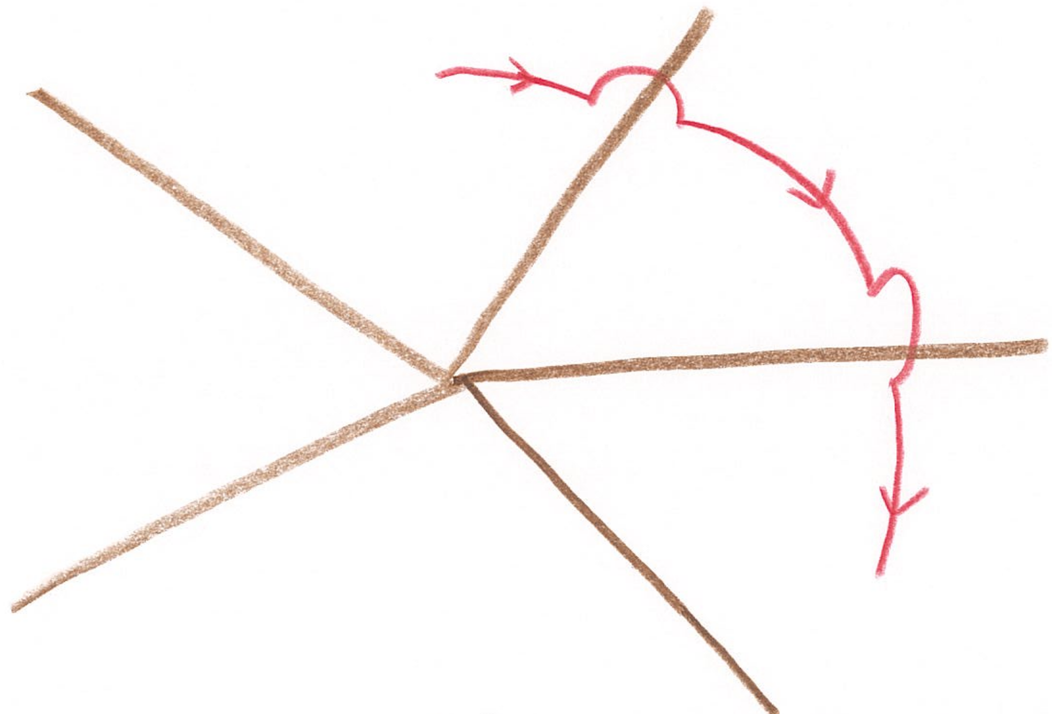
Witten Index & Wall Crossing



Kentaro Hori (IPMU)

Based on a joint work with

Heeyeon Kim (Seoul National Univ.) &

Piljin Yi (KIAS)

Background: a joint work on 2d index (elliptic genus)

with

Francesco Benini (Simons Center),

Richard Eager (IPMU) &

Yuji Tachikawa (Univ. Tokyo)

Related works :

- Chiung Hwang, Joonho Kim, Seok Kim, Jaemo Park
  (POSTECH & Seoul Nat'l U.)

- Clay Cordova & Shu-Heng Shao
  (Harvard Univ.)

Motivation

Counting BPS states in

- String Theory
- Field Theory

THE MODELS

1d gauge theory with

$\mathcal{N}=2, 4, \cdots$ Supersymmetry

vector

chiral

fermi

Wilson

vector	G	Compact Lie group
chiral	V_c	$\mathbb{C}$- representation (f.d.)
fermi	V_f	$\mathbb{C}$- representation (f.d.)
Wilson	M	$\mathbb{Z}_2$-graded $\mathbb{C}$- "representation" (f.d.)

vector $v_t,\ \sigma,\ \lambda,\ D$

chiral $\phi,\ \psi$

fermi $\eta,\ F$

Wilson

vector

$$\begin{cases} \delta v_t = -\delta\sigma = \frac{i}{2}\epsilon\bar{\lambda} + \frac{i}{2}\bar{\epsilon}\lambda \\ \delta\lambda = \epsilon\left(D_t\sigma + iD\right) \\ \delta D = \frac{1}{2}\epsilon D_t^{(+)}\bar{\lambda} - \frac{1}{2}\bar{\epsilon} D_t^{(+)}\lambda \end{cases}$$

chiral

fermi

Wilson

$$D_t^{(\pm)} \mathcal{O} := \underbrace{D_t \mathcal{O}}_{\text{usual}} \pm i \sigma \mathcal{O}$$

$$\partial_t \mathcal{O} + i v_t \mathcal{O}$$

vector

chiral

$$\begin{cases} \delta\phi = -\epsilon\psi \\ \delta\psi = i\bar{\epsilon} D_t^{(+)}\phi \end{cases}$$

fermi

Wilson

vector

chiral

fermi

Wilson

$$E : V_c \longrightarrow V_f$$

G-equivariant holomorphic

$$\begin{cases} \delta \eta = \epsilon F + \bar{\epsilon} E(\phi) \\ \delta F = \bar{\epsilon}\left(-i D_t^{(+)} \eta + \psi^i \partial_i E(\phi)\right) \end{cases}$$

vector

chiral

fermi

Wilson

$$\mathcal{L}_g = \frac{1}{2e^2} \mathrm{Tr}\left[(D_t \sigma)^2 + i \bar{\lambda} D_t^{(+)} \lambda + D^2 \right]$$

$$\mathcal{L}_{FI} = - \zeta (D)$$

$$\zeta \in (\mathfrak{g}^*)^{G} \cong (\mathfrak{t}^*)^{W}$$

Fayet-Iliopoulos parameter
(F.I.)

vector

chiral

$$L_c = D_t\bar{\phi} D_t\phi + i\bar{\psi} D_t^{(-)}\psi - \bar{\phi}(\sigma^2 - D)\phi - i\bar{\phi}\lambda\psi + i\bar{\psi}\bar{\lambda}\phi$$

fermi

Wilson

vector

chiral

fermi

$$\mathcal{L}_f = i\bar{\eta} D_t^{(+)} \eta + \bar{F}F - \overline{E(\phi)} E(\phi) - \bar{\eta}\, \partial_i E(\phi) \psi^i - \bar{\psi}^i \partial_i \overline{E(\phi)}\, \eta$$

Wilson

vector

$$J : V_c \longrightarrow V_f^*$$

G-equiv. holomorphic

chiral

$$J(\phi)\,E(\phi) = 0$$

fermi

$$\mathcal{L}_J = \Psi^i \partial_i J(\phi)\,\eta - J(\phi)\,F$$

$$+ \text{ c.c.}$$

Wilson

Superpotential term $W = J(\phi)\,\eta$

vector

chiral

fermi

Wilson

$$Q : V_c \to \mathrm{End}^{\mathrm{od}}(M)$$

G-equiv. holomorphic

$$Q(\phi)^2 = 0$$

$$\mathrm{Pexp}_M\left(-i\int \mathcal{A}_t \, dt\right)$$

$$\mathcal{A}_t = v_t + \sigma - \psi^i \partial_i Q(\phi) + \bar{\psi}^{\bar{i}} \partial_{\bar{i}} Q(\phi)^\dagger$$

$$+ \{Q(\phi), Q(\phi)^\dagger\}$$

Anomaly free condition :

$$\det(V_c)^{\frac{1}{2}} \otimes \det(V_f)^{\frac{1}{2}} \otimes M$$

must be a representation of G.

$\mathcal{N}=4$: G, V, $W(\phi)$

<u>vect</u> $(v_t, \sigma_3, \lambda_-, D)$

<u>chiral</u> $\mathfrak{g}_{\mathbb{C}} \oplus V \ni (\sigma_1 + i\sigma_2, i\bar{\lambda}_+), (\phi, \psi_+)$

<u>fermi</u> $V \ni (\psi_-, F)$

$$E = (\sigma_1 + i\sigma_2)\phi, \quad J(\phi) = -dW(\phi)$$

<u>Wilson</u> $\mathbb{C}$ (trivial)

FLAVOR SYMMETRY

compact $H \circlearrowright V_c, V_f, M$

- Commute with G
- E, J, Q H-equiv

$\rightsquigarrow$ Symmetry commuting with Supercharges

$\rightsquigarrow$ deformation by background $(v_t^H, \underset{\substack{\| \\ m}}{\sigma^H})$ $\leftarrow$ "real mass"

$$\boxed{(\partial_{v_t^H} - \partial_m) L = Q\text{-exact}}$$

SUSY correlator indep. of $v_t^H - m$

$\mathcal{N}=4$ system (G, V, W) with $W(\omega^R \phi) = \omega^2 W(\phi)$

has 2 $U(1)$ R-symmetries. One of them

	$\sigma_1 + i\sigma_2$	$\bar{\lambda}_+$	ϕ	ψ_+	ψ_-	F
J_-	-1	-1	$\frac{R}{2}$	$\frac{R}{2}$	$\frac{R}{2}-1$	$\frac{R}{2}-1$

is an $\mathcal{N}=2$ flavor symmetry

<u>NB</u> There may exist $\mathcal{N}=4$ flavor symmetry.

