SUSY Partition Sums of N=(4,4) GLSMs

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Exact Results in SUSY Gauge Theories in Various Dimensions
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Part I

Elliptic Genera of ALE and ALF Spaces

arXiv:1406.6342

[Harvey, **S.L.**,Murthy]

Motivation

Spectrum of monopole strings

- New predictions on the BPS spectrum of self-dual strings in 6d (2,0) theories

  e.g. 1/4 BPS state counting in 5d N=2 SYM [E.Koh,H.Kim,S.Kim,K.Lee,**S.L.**]

  Topological vertex technique [Highighat,Iqbal,Kozca,Lockart,Vafa]

  ABJM model on interval [Hosomichi,**S.L.**]

- Self-dual strings in (2,0) theories = monopole strings in 5d N=2 SYM theories

- In particular, SU(3) monopole string spectrum can be read off from the elliptic genus of (4,4) Taub-NUT CFT

Motivation

Generic features of non-compact CFT

- BPS Spectrum of the cigar CFT ($\frac{SL(2,\mathbb{R})}{U(1)}$) is well understood recently
  [Ashok,Troost]
  [Ashok,Doroud,Troost][Murthy]

- A few key results :

  Non-holomorphic elliptic genus: difference between density of states of B and F in continuum

  Separate the ellipitc genus into two pieces: $\mathcal{E} = \mathcal{E}_{\rm disc} + \mathcal{E}_{\rm rest}$

  Contribution from discrete states of CFT, $E_{\rm disc}$ has Mock Modularity

Motivation

¼-BPS counting in N=4 string theory [Dijkgraaf,Verlinde[2]]

O(22,6;Z)

$$\frac{1}{\Phi_{10}(\tau,z,\sigma)} = \sum_{\substack{m\geq -1\\ n\geq -1\\ l\in\mathbb{Z}}} e^{2\pi i(m\sigma+n\tau+lz)} D(m,n,l)$$

$$m = \frac{1}{2}Q_e^2$$
$$n = \frac{1}{2}Q_m^2$$
$$l = Q_e\cdot Q_m$$

- Degenercy D(m,n,l) : Fourier coefficient of $1/\Phi_{10}$

- An interesting interpretation of the formula : D1-D5 in IIB on K3xS[1]xTN

$$\frac{1}{\Phi_{10}(\tau,z,\sigma)} = \underbrace{\sum_{m=-1}^{\infty} e^{2\pi i\sigma m}\mathcal{E}(\tau,z;\mathrm{Sym}^{m+1}[\mathrm{K}_3])}_{\text{BMPV BH: D1-D5 on K3xS}^1}\cdot\underbrace{\frac{\eta(q)^2}{\vartheta_1(\tau,z)^2}}_{\text{(0,4) TN}}\cdot\underbrace{\eta(q)^4}_{\text{4 free left fermions}}\cdot\underbrace{\frac{1}{\Delta(\tau)}}_{\text{mom + KK monopole}}$$

[David,Sen]

- BUT, D(m,n,l) is **ambiguous** : charges + **moduli** needed to determine the contour for the Fourier coefficient (wall-crossing, space-time derivation)

[Cheng,Verlinde]

Q: CFT derivation to choose such a contour ?

Elliptic Genus of Compact CFTs

Definition : For N=(0,2) SUSY theories, ($q = e^{2\pi i\tau}$)

$$\mathcal{E}(\tau;\vec{z}) = \mathrm{Tr}_{\mathcal{H}_{\mathrm{RR}}}\left[(-1)^F q^{L_0}\bar{q}^{\bar{L}_0} e^{2\pi i \vec{z}\cdot\vec{J}}\right] \quad \text{with} \quad \{Q_+, \bar{Q}_+\} = \bar{l}_0 - c/24 \equiv \bar{L}_0$$

- H_{RR}: Hilbert space of SCFT in Ramond-Ramond sector

- J: Global charges that commutes with (0,2) supercharges collectively

Properties

- Modular and elliptic : Jacobi form of weight 0 and index n

$$\mathcal{E}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = e^{\frac{2\pi i n c z^2}{c\tau+d}}\mathcal{E}(\tau, z) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

$$\mathcal{E}(\tau, z + k\tau + l) = e^{-2\pi i n(k^2\tau + 2kz)}\mathcal{E}(\tau, z) \qquad k, l \in \mathbb{Z}$$

