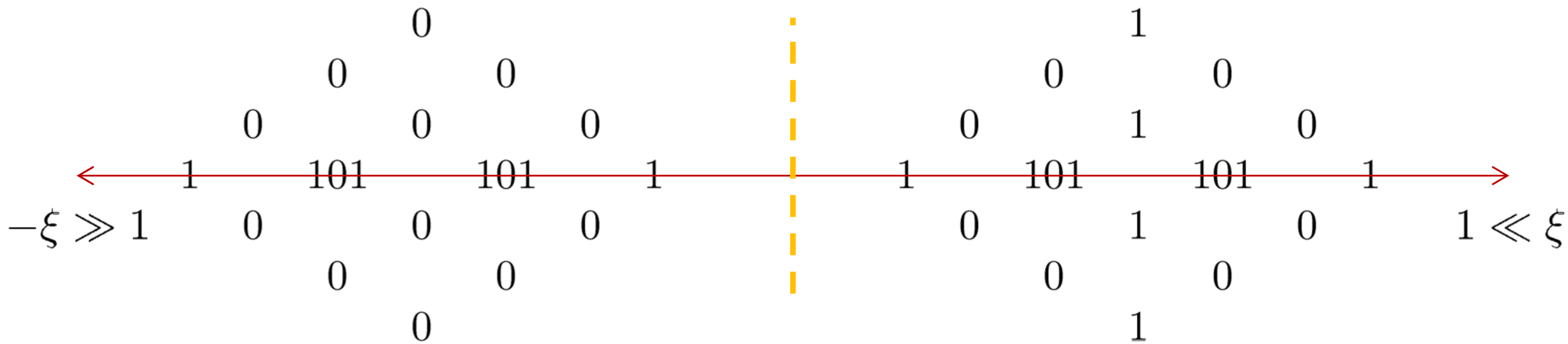
$0 < \xi$

$$\Omega = (-1)^{K(N-K)} \cdot \frac{\prod_{i=1}^{N}(\mathbf{y}^{-i} - \mathbf{y}^{i})}{\prod_{j=1}^{K}(\mathbf{y}^{-j} - \mathbf{y}^{j}) \prod_{l=1}^{N-K}(\mathbf{y}^{-l} - \mathbf{y}^{l})}$$

quintic CY3 hypersurface in CP4

$$G_5(X) = 0 \qquad\qquad \{X^{1,\dots,5}\}//U(1)$$

$$g_{quintic}(u;0) \quad = \quad \frac{1}{2\sinh[z/2]} \cdot \left[-\frac{\sinh[(u-z)/2]}{\sinh[u/2]}\right]^5 \cdot \left[-\frac{\sinh[(5u)/2]}{\sinh[(5u-z)/2]}\right]$$



Higgs, Coulomb, Landau-Ginzburg, Hybrid, …

N=4 rank 2 GLSM Q.M. for CY3 in WCP$_{(11222)}$

	P	$X_{1,2}$	$Y_{1,2,3}$	Z
$U(1)_1$	-4	0	1	1
$U(1)_2$	0	1	0	-2

hybrid

0
0 0
0 0 0
1 86 86 1
0 0 0
0 0
0

geometric

1
0 0
0 2 0
1 86 86 1
0 2 0
0 0
1

LG

0
0 0
0 0 0
1 83 83 1
0 0 0
0 0
0

orbifold

1
0 0
0 1 0
1 83 83 1
0 1 0
0 0
1

a noncompact N=4 GLSM Q.M.

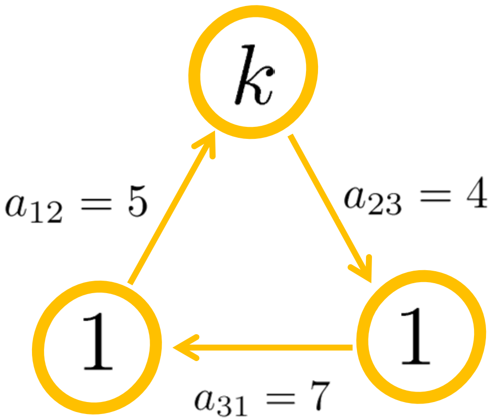
product of O(-1) line bundles over CP, gapped by twisted masses