But, when the theory is compact (discrete spectrum),

the SUSY ground states have flavor charge zero!

THE INDEX

$\xi, m \in \mathfrak{h}$

Consider $\mathrm{Tr}_{\mathcal{H}_m}\left((-1)^F e^{iG^H_m(\xi)} e^{-\beta H_m}\right)$

in the theory deformed by real mass m.

$\longrightarrow \quad v^H_t = i v^H_\tau = -i\xi/\beta$

depends on $-i\xi/\beta + m$ but not on $-i\xi/\beta - m$.

I.e. holomorphic in $\xi + i\beta m$

$$\overset{\text{compact}}{\overset{\downarrow}{=}} \quad \mathrm{Tr}_{\mathcal{H}}\left((-1)^F e^{iG^H(\xi + i\beta m)} e^{-\beta H}\right)$$

in undeformed theory.

In a compact $\mathcal{N}=4$ theory, $y \in \mathbb{C}^*$

$$I(y) = \mathrm{Tr}_{\mathcal{H}}\left((-1)^F y^{J_-} e^{-\beta H}\right)$$

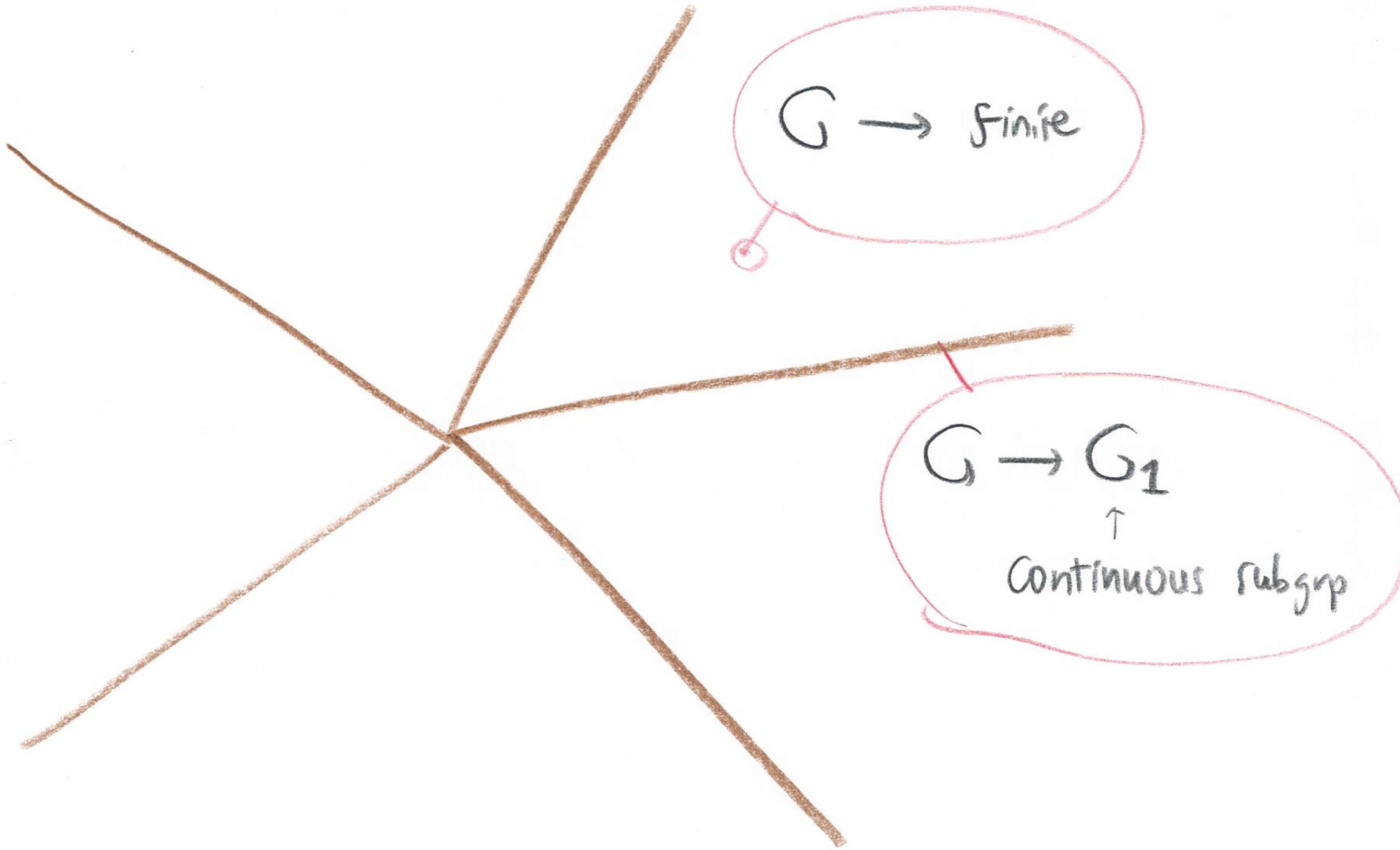
- No adisrional flavor twist

$=$ No ambiguity in choice of R.

In a non-compact $\mathcal{N}=4$ theory, we "can" have a flavor twist (rather, it is necessary).

PHASES

The F.I. parameter space $(t^*)^W$:



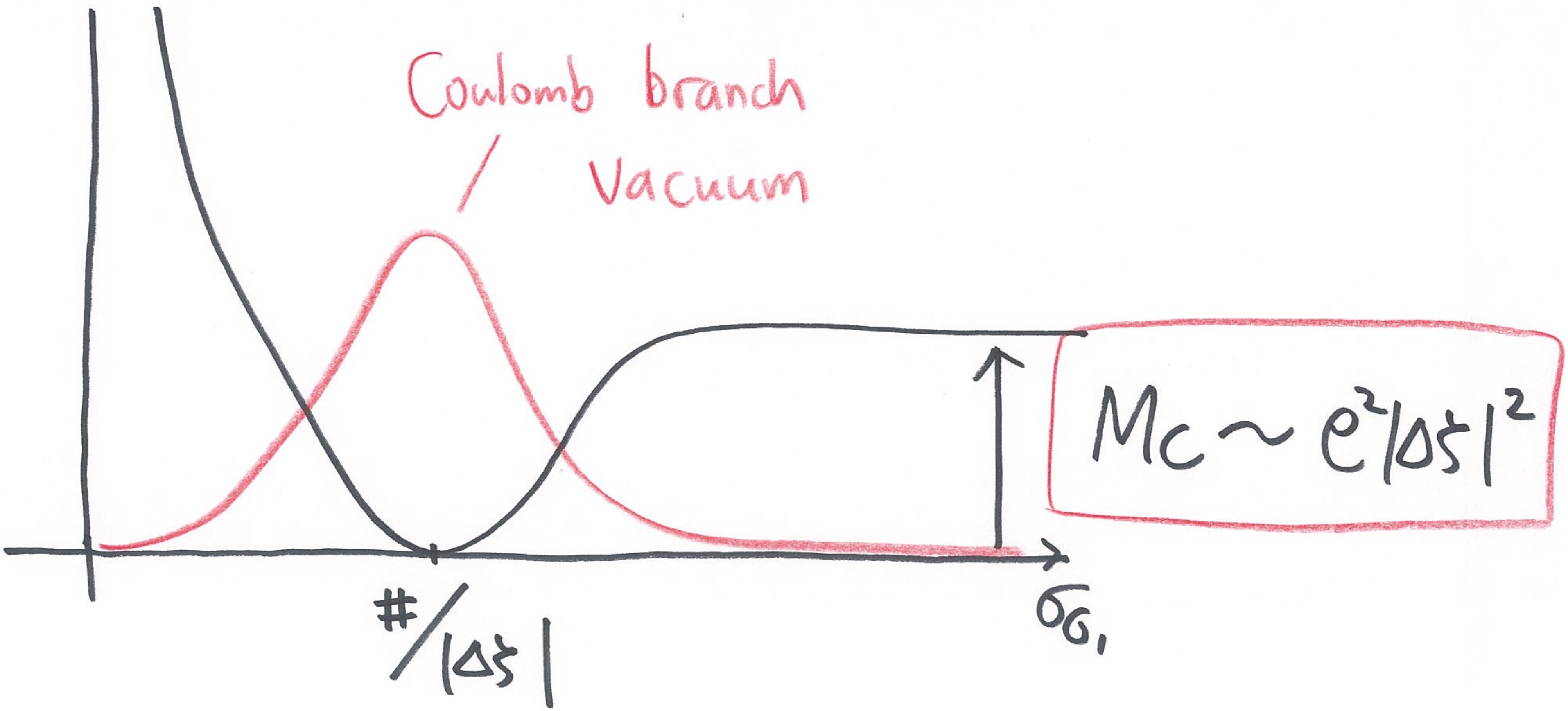
Inside a phase, vector has "mass"

$$\boxed{M_H \sim e\sqrt{|\xi|}}$$

$\dashrightarrow$ Effective theory on Higgs branch

e.g.
- Non-linear σ-model
- Landau-Ginzburg model / orbifold
- Hybrid

Near a phase bdry, G_1-vector is "light"



As $\zeta \to$ phase bdry ($\Delta\zeta \to 0$), it runs away to ∞!