Elliptic Genus of Compact CFTs

Definition : For N=(0,2) SUSY theories, $(q = e^{2\pi i\tau})$

$$\mathcal{E}(\tau;\vec{z}) = \mathrm{Tr}_{\mathcal{H}_{\mathrm{RR}}}\left[(-1)^F q^{L_0}\bar{q}^{\bar{L}_0} e^{2\pi i \vec{z}\cdot\vec{J}}\right] \quad \text{with} \quad \{Q_+,\bar{Q}_+\} = \bar{l}_0 - c/24 \equiv \bar{L}_0$$

- H_{RR}: Hilbert space of SCFT in Ramond-Ramond sector

- J: Global charges that commutes with (0,2) supercharges collectively

Properties

- Modular and elliptic : Jacobi form of weight 0 and index n

- Holomorphic in terms of q : only discrete states with $\bar{L}_0 \doteq 0$ can contribute (right-moving ground states)

GLSMs for ALE spaces

Focus on A_{N-1} ALE spaces: two different GLSM models

GLSM I: (2,2) GLSM from toric geometry

$$2\pi\mathcal{L} = \frac{1}{e^2}\mathcal{L}_{\rm vec} + \mathcal{L}_\Phi$$

$$\mathcal{L}_{\rm vec} = \int d^4\theta\ \bar{\Sigma}_a \Sigma_a$$

$$\mathcal{L}_\Phi = \int d^4\theta\ \bar{\Phi}_i e^{-2Q_a^i V^a}\Phi_i$$

- N-1 (2,2) U(1) vector multiplets V_a

- N+1 (2,2) chiral multiplets $\Phi^i = (X_1, Y_a, X_2)$:

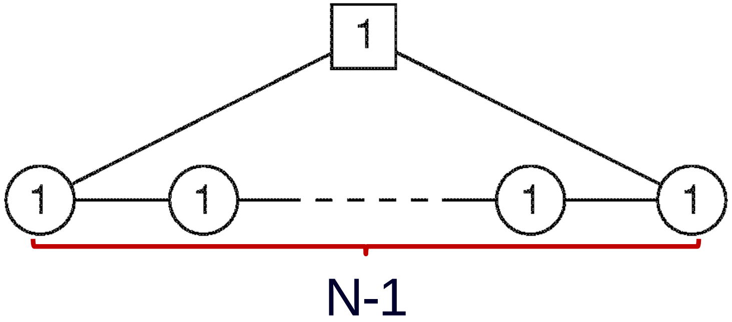
$$Q_a^i = \begin{pmatrix} 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & & \ddots & & \\ & & & 1 & -2 & 1 \end{pmatrix}$$

- Global symmetry:

	X_1	Y_a	X_2
$U(1)_1$	1	0	0
$U(1)_2$	0	0	1
$U(1)_R$	0	0	0

GLSMs for ALE spaces

GLSM II: (4,4) GLSM from Kronheimer hyperKahler construction



$$2\pi\mathcal{L} = \frac{1}{e^2}\left(\mathcal{L}_{\text{vec}} + \mathcal{L}_\Phi\right) + \mathcal{L}_{Q,\tilde{Q}} + \mathcal{L}_{\mathcal{W}}$$

(4,4) vector hyper

$$\mathcal{W} = \sqrt{2}\tilde{Q}\Phi Q$$

- Global symmetry: SO(4) x SU(2) R-symmetry & $U(1)_f$ flavor symmetry

(4,4) supercharges : $(\mathbf{2},\mathbf{1},\mathbf{2})_+ \oplus (\mathbf{1},\mathbf{2},\mathbf{2})_-$

Both GLSM I and II have the same Higgs branch = A_{N-1} ALE space, but they have different Coulomb branch

Elliptic Genus of ALE Spaces

Path-integral representation: $w \simeq w+1 \simeq w+\tau$

$$\mathcal{E}(\tau;\vec{z}) = \mathrm{Tr}_{\mathcal{H}_{\mathrm{RR}}}\left[(-1)^F q^{L_0}\bar{q}^{\bar{L}_0} e^{2\pi i \vec{z}\cdot\vec{J}}\right]$$

$$= \int \mathcal{D}\Phi\mathcal{D}V\ e^{-S[\Phi,V]}$$

with

$$\varphi(w+2\pi,\bar{w}+2\pi) = \varphi(w,\bar{w})$$

$$\varphi(w+2\pi\tau,\bar{w}+2\pi\bar{\tau}) = e^{2\pi i z J_\varphi}\varphi(w,\bar{w})$$

Localization:

- Choose two supercharges generating N=(0,2) SUSY

- Kinetic and superpotential terms are all Q-exact

$$\mathcal{E} = \lim_{t\to\infty}\int \mathcal{D}\Phi\mathcal{D}V e^{-t(S_{\mathrm{vec}}+S_\Phi)-S_{\mathcal{W}}}$$

