The Coulomb Branch Formula for Quiver Moduli Spaces

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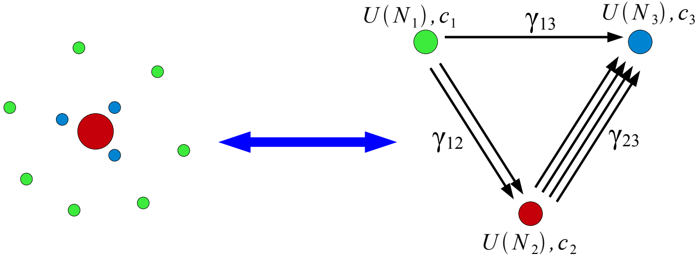
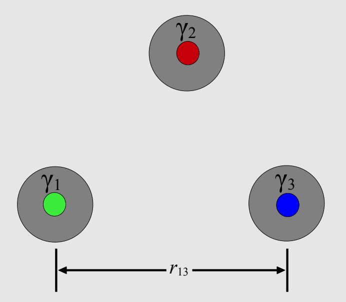
"BPS States, Hitchin Systems and Quivers"
September 3-5, 2014

- Lecture courses by: Philip Boalch, Greg Moore and Markus Reineke
- Seminars by: Iosif Bena, Frédéric Chapoton and Michele Del Zotto
- More info and registration: http://jmplyon2014.sciencesconf.org

BPS bound states of $d = 4, \mathcal{N} = 2$ theories $\Longleftrightarrow$ $\mathcal{N} = 4$ Quiver Quantum Mechanics



Black hole bound states



N black holes with charges γ_i located at $\vec{r}_i$ in $\mathbb{R}^3$

Static BPS solutions exist due to interplay between gravitational attraction and electro-magnetic repulsion

$\Longrightarrow$ Bound states are part of the 1-particle Hilbert space $\mathcal{H}_{\mathrm{BPS}}(\gamma; t)$

Black hole bound states resolve a puzzle for the "light" spectrum:

$$S_{\rm BH} \sim \pi \sqrt{\tfrac{2}{3} P^3 Q_0}$$

Puzzle: $Q_0 \geq -\frac{1}{24} P^3$ in microscopic CFT

Solution: The states with $-\frac{P^3}{24} \leq Q_0 \leq 0$ are realized as BH bound states: Gaiotto, Dabholkar, Nampuri (2007); Denef, Moore (2008); Sen (2011);...

More generally: Microscopic counting of $\mathcal{N} = 2$ BPS black holes does not accurately distinguish single center and multi-center solutions with charge γ

Strominger, Vafa (1996); Maldacena, Strominger, Witten (1997); Bena, Wang, Warner (2006); Strominger, Gaiotto, Yin (2006); Denef, Moore (2007); De Boer, Denef, El-Showk, Messamah, Van den Bleeken (2008);...

$\Rightarrow$ Understanding of bound states is crucial for precision tests of black hole entropy

Denef equations

$\mathcal{N} = 2$ BPS equations of motion require the distances $r_{ij} = |\vec{r}_i - \vec{r}_j| \in \mathbb{R}_+$ to satisfy:

$$\sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{\gamma_{ij}}{r_{ij}} = c_i(\{\gamma_k\}; t)$$

- $\gamma_{ij} = \langle \gamma_i, \gamma_j \rangle \in \mathbb{Z}$: Dirac-Schwinger-Zwanziger innerproduct
- $c_i(\{\gamma_j\}; t) \in \mathbb{R}$: stability parameters depending on vector multiplet scalars t

Denef (2000)

Phase space $M(\{\gamma_i\}, \{c_i\})$:

- parametrizes $\vec{r}_i \in \mathbb{R}^3, i = 1, \ldots, N$
- has dimension $2N - 2$

De Boer, El-Showk, Messamah, Van den Bleeken (2008)

Denef equations: Two aspects

Wall-crossing:

Solutions might decay or recombine upon varying $c_i \in \mathbb{R}$:

Denef (2000); Denef, Moore (2007),...

For example $N = 2$: $\lim_{c_1 \to 0} r_{12} = \lim_{c_1 \to 0} \frac{\gamma_{12}}{c_1} = \pm\infty$

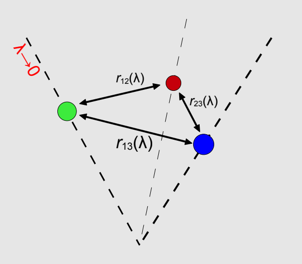
Scaling solutions:

Centers could get arbitrarily close, depending on $\{\gamma_i\}$

Bena, Wang, Warner (2006); Denef, Moore (2007),...