chirals	$U(1)$	$[U(N)\times U(K)]_{\mathrm{F}}$
X	$+1$	$(N,1)$
Y	-1	$(1,K)$

$$\Omega^{\xi>0}-\Omega^{\xi<0}=(-1)^{N+K-1}\times\frac{\mathbf{y}^{N-K}-\mathbf{y}^{K-N}}{\mathbf{y}-\mathbf{y}^{-1}}$$

$$\Omega^{\xi<0}\Big|_{\mathbf{y}=1}=(-1)^{N+K-1}K \qquad \Omega^{\xi>0}\Big|_{\mathbf{y}=1}=(-1)^{N+K-1}N$$

N = 4 (2,1,1) triangle quiver

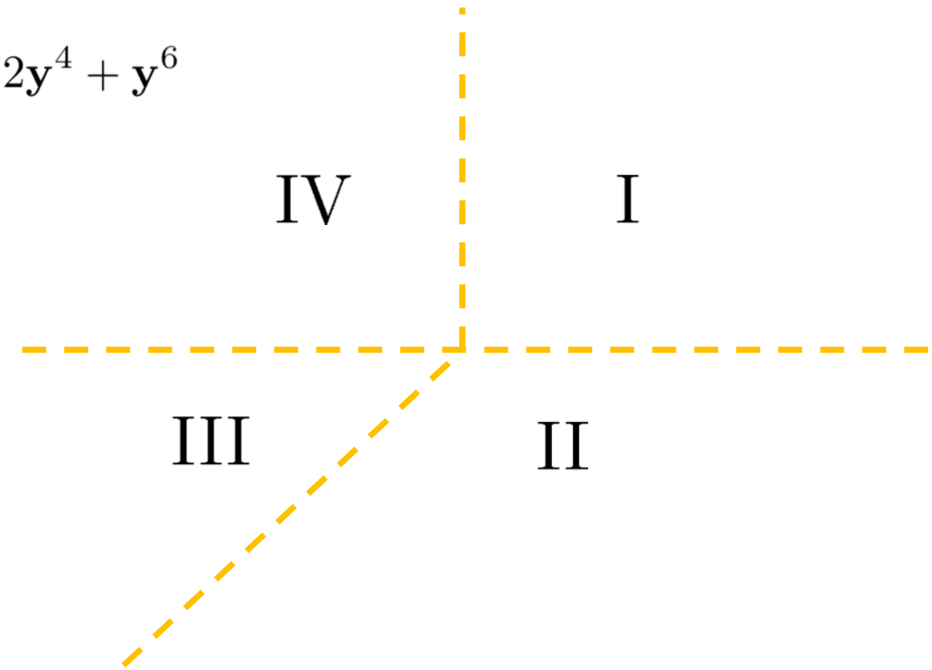


$$\Omega_{\rm I}^{(k=2)} = 50$$

$$\Omega_{\rm II}^{(k=2)} = \frac{1}{\mathbf{y}^4} + \frac{2}{\mathbf{y}^2} + 87 + 2\mathbf{y}^2 + \mathbf{y}^4$$

$$\Omega_{\rm III}^{(k=2)} = \frac{1}{\mathbf{y}^6} + \frac{2}{\mathbf{y}^4} + \frac{4}{\mathbf{y}^2} + 89 + 4\mathbf{y}^2 + 2\mathbf{y}^4 + \mathbf{y}^6$$

$$\Omega_{\rm IV}^{(k=2)} = \frac{1}{\mathbf{y}^6} + \frac{2}{\mathbf{y}^4} + \frac{4}{\mathbf{y}^2} + 54 + 4\mathbf{y}^2 + 2\mathbf{y}^4 + \mathbf{y}^6$$