- Path-integral can localize onto the space of flat connections on T^2

$$u = \frac{1}{2\pi}\oint_{\sigma_2} A - \frac{\tau}{2\pi}\oint_{\sigma_1} A$$

$$u \simeq u+1 \simeq u+\tau$$

$$(w = \sigma_1+\tau\sigma_2)$$

Elliptic Genus of ALE Spaces

Formula: careful analysis on boson and fermion zero modes are required

$$\mathcal{E} = \frac{1}{|W|} \sum_{u_* \in \mathfrak{M}_\eta} \underset{u_*}{\text{JK-Res}}(Q_*, \eta) Z_{\text{1-loop}}(u)$$

[Benini,Eager,Hori,Tachikawa]
[Gadde,Gukov]

- $Z_{\text{1-loop}}(u)$: 1-loop determinant around the SUSY saddle points containing singular points on u-plane

- JK-Res is a residue operation associated with a rk[G]-component vector η

Example: Apply the above formula to **GLSM II**

$$\mathcal{E}(\tau; \xi_1, \xi_2, z) = \mathrm{Tr}_{\mathcal{H}_{\rm RR}}\Big[(-1)^F q^{L_0} \bar{q}^{\bar{L}_0} e^{2\pi i z J_R} e^{2\pi i \xi_1 J_f} e^{2\pi i \xi_2 (J_1 - J_2)}\Big]$$

- J_1, J_2, J_3 are Cartans of $SU(2)_1$ x $SU(2)_2$ x $SU(2)_3$ = SO(4) x SU(2) R-symmetry and J_f is the $U(1)_f$ charge

Elliptic Genus of ALE Spaces

Example: Apply the above formula to **GLSM II**

$$\mathcal{E}_{A_{N-1}} = \frac{1}{N}\sum_{a,b=0}^{N-1} \frac{\vartheta_1(\tau, \frac{a+b\tau}{N}+\xi_1+z)\vartheta_1(\tau, \frac{a+b\tau}{N}+\xi_1-z)}{\vartheta_1(\tau, \frac{a+b\tau}{N}+\xi_1+\xi_2)\vartheta_1(\tau, \frac{a+b\tau}{N}+\xi_1-\xi_2)}$$

- Holomorphic elliptic genus, unlike the cigar CFT

- Both GLSM I and GLSM II give us the same answer, which implies that there is no contribution from the Coulomb branch

- Terms with nonzero b are from twisted sectors in the orbifold space

$$\sum_{a=0}^{N-1} \frac{\vartheta_1(\tau, \frac{a+b\tau}{N}+z)\vartheta_1(\tau, \frac{a+b\tau}{N}-z)}{\vartheta_1(\tau, \frac{a+b\tau}{N})^2} \quad \text{when} \quad \xi_1=\xi_2=0$$

a : projection

b : twisted bd. condition

GLSM for ALF Spaces

Focus on A_{N-1} multi Taub-NUT space with N coincident centers

GLSM : 2d N=(4,4) QED + 1 charged + 1 Stuekelberg hypermultiplets

$$(V, \Phi) \qquad (Q, \tilde{Q}) \qquad \Psi = (r_1, r_2, \chi_\pm) \qquad \Gamma = (r_3, \gamma, \tilde{\chi}_\pm)$$

$$\gamma \simeq \gamma + 2\pi$$

$$2\pi\mathcal{L} = \frac{1}{e^2}\mathcal{L}_{\text{vec}} + \mathcal{L}_{Q,\tilde{Q}} + \frac{1}{g^2}(\mathcal{L}_\Psi + \mathcal{L}_{\text{st}}) + \mathcal{L}_\mathcal{W}$$

- Stueckelberg formalism: a theory for massive photon (Higgs mechanism)

$$\mathcal{L}_{\text{St}} = \frac{1}{2}\int d^4\theta \left(\Gamma + \bar{\Gamma} - \sqrt{2}NV\right)^2 = -\frac{1}{2}(\partial_\mu\gamma - NA_\mu)^2 + \cdots$$

with U(1) gauge invariance

$$A_\mu \to A_\mu + \partial_\mu \alpha(x)$$

$$\gamma \to \gamma - N\alpha(x)$$

GLSM for ALF Spaces

GLSM : 2d N=(4,4) QED + 1 charged + 1 Stuekelberg hypermultiplets

$$2\pi\mathcal{L} = \frac{1}{e^2}\mathcal{L}_{\mathrm{vec}} + \mathcal{L}_{Q,\tilde{Q}} + \frac{1}{g^2}(\mathcal{L}_{\Psi} + \mathcal{L}_{\mathrm{st}}) + \mathcal{L}_{\mathcal{W}}$$