.... WALL CROSSING.

Denef 2002
(for $\mathcal{N}=4$)

COMPUTATION

SUSY configuration

$$D_\tau \sigma = 0 , \quad D = 0 \qquad \text{for} \quad \underline{\text{vector}}$$

$$\rightsquigarrow \quad \mathcal{M} = \{ (g, \sigma) \in G \times \mathfrak{g} \mid g^{-1} \sigma g = \sigma \} / G$$

$$\cong (T \times \mathfrak{t}) / W$$

$$\boxed{u := \frac{\beta}{2\pi} (-v_\tau^0 + i \sigma^0) \in \mathfrak{t}_{\mathbb{C}}}$$

$$z := \frac{\beta}{2\pi} (-v_\tau^H + i m) \in \mathfrak{t}_{\mathbb{C}}^H$$

$\exists\, \phi$ - zero mode when $e^{2\pi i(u+z)} \circlearrowright V_c$ has e.v. $=1$.

$\longrightarrow$ union of hyperplanes $\underline{\widetilde{\mathcal{M}}_{\text{sing}}} \subset \widetilde{\mathcal{M}} = \underset{\substack{\| \\ \{u\}/Q^\vee}}{T} \times t$

Under suitable assumption $\qquad (\ell := \operatorname{rank} G)$

$$I = \frac{1}{|W|} \lim_{\substack{e \to 0 \\ \varepsilon \to 0}} \int_{\widetilde{\mathcal{M}} \setminus \Delta_\varepsilon} d^{2\ell} u \; Z_e(u) \Big|_{\text{matter 1-loop}}$$

$\underline{\Delta_\varepsilon \subset \widetilde{\mathcal{M}} \quad \varepsilon\text{-nhd of } \widetilde{\mathcal{M}}_{\text{sing}} \subset \widetilde{\mathcal{M}}}$

$u, \bar{u}$ not SUSY-closed, but is a part of a

supermultiplet $(u, \bar{u}, \lambda_0, \bar{\lambda}_0, D_0)$ $\mathfrak{t}_{(\mathbb{C})}$-valued constant modes

$$Z'_{\text{vector}} = \det{}' D_\tau^{(+)}$$

$$Z_{\text{chiral}} = \frac{\det \tilde{D}_\tau^{(-)}}{\det\left(-\tilde{D}_\tau^2 + \tilde{\sigma}^2 - iD_0\right)} \left\langle \frac{1}{(\ell!)^2} \left(\int_0^\beta \bar{\phi} \lambda_0 \psi \, d\tau \int_0^\beta \bar{\psi} \bar{\lambda}_0 \phi \, d\tau \right)^\ell \right\rangle$$

$$Z_{\text{fermi}} = \det \tilde{D}_\tau^{(+)}$$

$$Z_{\text{Wilson}} = \operatorname{Str}_M e^{2\pi i (u+z)}$$

$$\xrightarrow{\int d^{2\ell}\chi_0} I =$$

$$= \frac{1}{|W|} \lim_{\substack{e\to 0\\ \varepsilon\to 0}} \int_{\tilde{M}\setminus\Delta_\varepsilon} d^{2\ell}u \int_t d^\ell D_0 \; g(u, D_0) \det h(u, D_0)$$

$$\cdot \exp\left(-\frac{\beta}{2e^2} D_0^2 - i\beta\zeta(D_0)\right)$$

$$\cdot \frac{\partial}{\partial \bar{u}_a} g(u, D_0) = -i h^{ab}(u, D_0)(D_0)_b \, g(u, D_0)$$

$$\frac{\partial}{\partial \bar{u}_c} h^{ab}(u, D_0) = (a, b, c) \text{ symmetric.}$$

The case $G = U(1)$ $\quad l=1 \quad \widetilde{\mathcal{M}} = \mathcal{M} = \mathbb{C}/\mathbb{Z} \quad D_0 \to D$

$$I = \lim \int_{\mathcal{M}\setminus\Delta_\varepsilon} d^2u \int_{\Gamma} dD \, \frac{i}{D} \frac{\partial}{\partial \bar{u}} g(u,D) \exp\left(-\frac{\beta}{2e^2} D^2 - i\beta\zeta D\right)$$

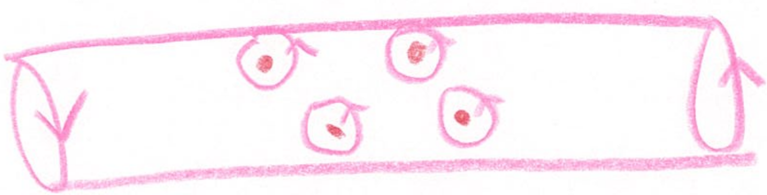
deform in advance

$$= \frac{1}{2} \lim \int_{\Gamma} dD \oint_{\partial(\mathcal{M}\setminus\Delta_\varepsilon)} du \, \frac{1}{D} g(u,D) \exp\left(-\frac{\beta}{2e^2} D^2 - i\beta\zeta D\right)$$

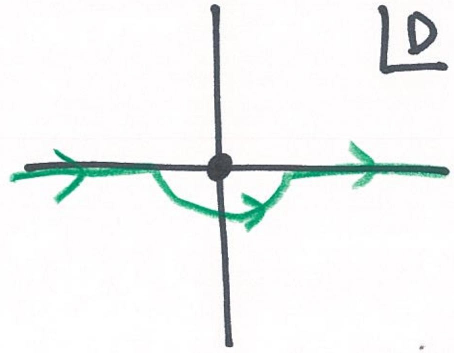
$$\Delta_\varepsilon = \Delta_\varepsilon^{(+)} \sqcup \Delta_\varepsilon^{(-)} \qquad \left(\pm \ \begin{matrix}\text{pos}\\ \text{neg}\end{matrix} \text{ charged fields}\right)$$