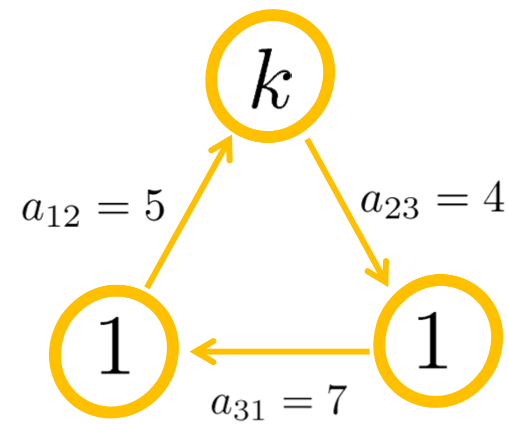
when combined with the augmented Coulombic index

$$\bar{\Omega}^{-}\,(\text{quiver}) \;\sim \sum_{\text{partition}} (-1)^{\#} \left[\int_{\mathcal{M}} ch \wedge \mathcal{A}\right] \times \frac{\prod_{\text{subquivers}} \bar{\Omega}_{\text{Inv}}}{|\Gamma|}$$

Manchot+Pioline-Sen 2013

we may infer the quiver invariants recursively

N = 4 (2,1,1) triangle quiver

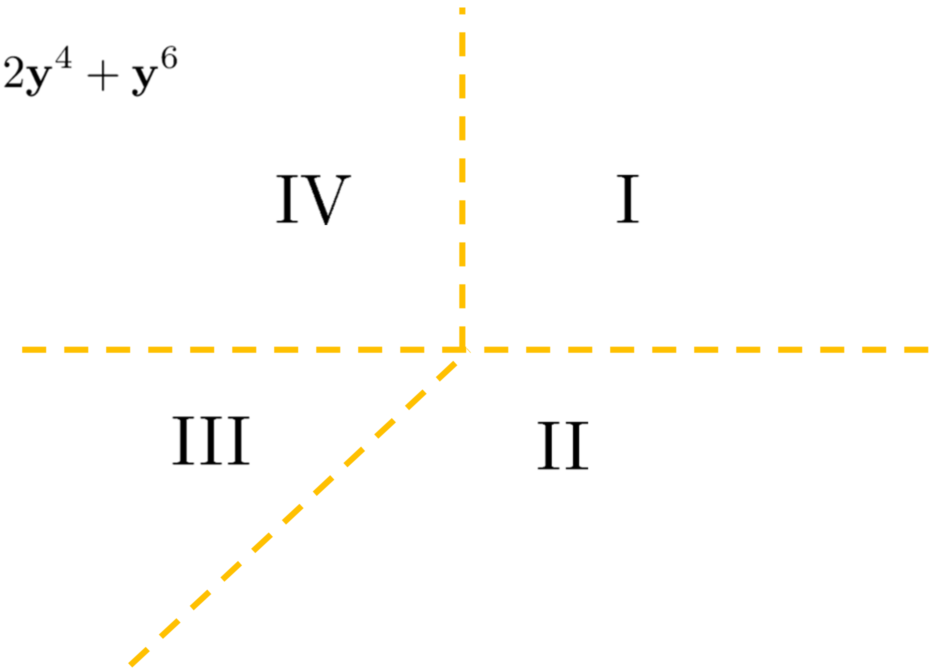


$$\Omega_{\mathrm{I}}^{(k=2)} = 50$$

$$\Omega_{\mathrm{II}}^{(k=2)} = \frac{1}{\mathbf{y}^4} + \frac{2}{\mathbf{y}^2} + 87 + 2\mathbf{y}^2 + \mathbf{y}^4$$

$$\Omega_{\mathrm{III}}^{(k=2)} = \frac{1}{\mathbf{y}^6} + \frac{2}{\mathbf{y}^4} + \frac{4}{\mathbf{y}^2} + 89 + 4\mathbf{y}^2 + 2\mathbf{y}^4 + \mathbf{y}^6$$

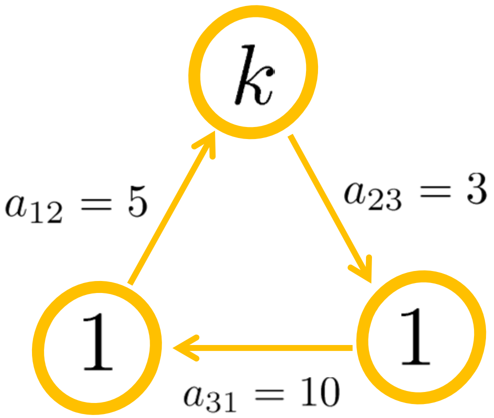
$$\Omega_{\mathrm{IV}}^{(k=2)} = \frac{1}{\mathbf{y}^6} + \frac{2}{\mathbf{y}^4} + \frac{4}{\mathbf{y}^2} + 54 + 4\mathbf{y}^2 + 2\mathbf{y}^4 + \mathbf{y}^6$$