- Superpotential W : $\mathcal{W} = \sqrt{2}\tilde{Q}\Phi Q - \Psi\Phi$

- Global symmetry : SO(4) x SU(2) R-symmetry + $U(1)_f$ flavor symmetry

rotate this angle

GLSM flows to NLSM on A_{N-1} multi Taub-NUT space :

$$\mathcal{L}_{\mathrm{b}} = -\frac{1}{2}\left[H(r)\partial_\mu\vec{r}\cdot\partial_\mu\vec{r} + H(r)^{-1}\left(\partial_\mu\kappa + \frac{N}{2}\vec{w}\cdot\partial_\mu\vec{r}\right)^2\right]$$

$$H(r) = \frac{1}{g^2 N} + \frac{N}{2|\vec{r}|}$$

$$\vec{\nabla}\times\vec{w} = \vec{\nabla}(1/r)$$

$$\kappa = \gamma + \arg q$$

Elliptic Genus of ALF Space

Definition : choosing two supercharge, one can define

$$\mathcal{E}(\tau;\xi_1,\xi_2,z) = \mathrm{Tr}_{\mathcal{H}_{\mathrm{RR}}}\Big[(-1)^F q^{L_0}\bar{q}^{\bar{L}_0} e^{2\pi i z J_R} e^{2\pi i \xi_1 J_f} e^{2\pi i \xi_2 (J_1 - J_2)}\Big]$$

- J_1, J_2, J_3 : Cartans of $SU(2)_1$ x $SU(2)_2$ x $SU(2)_3$ = SO(4) x SU(2) R-symmetry
  J_f : $U(1)_f$ charge

Path integral formulation:

$$\mathcal{E} = \int \mathcal{D}(\text{fields}) e^{-t(S_{\mathrm{vec}}+S_{Q,\tilde{Q}}+S_{\Psi}) - \frac{1}{g^2}S_{\mathrm{St}} - S_{\mathcal{W}}}$$

- Kinetic term for the Stueckelberg field cannot be treated as a Q-exact term [Hori,Kapustin]
- Can perform honest path-integral over Stueckelberg fields

$$\mathcal{L}_{\mathrm{St}} = \left|\partial_w \gamma + \frac{\bar{u}}{2i\tau_2}\right|^2 + \cdots + \lambda_-^0 \bar{\tilde{\chi}}_+^0 - \bar{\lambda}_-^0 \tilde{\chi}_+^0 + \mathcal{O}(1/\sqrt{t})$$

Elliptic Genus of ALF Space

Result : (need careful analysis on fermion zero modes)

$$E(\tau) = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$$
$$u \simeq u+1 \simeq u+\tau$$

$$\mathcal{E}_N(\tau;\xi_1,\xi_2,z) = \frac{g^2}{N}\int_{E(\tau)} \frac{du d\bar{u}}{\tau_2} \frac{\vartheta_1(\tau,u+\xi_1+z)\,\vartheta_1(\tau,u+\xi_1-z)}{\vartheta_1(\tau,u+\xi_1+\xi_2)\,\vartheta_1(\tau,u+\xi_1-\xi_2)} \sum_{(p,\omega)\in\mathbb{Z}^2} e^{-\frac{g^2\pi}{\tau_2}\left|u+\frac{p+\tau\omega}{N}\right|^2}$$

- Non-holomorphic elliptic genus which depends on the size parameter g

- Simple poles of the integrand are mild enough, and the result is finite

- **(p,w)** : momentum and winding number along TN circle

$$\gamma(w,\bar{w}) = \sum_{(p,w)\in\mathbb{Z}^2} \left(w\sigma_1 - p\sigma_2\right) + \text{ oscillator modes}$$

- Modular and elliptic : Jacobi form of weight 0 and index (1,0,-1) w.r.t. $(z,\ \xi_1,\ \xi_2)$

Consistency Checks

[1] Witten index = Euler number ? : Yes

$$\mathcal{E}_N(\tau; z = \xi_i = 0) = N$$

[2] Large radius limit $g^2 \to \infty$: ALF to ALE ?

$$\mathcal{E}_N(\tau;\xi_1,\xi_2,z) = \frac{g^2}{N}\int_{E(\tau)} \frac{du d\bar{u}}{\tau_2} \frac{\vartheta_1(\tau,u+\xi_1+z)\,\vartheta_1(\tau,u+\xi_1-z)}{\vartheta_1(\tau,u+\xi_1+\xi_2)\,\vartheta_1(\tau,u+\xi_1-\xi_2)} \sum_{(p,\omega)\in\mathbb{Z}^2} e^{-\frac{g^2\pi}{\tau_2}\left|u+\frac{p+\tau\omega}{N}\right|^2}$$