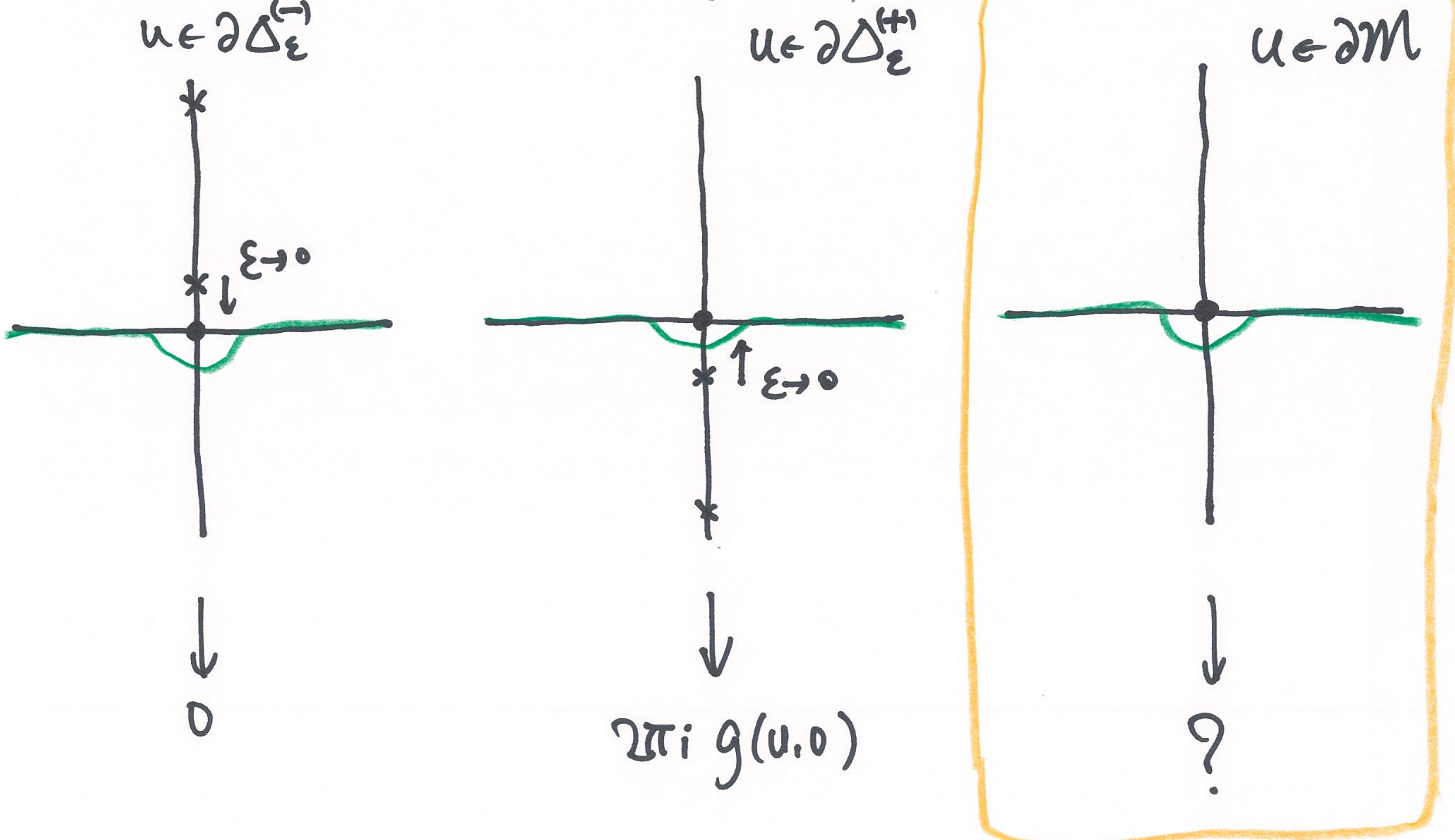
$$\partial(\mathcal{M}\setminus\Delta_\varepsilon) = -\partial\Delta_\varepsilon^{(-)} - \partial\Delta_\varepsilon^{(+)} + \partial\mathcal{M}$$

at infinity



Take, say, $\Gamma = \Gamma_-$





Higgs Scaling

hold $\zeta' = \beta^3 e^2 \zeta$ fixed as $e \to 0$

$$\Big[\ \underline{M_H \beta \text{ fixed}},\ \ \underline{M_C \beta \to \infty}\ \Big]$$

$$\int_{\Gamma_-} \frac{dD}{D}\, g(u,D) \exp\left(-\frac{\beta}{2e^2} D^2 - i\beta\zeta D\right)$$

$D = \beta e^2 D'$

$$= \int_{\Gamma_-'} \frac{dD'}{D'}\, g(u, \beta e^2 D') \exp\left(-\frac{\beta^3 e^2}{2} (D')^2 - i\zeta' D'\right)$$

$$\xrightarrow{e \to 0} \int_{\Gamma_-'} \frac{dD'}{D'}\, g(u,0) \exp\left(-i\zeta' D'\right) = \begin{cases} 0 & \zeta' > 0 \\ 2\pi i\, g(u,0) & \zeta' < 0 \end{cases}$$

Coulomb scaling

hold $\zeta'' = \sqrt{\beta}\, e\zeta$ fixed as $e \to 0$

$$\left[\ M_H \beta \to 0 , \quad M_c \beta \ \text{fixed} \ \right]$$

$$\int_{\Gamma_-} \frac{dD}{D}\, g(u,D) \exp\left(-\frac{\beta}{2e^2} D^2 - i\beta\zeta D \right)$$

$D = \frac{e}{\sqrt{\beta}} D''$

$$= \int_{\Gamma''_-} \frac{dD''}{D''}\, g\Big(u, \underbrace{\frac{e}{\sqrt{\beta}} D''}_{\substack{\downarrow e\to 0 \\ 0}}\Big) \exp\left(-\frac{1}{2}(D'')^2 - i\zeta'' D'' \right)$$

$\cdots$ continuous fcn of ζ'' $\longrightarrow \begin{cases} 0 & \zeta'' \to +\infty \\ 2\pi i\, g(u,0) & \zeta'' \to -\infty \end{cases}$

In Higgs scaling

$$I = \frac{1}{2}\left[\ 0 \quad - 2\pi i \oint_{\partial\Delta_\varepsilon^{(+)}} du\, g(u,0) + \boxed{\begin{cases} 0 & \zeta' > 0 \\ 2\pi i \oint_{\partial M} du\, g(u,0) & \zeta' < 0 \end{cases}}\ \right]$$

$$= \begin{cases} -\pi i \oint_{\partial\Delta_\varepsilon^{(+)}} du\, g(u,0) & \zeta' > 0 \\ \pi i \oint_{\partial\Delta_\varepsilon^{(-)}} du\, g(u,0) & \zeta' < 0 \end{cases}$$

general G

$$I = \mathrm{JKres}_{\zeta} \left(g \, d^l u \right) \cdot \frac{1}{|W|}$$

$$g = \prod_{\alpha} \left(x^{\frac{\alpha}{2}} - x^{-\frac{\alpha}{2}} \right) \frac{\prod_j \left(x^{-\frac{q_j}{2}} y^{-\frac{q_j^H}{2}} - x^{\frac{q_j}{2}} y^{\frac{q_j^H}{2}} \right)}{\prod_i \left(x^{\frac{Q_i}{2}} y^{\frac{Q_i^H}{2}} - x^{-\frac{Q_i}{2}} y^{-\frac{Q_i^H}{2}} \right)}$$

$$\cdot \sum_k (-1)^{r_k} x^{q_k} y^{q_k^H}$$

$$V_c \big|_{T \times T^H} = \bigoplus_i \mathbb{C}(Q_i, Q_i^H)$$

$$V_f \big|_{T \times T^H} = \bigoplus_j \mathbb{C}(q_j, q_j^H)$$

$$M \big|_{T \times T^H} = \bigoplus_k \mathbb{C}(q_k, q_k^H)[r_k]$$

$\mathbb{Z}_2$ grading

$$x = e^{2\pi i u}$$

$$y = e^{2\pi i z}$$

$\mathcal{N}=4$ system (G, V, W)

$$V|_{T\times U(1)_V} = \bigoplus_i \mathbb{C}(Q_i, R_i)$$

$$I(y) = \mathrm{Tr}_{\mathcal{H}} (-1)^F y^{J_-} e^{-\beta H}$$

$$= \frac{1}{|W|} \text{JK-res}_{\zeta} \left(g \, d^l u \right) \quad ;$$

$$g = \left(\frac{1}{y^{\frac{1}{2}} - y^{-\frac{1}{2}}} \right)^l \prod_{\alpha} \frac{x^{-\frac{\alpha}{2}} - x^{\frac{\alpha}{2}}}{x^{\frac{\alpha}{2}} y^{-\frac{1}{2}} - x^{-\frac{\alpha}{2}} y^{\frac{1}{2}}} \prod_i \frac{x^{-\frac{Q_i}{2}} y^{-\left(\frac{R_i}{4} - \frac{1}{2}\right)} - x^{\frac{Q_i}{2}} y^{\frac{R_i}{4} - \frac{1}{2}}}{x^{\frac{Q_i}{2}} y^{\frac{R_i}{4}} - x^{-\frac{Q_i}{2}} y^{-\frac{R_i}{4}}}$$

$$\text{JK-res}_{\zeta}(\omega) = \sum_{p} \text{JK-Res}(\underbrace{Q(p)}, \zeta)[\omega]$$

isolated intersection of singular hyperplanes

The set of charges defining the hyperplanes meeting at p

$$\text{JK-Res}(Q(p), \zeta)\left[\frac{d^l u}{Q_1(u)\cdots Q_l(u)}\right]$$

$Q_1, \cdots, Q_l \in Q(p)$ linearly indep

$$= \begin{cases} \dfrac{1}{|\det(Q_1 \cdots Q_l)|} & \text{if } \zeta \in \text{Cone}(Q_1, \cdots, Q_l) \\ 0 & \text{else} \end{cases}$$