$$\Omega^{(k=2)}\Big|_{\mathrm{Inv}} = 49$$

$$\Omega^{(k=1)}\Big|_{\mathrm{Inv}} = 34$$

N = 4 (3,1,1) triangle quiver

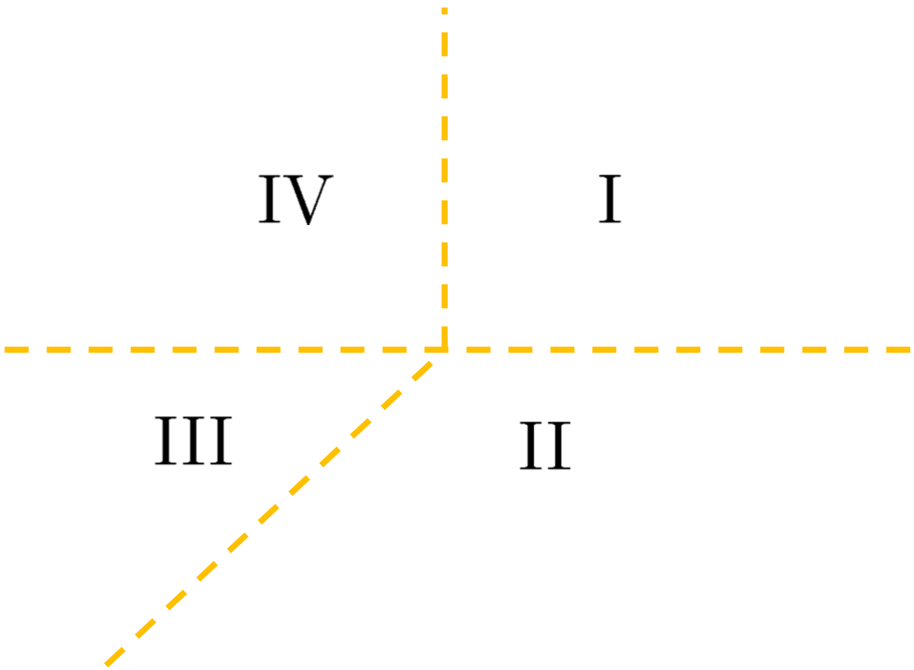


$$\Omega_{\mathrm{I}}^{(k=3)} = \frac{1}{\mathbf{y}^6} + \frac{2}{\mathbf{y}^4} - \frac{2}{\mathbf{y}^2} - 7 - 2\mathbf{y}^2 + 2\mathbf{y}^4 + \mathbf{y}^6$$

$$\Omega_{\mathrm{II}}^{(k=3)} = 0$$

$$\Omega_{\mathrm{III}}^{(k=3)} = \frac{1}{\mathbf{y}^4} + \frac{1}{\mathbf{y}^2} + 1 + \mathbf{y}^2 + \mathbf{y}^4$$

$$\Omega_{\mathrm{IV}}^{(k=3)} = \frac{1}{\mathbf{y}^4} - \frac{4}{\mathbf{y}^2} - 9 - 4\mathbf{y}^2 + \mathbf{y}^4$$



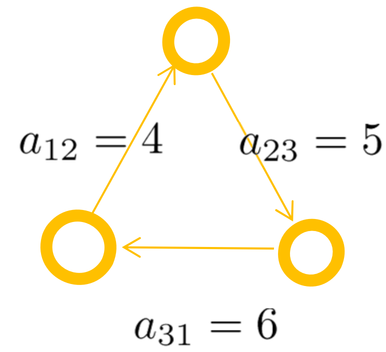
$$\Omega^{(k=2)}\Big|_{\mathrm{Inv}} = \frac{6}{\mathbf{y}} + 6\mathbf{y}$$