For example $N = 3$: If $\gamma_{12} + \gamma_{23} \geq \gamma_{31}$, and cyclic perm. $\Rightarrow$

$$\lim_{\lambda \to 0} r_{ij}(\lambda) = \lambda \gamma_{ij} + \mathcal{O}(\lambda^2) \in M(\{\gamma_i\}, \{c_i\})$$



BPS index

$\mathcal{H}(\gamma, t)$: Hilbert space of $\mathcal{N} = 2$ supergravity with fixed γ and vector multiplet scalars t

BPS index: $$\Omega(\gamma; t) = -\frac{1}{2}\mathrm{Tr}_{\mathcal{H}(\gamma,t)}\,(2J_3)^2\,(-1)^{2J_3},$$
with J_3 generator of $SU(2)_{spin}$ in $\mathbb{R}^3$

Single-centered index: $\Omega_{\mathrm{S}}(\gamma)$= # of states associated to the single center BH with charge γ. Expected to be independent of t and y Sen (2009)

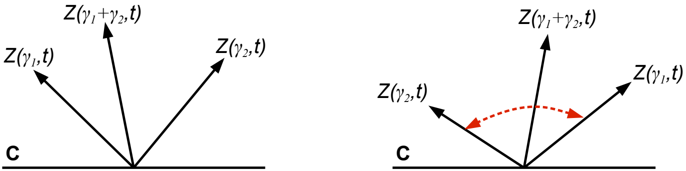
Generically a protected quantity $\Rightarrow$ independent of

- coupling constant
- hypermultiplet scalars

Wall-crossing

Dependence of $\Omega(\gamma; t)$ on vector multiplet scalars t

A central charge $Z(\gamma, t) \in \mathbb{C}$ is associated to every BPS state



- If $Z(\gamma_1, t) || Z(\gamma_2, t)$: $M(\gamma_1) + M(\gamma_2) = M(\gamma_1 + \gamma_2)$
  $\Rightarrow$ bound states are marginally stable
- $\Omega(\gamma_1 + \gamma_2; t)$ is only locally constant as function of t; it might jump across walls where $Z(\gamma_1, t) || Z(\gamma_2, t)$.

Wall-crossing

The jump $\Delta\Omega(\gamma; t \to t') = \Omega(\gamma; t') - \Omega(\gamma; t')$ is given by the wall-crossing formula

Kontsevich, Soibelmann (2008); Joyce, Song (2008); ...

Put strong constraints on the BPS indices $\Omega(\gamma; t)$ and the physics of bound states.

BPS index

Decomposition formula:

$$\bar{\Omega}(\gamma; t) = \sum_{\substack{\sum_i N_i\gamma_i = \gamma, \\ \gamma_i \neq \gamma_j, i\neq j}} g(\{N_i\gamma_i\}; \{c_i(t)\}) \prod_j \frac{\bar{\Omega}_{\rm S}(\gamma_j)^{N_j}}{N_j!}$$

- $\bar{\Omega}_{\rm S}(\gamma_j) = \sum_{n|\gamma_j} \frac{\Omega_{\rm S}(\gamma_j/n)}{n^2}$: rational invariant associated to center j Familiar from Schwinger pair creation and D-instanton measure. Gopakumar, Vafa (1998); Kontsevich, Soibelman (2008); Joyce, Song (2008); Kim, Park, Wang, Yi (2011),...
- $\frac{\bar{\Omega}_{\rm S}(\gamma)^N}{N!}$: Maxwell-Boltzmann distribution
- $g(\{N_i\gamma_i\}; \{c_i\}) \in \mathbb{Z}$: # of "binding" states

JM, Pioline, Sen (2010)

Main question: How to interpret and determine $g(\{N_i\gamma_i\}; \{c_i\})$?