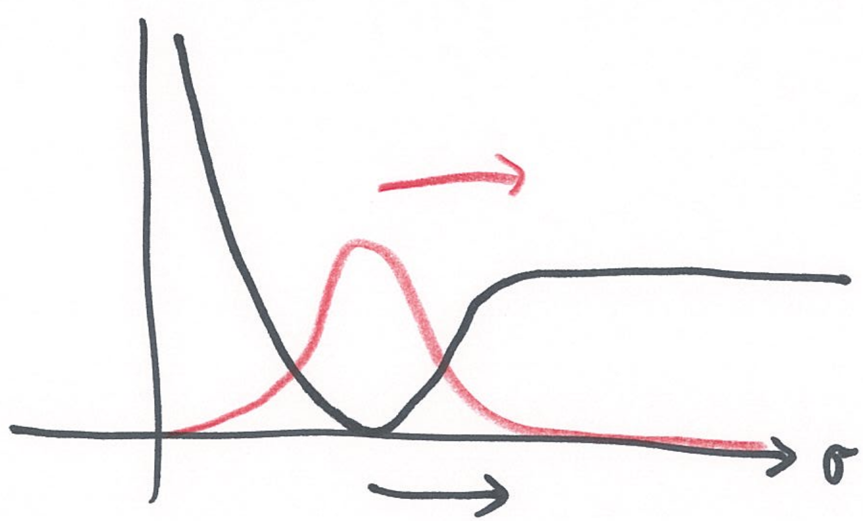
A WALL CROSSING FORMULA

U(1) theory :

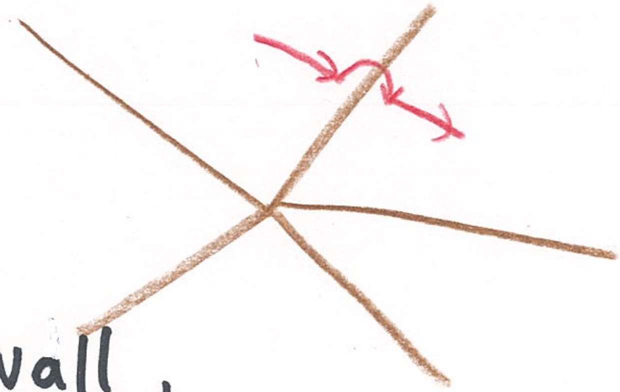
$$\Delta I := I\big|_{\zeta \ll 0} - I\big|_{\zeta \gg 0}$$

$$= \frac{1}{2\pi i}\left[\oint_0 - \oint_\infty\right]\frac{dx}{x}\, g(x,y)$$

… This agrees with the Coulomb branch analysis



general G :



Flat directions at the wall, responsible for the wall crossing, are very complicated in general :

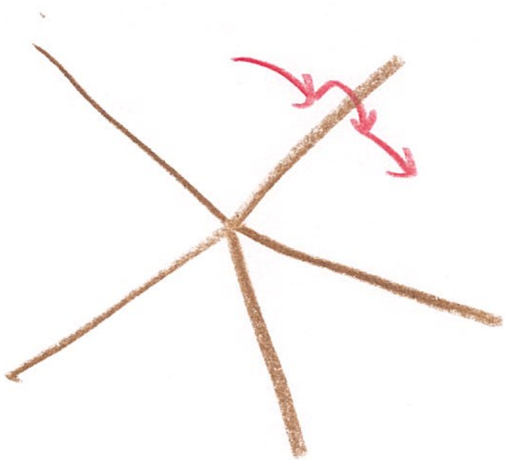
$$\bigcup_i \underline{\mathrm{Higgs}^{(i)}} \rtimes \underline{\mathrm{Coulomb}^{(i)}}$$

"union" of several mixed branches

A Simple wall : $\exists!$ mixed branch

$\text{Higgs}^{(1)} \rtimes \text{Coulomb}^{(1)}$ with $G_1 \cong U(1)$.

(e.g. Any wall is simple if G is Abelian)



Then

$$\Delta I = I_{\text{Higgs}^{(1)}} * \Delta I_{\text{Coulomb}^{(1)}}$$

the one for $G_1 = U(1)$ theory.

$\mathcal{N}=4$ system

Normalizable SUSY ground states have trivial flavor charge

$$\Rightarrow \quad \Delta I = I_{\text{Higgs}^{(1)}} \cdot \Delta I_{\text{Coulomb}^{(1)}} \quad \text{on a simple wall}$$

∘ Also, in $U(1)$ theory, the W.C. states form an $SU(2)$ repr. of spin $j = \frac{1}{2}\left(|N_{eff}^{(4)}| - 1\right)$ where

$$N_{eff}^{(4)} := \#(\text{positive charge chirals}) - \#(\text{negative charge chirals})$$

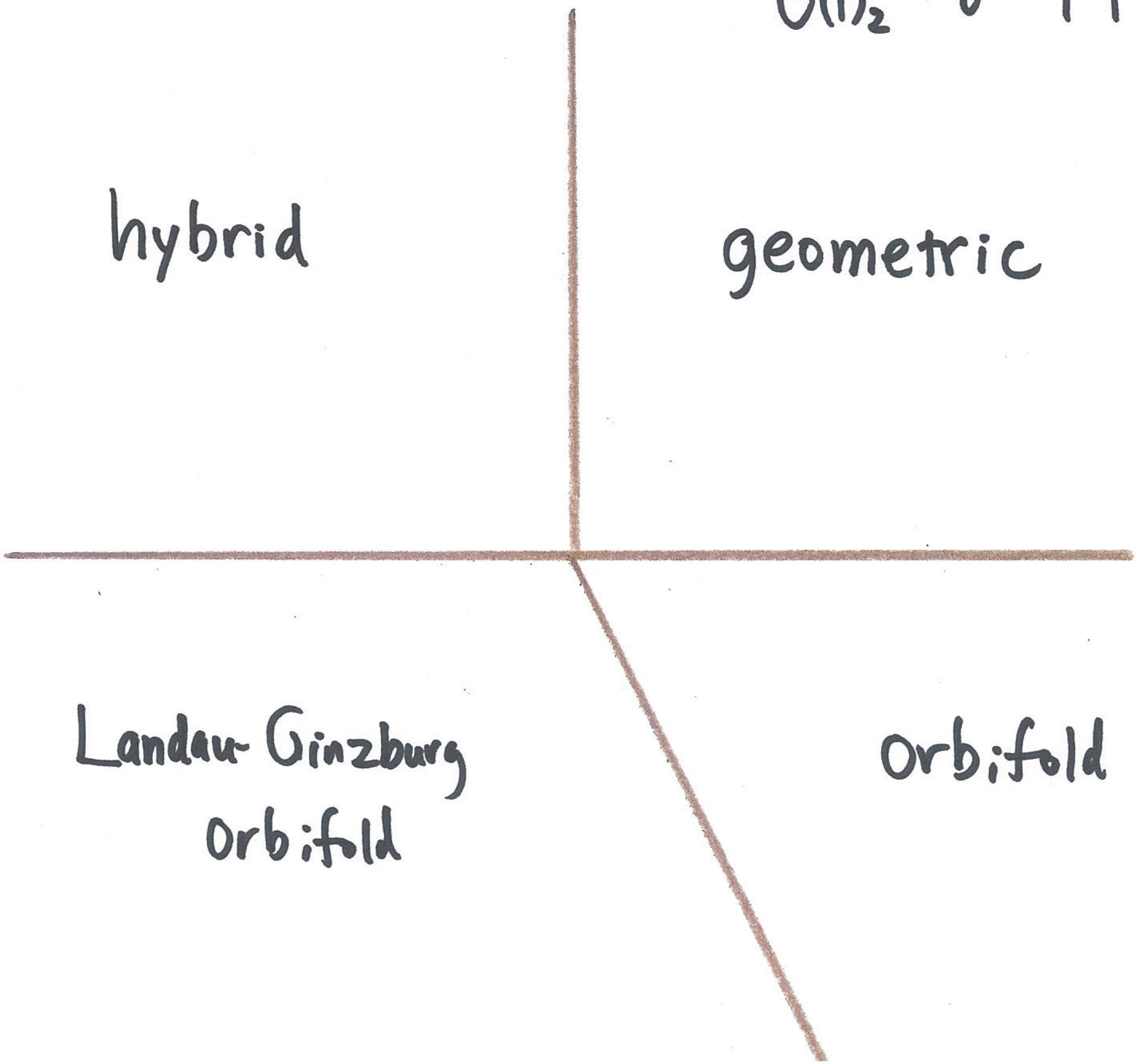
$$\therefore \Delta I = (-1)^{N_{eff}^{(4)}} \cdot \operatorname{sgn}(N_{eff}^{(4)}) \cdot \left(y^{-\frac{|N_{eff}^{(4)}|-1}{2}} + \cdots + y^{\frac{|N_{eff}^{(4)}|-1}{2}} \right)$$

e.g. "The two parameter model"