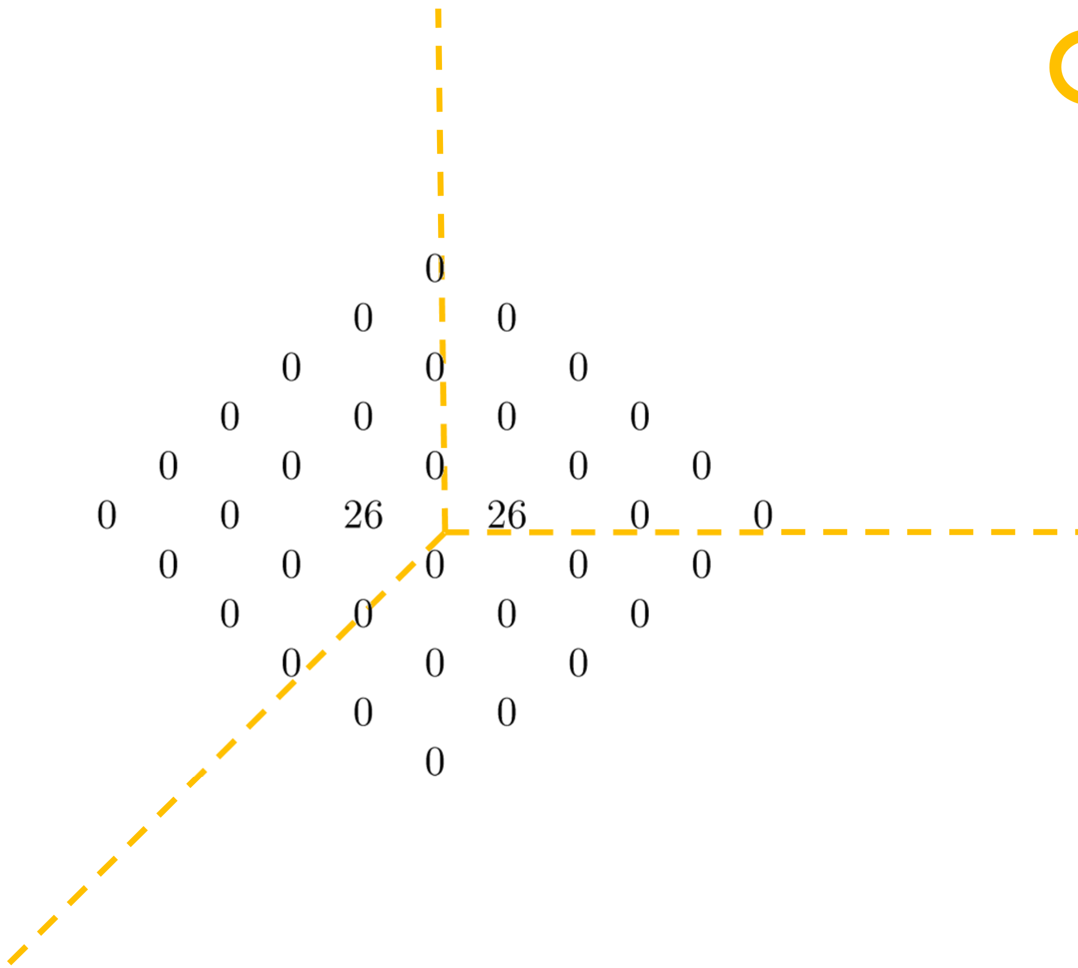
$$\Omega^{(k=3)}\Big|_{\mathrm{Inv}} = 0 = \Omega^{(k=1)}\Big|_{\mathrm{Inv}}$$