Quiver quantum mechanics: Field content

Low energy excitations of the black hole bound state

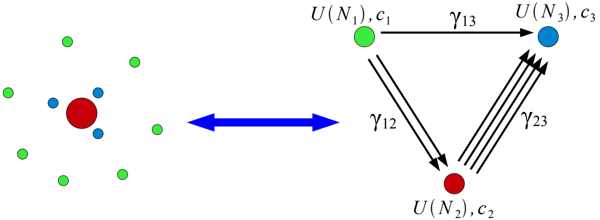
$$\Downarrow$$

$\mathcal{N} = 4$ quiver quantum mechanics

Denef (2002); Denef, Moore (2007); ... Familiar from BPS monopoles: Cederwall, Ferretti, Nilsson, Salomonson (1995); Sethi, Stern, Zaslow (1995); Gauntlett, Harvey (1996); Gauntlett, Kim, Park, Yi (2000),...

Field content is:

- determined by multiplicities $\{N_i\}$ and innerproducts $\{\gamma_{ij}\}$:
  - vector multiplets $(\vec{r}_i, A_i, \lambda_i)$ with gauge group $U(N_i)$
  - $|\gamma_{ij}|$ bifundamental chiral multiplets $(\phi^a_{ij}, F^a_{ij}, \psi^a_{ij}), a = 1, \dots, |\gamma_{ij}|$
- parametrized by a quiver $(\vec{N}, \vec{c})$:



Coulomb branch

$g_s \gg 1$: Coulomb branch

- vector multiplets: $$\sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{\gamma_{ij}}{r_{ij}} = c_i(\{\gamma_k\}; t)$$

  Denef (2002); Kim, Park, Wang, Yi (2011),. . .

N.B.: BPS solutions require more conditions to be physical, in particular regularity of metric

Coulomb branch

Goal: determine $g(\{\gamma_i\};\{c_i\})$ in space-time

1. $g(\{\gamma_i\},\{c_i\})$ is the (twisted) Dirac index of the space $M(\{\gamma_i\},\{c_i\})$
2. $\exists$ symplectic form on $\mathcal{M}_n$:

$$\omega = \frac{1}{2}\sum_{i<j}\gamma_{ij}\,\epsilon^{abc}\frac{\mathrm{d}r_{ij}^a\wedge\mathrm{d}r_{ij}^b\,r_{ij}^c}{|r_{ij}|^3}$$

3. $g(\{\gamma_i\},\{c_i\})$ can be determined using geometric quantization in special cases

   De Boer, El-Showk, Messamah, Van den Bleeken (2008); ...

Coulomb branch

General computation is feasible by refining the index:

$$\Omega(\gamma, y; t) = \frac{\mathrm{Tr}_{\mathcal{H}_{\mathsf{BPS}(\gamma,t)}} (-y)^{2J_3}}{-y^{-2} + 2 - y^2}$$

$\Rightarrow$ $g(\{\gamma_i\}, y; \{c_i\})$: equivariant Dirac index of $M(\{\gamma_i\}, \{c_i\})$

Index theorem: $g(\{\gamma_i\}, y; \{c_i\}) = \int_M \mathrm{Ch}(\mathcal{L}, \nu)\, \hat{A}(M, \nu)|_{2N-2}$
with $\nu = \log(y)$, $\mathrm{Ch}(\mathcal{L}, \nu)$ = equivariant Chern character of $\mathcal{L}$,
$\hat{A}(M, \nu)$ = equivariant $\hat{A}$−genus of M

Berline, Vergne (1985)

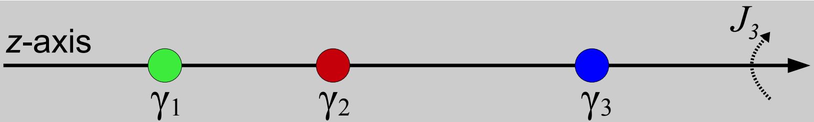
Coulomb branch: Localization

Evaluate integral by localization with respect to J_3
Duistermaat, Heckman (1982); Berline, Vergne (1985);...

$\Downarrow$

Sum over isolated fixed points $\in M(\{\gamma_i\}, \{c_i\})$ of J_3

The solutions which contribute are of the form:



JM, Pioline, Sen (2011)

Coulomb branch formula

Fixed point formula:

$$g(\{\gamma_i\}, y; \{c_i\}) = \frac{(-1)^{\sum_{i<j} \gamma_{ij} + N - 1}}{(y - y^{-1})^{N-1}} \sum_{p \in \{\text{f.p. of } J_3\}} s(p)\, y^{2J_3(p)}$$