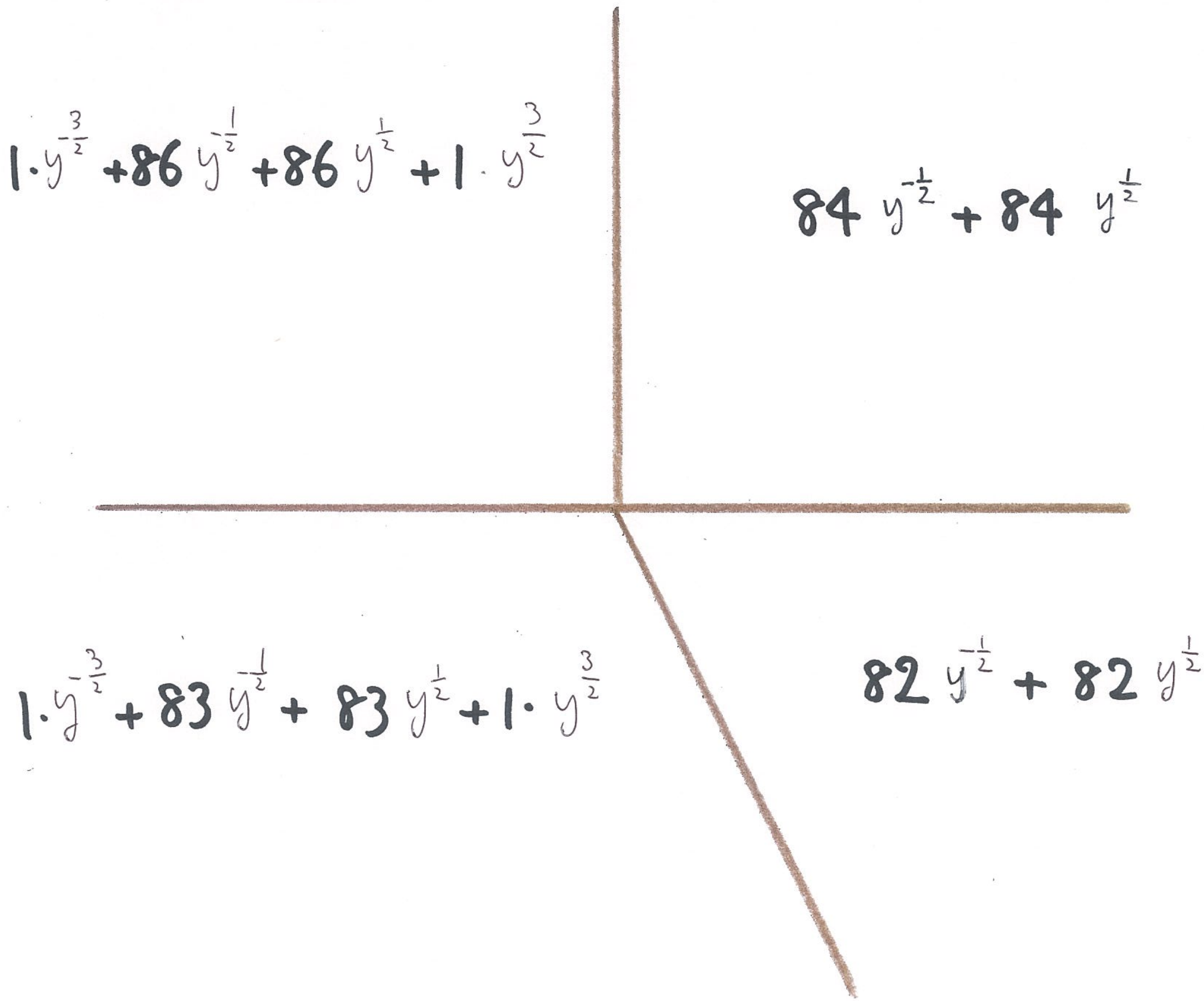
$\mathcal{N}=4$

	P	X_{12}	Y_{123}	Z
$U(1)_1$	-4	$0\ 0$	$1\ 1\ 1$	1
$U(1)_2$	0	$1\ 1$	$0\ 0\ 0$	-2

$$W = p f(x, Y, z)$$

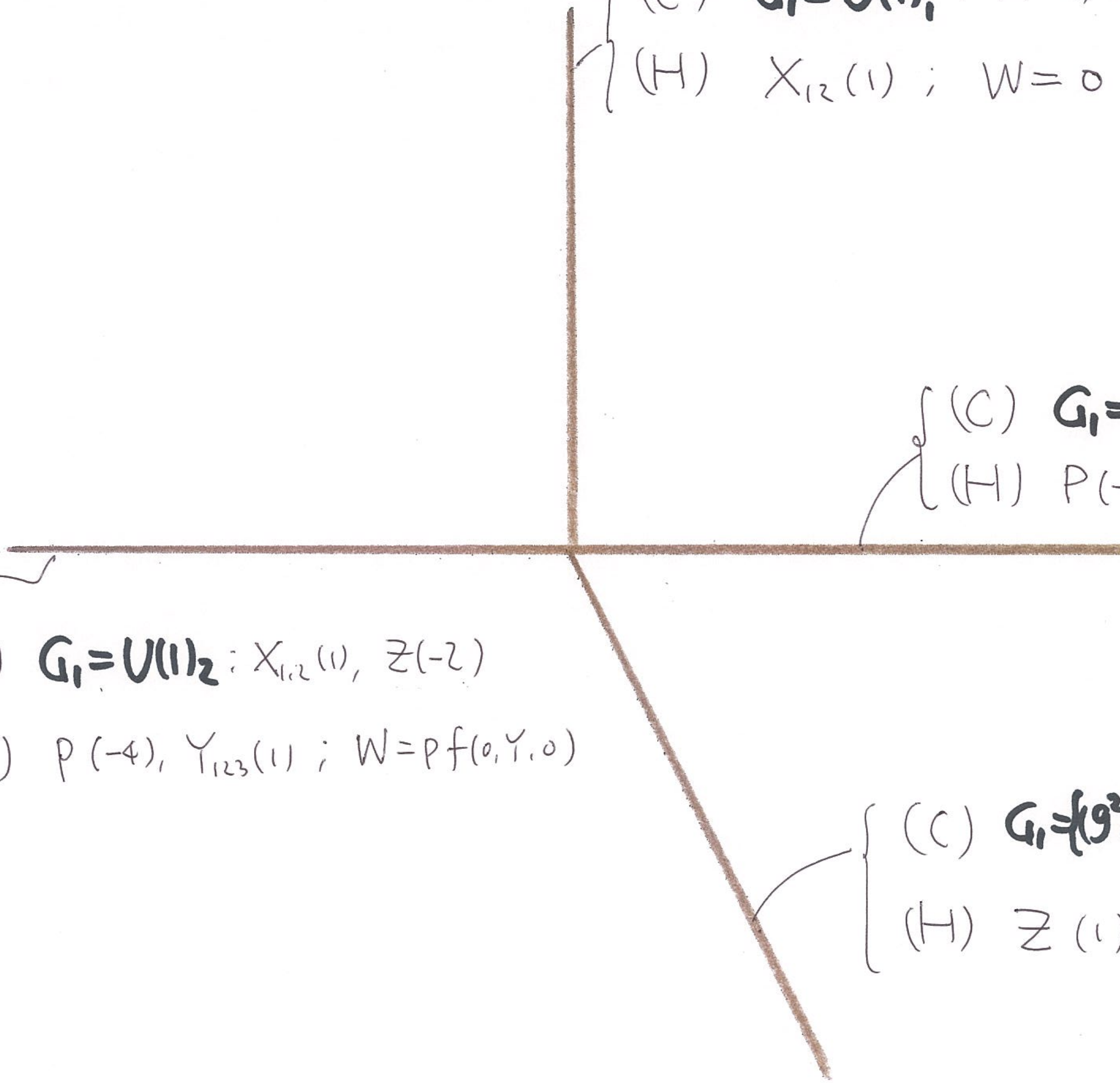


The index $I(y)$



$1 \cdot y^{-\frac{3}{2}} + 86\, y^{-\frac{1}{2}} + 86\, y^{\frac{1}{2}} + 1 \cdot y^{\frac{3}{2}}$
$84\, y^{-\frac{1}{2}} + 84\, y^{\frac{1}{2}}$
$1 \cdot y^{-\frac{3}{2}} + 83\, y^{-\frac{1}{2}} + 83\, y^{\frac{1}{2}} + 1 \cdot y^{\frac{3}{2}}$
$82\, y^{-\frac{1}{2}} + 82\, y^{\frac{1}{2}}$