wall-crossing-safe invariants for general d=1 GLSM

$$\Omega\Big|_{\mathrm{Inv}} \equiv \mathrm{Tr}_{L^2}^{\xi=0}(-1)^F y^J$$

an Abelian triangle quiver





quintic CY3 hypersurface in CP4

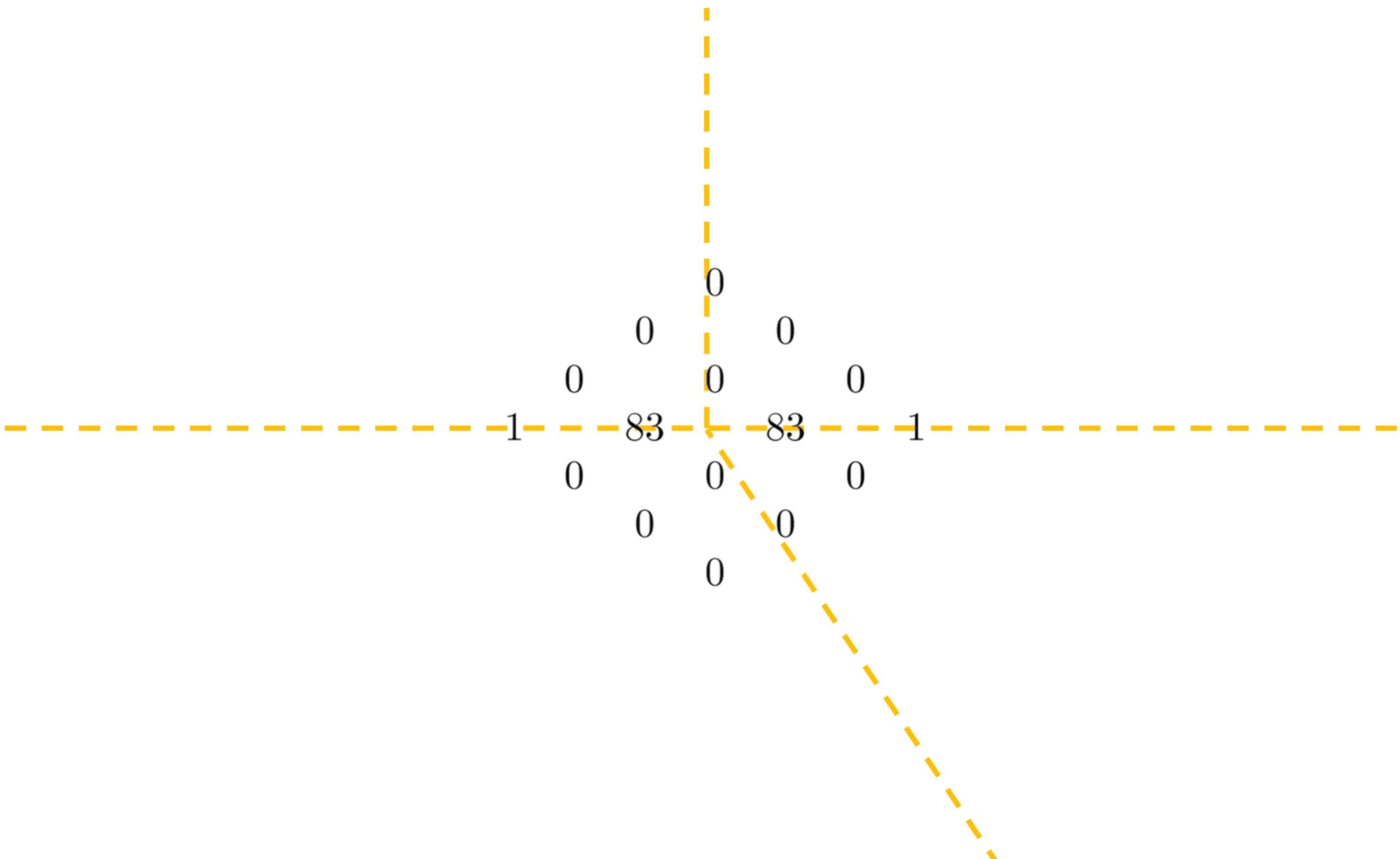
$G_5(X) = 0$ $\{X^{1,\ldots,5}\}//U(1)$

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 0 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{array}$$

$-\xi \gg 1$ $1 \ll \xi$

N=4 rank 2 GLSM Q.M. for CY3 in WCP$_{(11222)}$

	P	$X_{1,2}$	$Y_{1,2,3}$	Z
$U(1)_1$	-4	0	1	1
$U(1)_2$	0	1	0	-2



summary & outstanding questions

d=4 N=2 BPS States via d=1 N=4 Quiver GLSM

Wall-Crossing Coulomb/Multi-Center Index

Wall-Crossing-Safe Quiver Invariants

Index of d=1 N=4 GLSM: *Localization with Wall-Crossing*

Large Rank Limit?
d=4 N=2 BPS Black Hole Microstates?
Wall-Crossing-Safe Invariants directly from Path Integral?
……