- angular momentum:

$$J_3(p) = \frac{1}{2} \sum_{i<j} \gamma_{ij} \operatorname{sign}(z_j - z_i)$$

- sign:

$$s(p) = \operatorname{sign}\left(\det\left(\frac{\partial^2 W}{\partial z_i \partial z_j}\right)\right)$$

with $W(\{z_i\}) = -\sum_{i<j} \gamma_{ij} \operatorname{sign}(z_j - z_i) \log |z_i - z_j| - \sum_{i=1}^{N} c_i z_i$

Coulomb branch formula: Example

Example: $\gamma_i, i = 1, \ldots, 3$, $c_3 < c_2 < 0 < c_1$:

- Fixed points have orderings:
  $\{1, 2, 3; +\}, \{2, 1, 3; -\}, \{3, 1, 2; -\}, \{3, 2, 1; +\}$,
- Enumerate:

$$g(\{\gamma_i\}, y; \{c_i\}) = (-1)^{\gamma_{12}+\gamma_{23}+\gamma_{13}} (y - y^{-1})^{-2} \left(y^{\gamma_{12}+\gamma_{13}+\gamma_{23}} - y^{\gamma_{12}-\gamma_{23}-\gamma_{13}} - y^{\gamma_{13}+\gamma_{23}-\gamma_{12}} + y^{-\gamma_{12}-\gamma_{13}-\gamma_{23}}\right)$$

agreement with wall-crossing formula's proven by A. Sen (2011)

NB: Hard to find fixed points numerically.

Algorithm: Deforming γ_{ij}

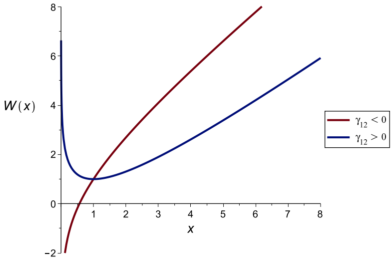
Problem: (numerical) determination of fixed points is tedious and time-consuming

Resolution: Recursive determination of $g(\{\gamma_i\}, y; \{c_i\})$ using:

1. γ_{ij} can be deformed from $\mathbb{Z}$ to $\mathbb{R}$ in Denef equations
2. take a convenient choice $\gamma_{0,ij}$ for γ_{ij}
3. determine $s_0(p)$
4. study the (dis)appearance of extrema of $W(\{z_i\})$ during the reverse deformation: $s(p) = s_0(p) + \sum_A s_A(p)$

   For example:

   $N = 2$, $x = z_2 - z_1$, $c_1 < 0$:



$$W(x) = -\gamma_{12} \ln(x) - c_1 x$$

JM, Pioline, Sen (2013)

Algorithm: Results

For quivers without closed loops:

- explicit expression for $s(p)$ is obtained:

$$s(p) = \prod_{k=1}^{N} \Theta(\tilde{\gamma}_{k,k+1}\, \tilde{c}_k)\, (-1)^{\sum_{k=1}^{N-1} \Theta(-\tilde{\gamma}_{k,k+1})}$$

where $\Theta(x) =$ step function

- agrees with Higgs branch result

$$\mathrm{Tr}_{\mathcal{H}_{\mathrm{BPS,Higgs}}(\vec{1}_N,\vec{c})}(-y)^{2J_3} = P(\mathcal{M}(\vec{1}_N,\vec{c}),-y)$$

Reineke (2002); JM, Pioline, Sen (2013)

Algorithm: Minimal modification hypothesis

With loops:

- scaling solutions are possible
- explicit algorithm, recursive in the number of centers
- sum over regular fixed points $\neq$ SU(2) character

Problem: What is the contribution of the scaling fixed point?

1. y-dependent: Minimal modification hypothesis:

$$g(\{\gamma_i\}, y; \{c_i\}) = \frac{(-1)^{\sum_{i<j}\gamma_{ij}+N-1}}{(y-y^{-1})^{N-1}} \left({\sum_p}' s(p)\, y^{2J_3(p)} + p_{\text{scal}}(y) \right)$$

   Determine $p_{\text{scal}}(y)$ iteratively by:
   - $g(\{\gamma_i\}, y; \{c_i\})$ is an SU(2) character
   - classically $J_3(p_{\text{scal}}) = 0 \quad \Rightarrow \quad \lim_{y\to\infty} \frac{p_{\text{scal}}(y)}{(y-y^{-1})^{N-1}} = 0$

2. y-independent: scaling point not distinguishable from single center black hole $\Rightarrow$ include $\Omega_{\rm S}(\sum_i \gamma_i) \neq 0$