The mixed branches



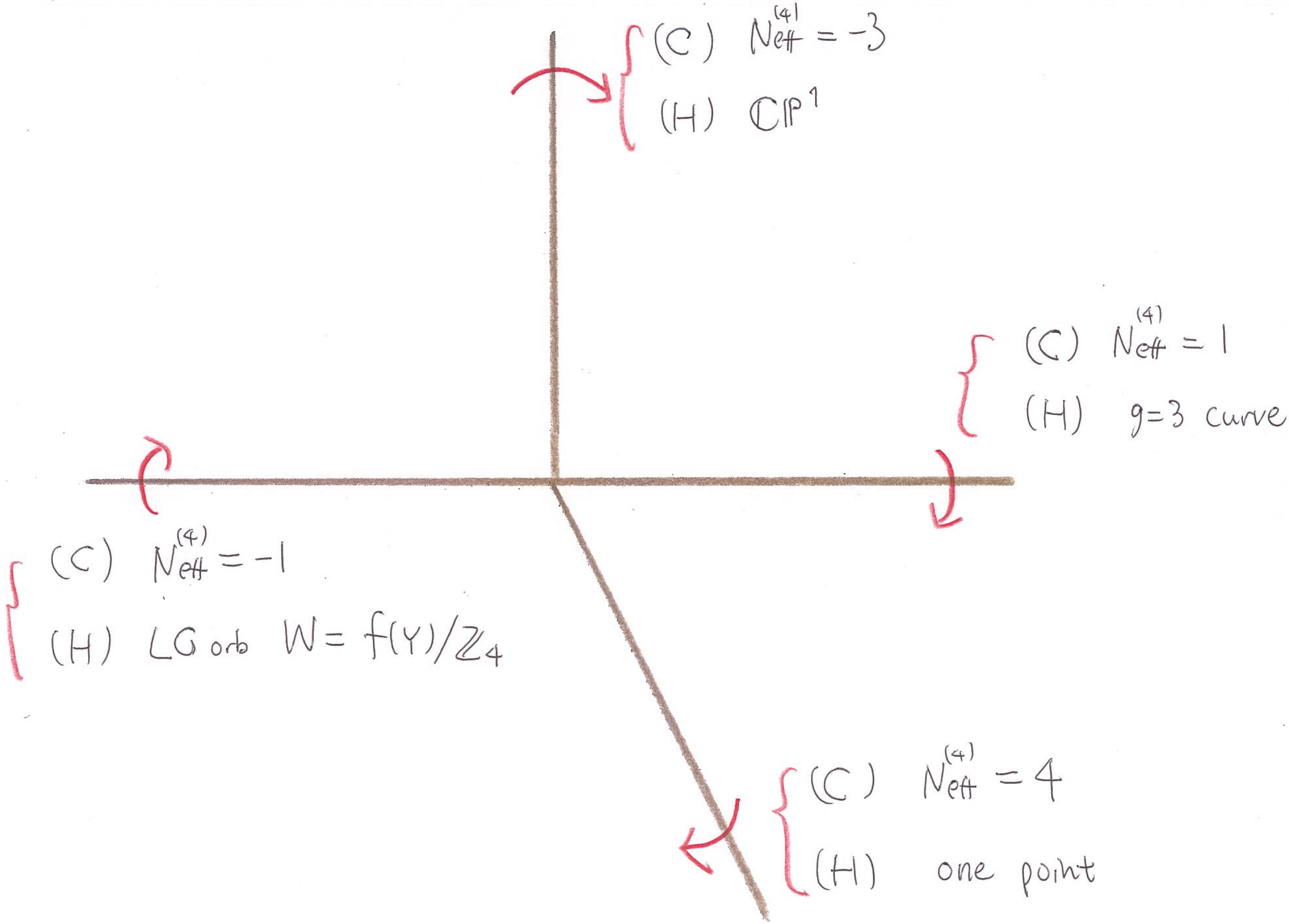
$\begin{cases} (C) & \mathbf{G_1 = U(1)_1} : P(-4),\ Y_{123}(1),\ Z(1) \\ (H) & X_{12}(1)\ ;\ W = 0 \end{cases}$

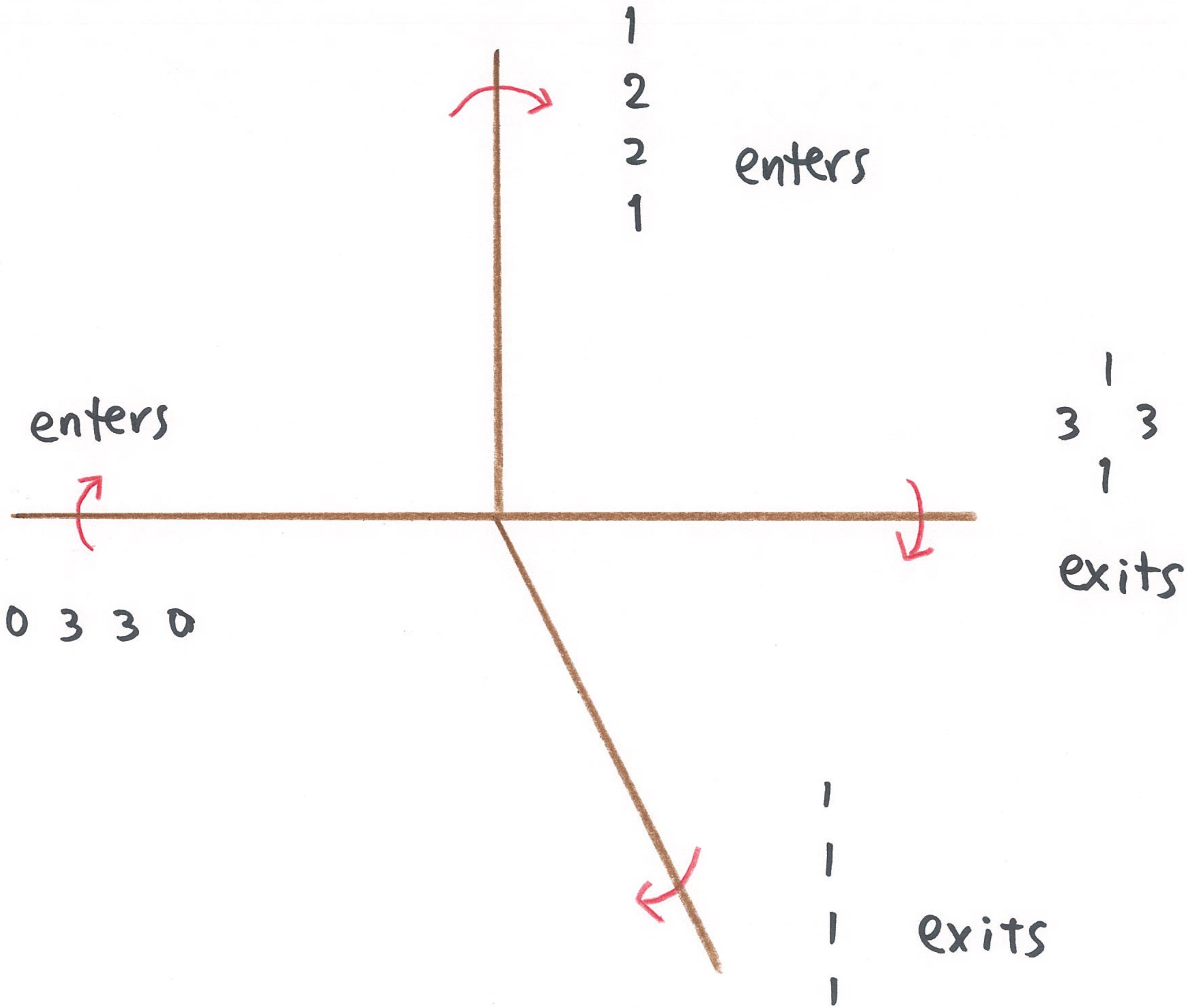
$\begin{cases} (C) & \mathbf{G_1 = U(1)_2} : X_{1,2}(1),\ Z(-2) \\ (H) & P(-4),\ Y_{1,2,3}\ ;\ W = p f(0, Y, 0) \end{cases}$

$\begin{cases} (C) & \mathbf{G_1 = U(1)_2} : X_{1,2}(1),\ Z(-2) \\ (H) & P(-4),\ Y_{123}(1)\ ;\ W = p f(0, Y, 0) \end{cases}$