General applicability

Coulomb branch formula computes the invariant $\Omega(\gamma, y; t)$ with as input the $\Omega_{\rm S}(\gamma, y)$

Different physical systems lead to different $\Omega_{\rm S}(\gamma, y)$:

- Quiver quantum mechanics: $\Omega_{\rm S}(\ell\gamma_i) = \delta_{1,\ell}$, $i \in V$
  Abelian quivers: Lefshetz hyperplane theorem $\Rightarrow$
  $\Omega_{\rm S}(\gamma, y) \in \mathbb{N}$
  Bena, Berkooz, De Boer, El-Showk, Van den Bleeken (2012); Lee, Wang, Yi (2012); JM, Pioline, Sen (2013), ...
- Quantum field theories (with line operators) Cordova, Neitzke (2013)
- Supergravity

$g_s \ll 1$: Higgs branch

- D-term eqs: $\sum_{j,i\to j}\sum_{a=1}^{\gamma_{ij}} \phi_{ij}^a(\phi_{ij}^a)^\dagger - \sum_{j,j\to i}\sum_{a=1}^{\gamma_{ji}} (\phi_{ij}^a)^\dagger \phi_{ij}^a = c_i \mathbf{1}_{N_i}$

  F-term eqs: $\frac{\partial W(\{\phi_{ij}^a\})}{\partial \phi_{kl}^b} = 0$

  $\Rightarrow$ equations for the Quiver Moduli Space (Kähler quotient) $\mathcal{M}(\vec{N};\vec{c})$
- Witten index: $\mathrm{Tr}_{\mathcal{H}_{\mathrm{Higgs}}(\vec{N},\vec{c})}(-1)^F = \chi(\mathcal{M}(\vec{N};\vec{c}))$

No oriented loops: $\chi(\mathcal{M}(\vec{d},\vec{\theta}))$ (and $p(\mathcal{M}(\vec{d},\vec{\theta}),y)$) are given by the Reineke formula Reineke (2002)

Higgs branch: Abelianization formula

Specialization of Coulomb branch formula:

$$\Omega_S(\ell\gamma_i) = \delta_{\ell,1}, \quad i \in \{\text{nodes}\}$$

Maxwell-Boltzmann distribution

$\Downarrow$

$g(\{N_i\gamma_i\};\{c_i\})$: # of ground states of Abelian quiver theory

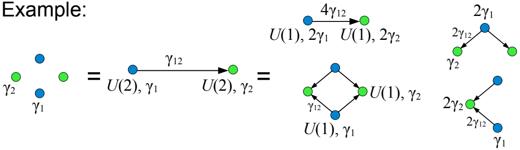
$\Rightarrow$ non-Abelian from Abelian:

Decomposition formula implies:

$$\chi(\mathcal{M}_Q(\vec{N};\vec{c})) \sim \sum_{Q'} \chi(\mathcal{M}_{Q'}(\vec{1}_{N'};\vec{c}'))$$

JM, Pioline, Sen (2010)

Example:



Mathematical studies by Mozgovoy, Okada, Reineke, Stoppa, Weist,...

Pure Higgs states

Higgs-Coulomb map $B : \mathcal{H}_{\text{Higgs}}(\gamma, t) \to \mathcal{H}_{\text{Coulomb}}(\gamma, t)$
surjective map with kernel $\ker(B)$: pure Higgs states
Berkooz, Verlinde (1999); Bena, Berkooz, De Boer, El-Showk, Van den Bleeken (2012); Lee, Wang, Yi (2012);...

$$\text{If } \ker(B) \neq \emptyset \quad \Rightarrow \quad P(\mathcal{M}(\vec{1}_N, \vec{c}), -y) - g(\{\gamma_i\}, y; \{c_i\}) \neq 0$$

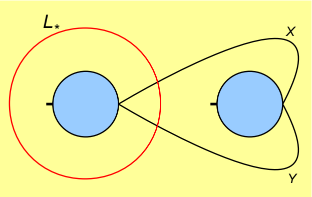
- $\ker(B) \neq 0$ only occurs with loops
- Extra states are associated to scaling fixed points
- $\Rightarrow$ Explicit models where $\Omega_{\mathrm{S}}(\gamma) \in \mathbb{N}$ can be tested and studied

Line operators

Line operators L in 4d $\mathcal{N} = 2$ gauge theory:

- Extended along the time direction
- Vacuum expectation value: $\langle L \rangle = \sum_\gamma \overline{\underline{\Omega}}(L, \gamma) \Upsilon_\gamma$
- $\overline{\underline{\Omega}}(L, \gamma)$ are determined using traffic rules and satisfy interesting recursion relations

  Gaiotto, Moore, Neitzke (2009/10)



2d line operator on triangulation for $SU(2)$ gauge theory

- $\overline{\underline{\Omega}}(L, \gamma)$ are conjectured to be computed by the Coulomb branch formula with $\Omega_{\mathrm{S}}(\gamma, y) = 0$ for $\gamma \notin \{\mathrm{nodes}\}$ Cordova, Neitzke (2013)

Quiver mutations

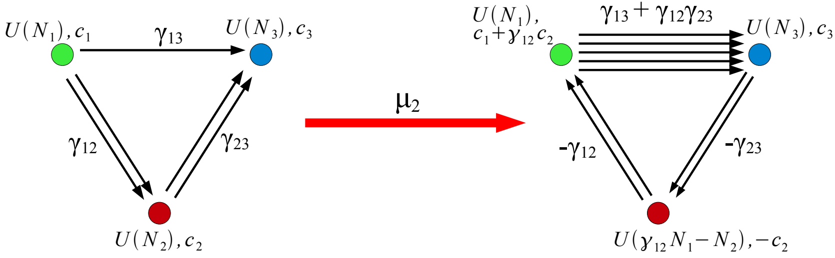
Quiver mutation ⇔ Seiberg duality

Change of gauge groups $U(N_i)$ and number of hypermultiplets γ_{ij}

Math: Bernstein, Gelfand, Ponomarev (1978); Derksen, Weyman, Zelevinsky (2008); Kontsevich, Soibelman (2008); Keller, Yang (2011), ... Physics: Seiberg (1994); Feng, Hanany, He, Uranga (2001); Berenstein, Douglas (2002); Mukhopadhya, Ray (2004),...

Quiver quantum mechanics: $\Omega_{\mathrm{S}}(\ell\gamma_i) = \delta_{1,\ell},\ i \in V$

Example: mutation on node 2 of 3-node quiver with $c_2 < 0$, $\sum_i c_i = 0$.



$$\chi(\mathcal{M}(\vec{N};\vec{c})) = \chi(\mathcal{M}'(\vec{N}';\vec{c}'))$$

Generalized quiver mutations

Coulomb branch formula has more general input
$\Omega_{\rm S}(\gamma, y) = \sum_n \Omega_n(\gamma)\, y^n \in \mathbb{Z}[y, y^{-1}]$

Generalized mutation symmetry motivated by:

- quiver mutation
- Fermi flip (particle-hole duality)

  Andriyash, Jafferis, Denef, Moore (2010)

Conditions:

1. Fermionic particle: $\Omega_n(\gamma_2) \geq 0$
2. $M(\gamma_2) = \sum_{\ell \geq 1} \sum_n \ell^2 \Omega_{\rm S}(\ell\gamma_2) < 0$
3. $\Omega_{\rm S}(\alpha, y; c) = \begin{cases} \Omega'_{\rm S}(\alpha + M(\gamma_2)\,\langle \alpha, \gamma_2 \rangle\, \gamma_2, y; c) & \text{for } \alpha \not\parallel \gamma_k \\ \Omega'_{\rm S}(-\alpha) & \text{for } \alpha \parallel \gamma_k \end{cases}$

Generalized quiver mutations

Charges transform as:

$$(\gamma_1, \gamma_2, \gamma_3) \longrightarrow (\gamma_1 + M\gamma_{12}\gamma_2, -\gamma_2, \gamma_3)$$

This induces transformations:

$(N_1, N_2, N_3) \longrightarrow (N_1, M\gamma_{12}N_1 - N_2, N_3)$
$(c_1, c_2, c_3) \longrightarrow \dots$

Proposal:

$$\Omega(\gamma, y; c) = \Omega'(\gamma', y; c')$$

Verified in many cases, but the generalized mutation symmetry remains to be proven.

Puts strong constraints on the $\Omega_{\rm S}(\gamma, y)$ JM, Pioline, Sen (2013)

Conclusion

BPS bound states provide many non-trivial insights for quiver quantum mechanics:

- Coulomb branch formula
- Abelianization formula
- Single centered invariants
- ...

Program `CoulombHiggs.m`:

- MATHEMATICA package for Coulomb and Higgs computations
- available at `arXiv:1302.5498`