$\begin{cases} (C) & \mathbf{G_1 = \{(g^2, g)\}} : P(-8),\ X_{12}(1),\ Y_{123}(2) \\ (H) & Z(1)\ ;\ W = 0 \end{cases}$

The mixed branches

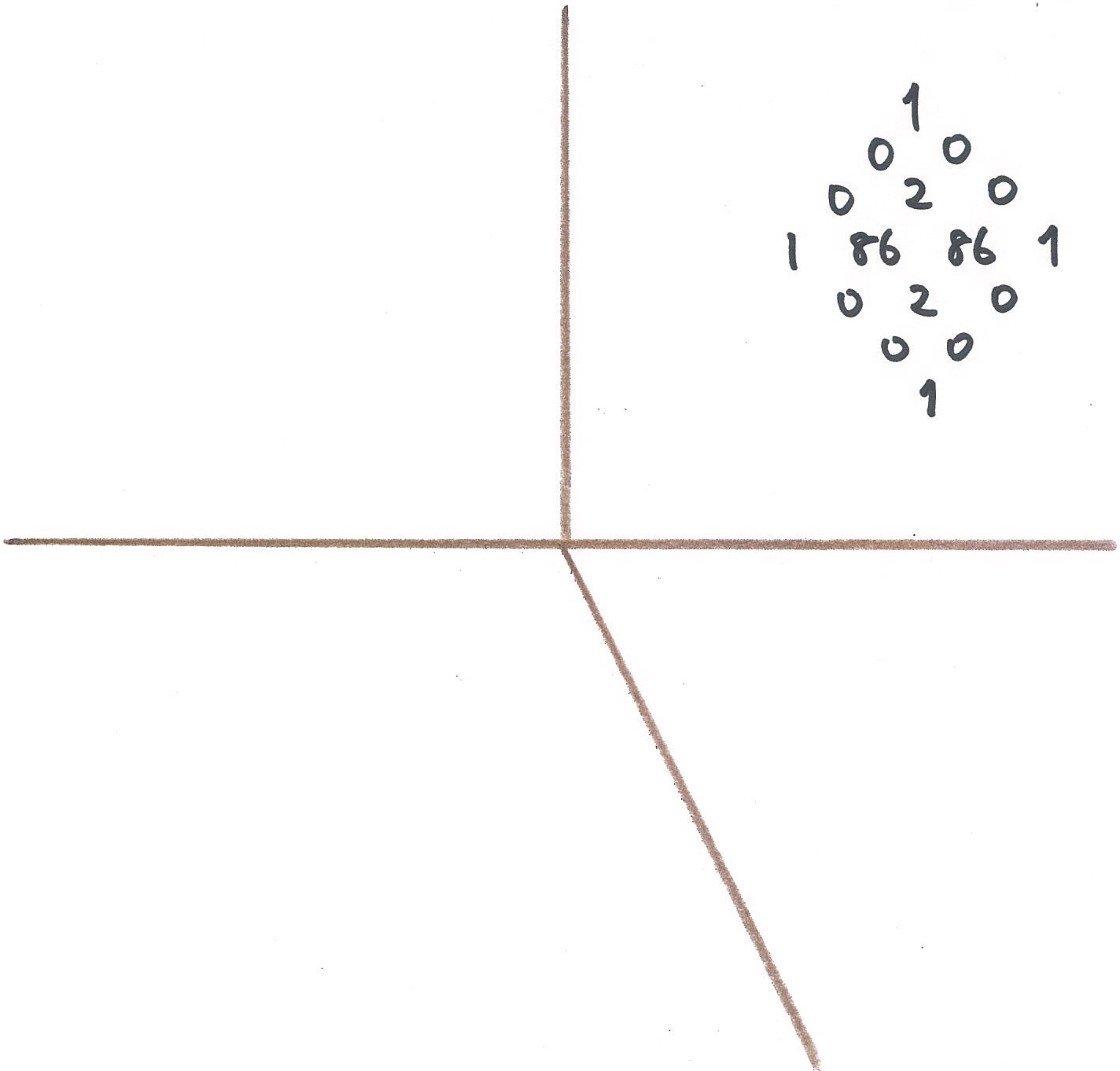


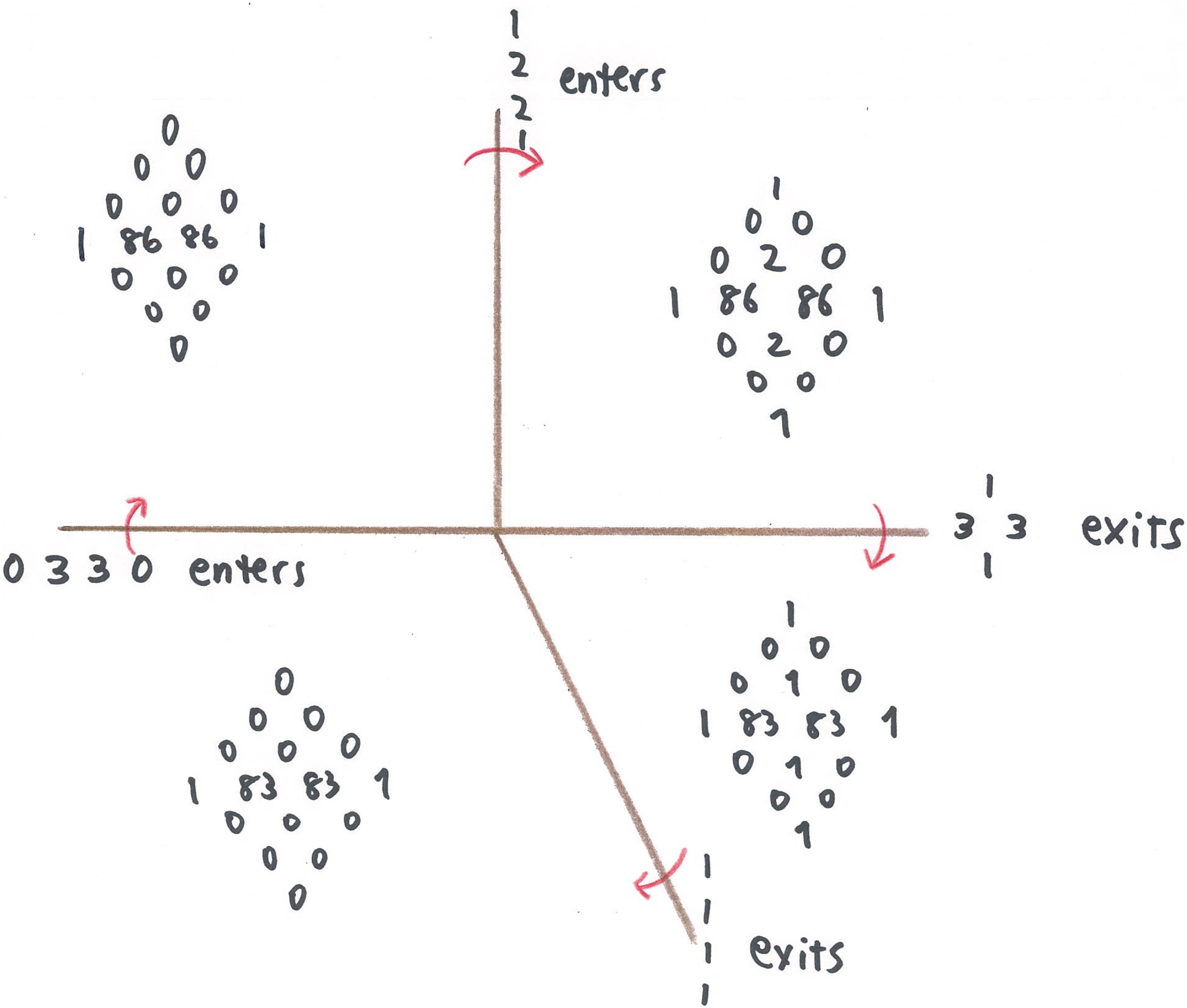


1
2
2
1
enters
enters
1
3 3
1
exits
0 3 3 0
1
1
1
1
exits

Lefschetz hyperplane Theorem

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 2 & & 0 & \\
1 & & 86 & & 86 & & 1 \\
 & 0 & & 2 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$




1 2 2 1 enters
0 3 3 0 enters
1 3 3 1 exits
1 1 1 1 exits
0 0 0 0 0 0 1 86 86 1 0 0 0 0 0 0
1 0 0 0 2 0 1 86 86 1 0 2 0 0 0 1
0 0 0 0 0 0 1 83 83 1 0 0 0 0 0 0
1 0 0 0 1 0 1 83 83 1 0 1 0 0 0 1