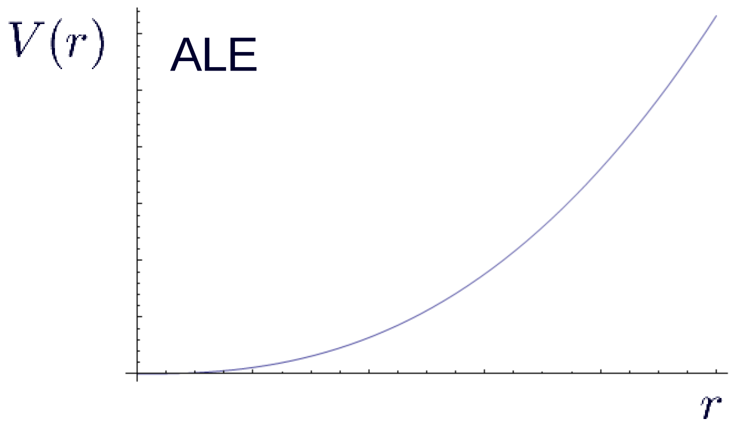
A_{N-1} multi Taub-NUT

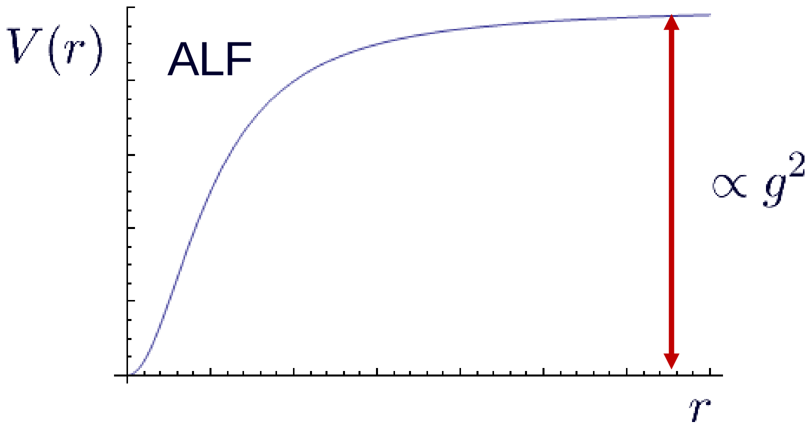
$\downarrow$

$$\mathcal{E}_{A_{N-1}} = \frac{1}{N}\sum_{a,b=0}^{N-1} \frac{\vartheta_1(\tau,\frac{a+b\tau}{N}+\xi_1+z)\vartheta_1(\tau,\frac{a+b\tau}{N}+\xi_1-z)}{\vartheta_1(\tau,\frac{a+b\tau}{N}+\xi_1+\xi_2)\vartheta_1(\tau,\frac{a+b\tau}{N}+\xi_1-\xi_2)}$$

A_{N-1} orbifold

Holomorphic vs Non-Holomorphic





[1] Continuum states with $E \geq V(r=\infty)$ contribute to the elliptic genus
Explains both non-holomorphicity and the dependence on g^2

[2] As g^2 varies, additional states contributing to the elliptic genus kick in/out
Explains why we shouldn't treat $\mathbf{L_{st}}$ as a Q-exact term

[3] In the large radius limit, the potential barrier becomes infinity
Explains why a non-holomorphic elliptic genus becomes holomorphic

Spectrum of Taub-NUT CFT

Separation of discrete states: $\mathcal{E} = \mathcal{E}_{\rm disc} + \mathcal{E}_{\rm rest}$

When all the chemical potentials are real,

$$\mathcal{E}_{\rm disc} = \sum_p e^{2\pi i p \xi_1} \int_{\gamma_0}^{\gamma_0+1} d\gamma \; e^{-2\pi i \gamma p} \frac{\vartheta_1(\tau, \gamma + z)\vartheta_1(\tau, \gamma - z)}{\vartheta_1(\tau, \gamma + \xi_2)\vartheta_1(\tau, \gamma - \xi_2)}$$

$$\text{“}=\text{”} \frac{\vartheta_1(\tau, \xi_1 + z)\vartheta_1(\tau, \xi_1 - z)}{\vartheta_1(\tau, \xi_1 + \xi_2)\vartheta_1(\tau, \xi_1 - \xi_2)}$$

holomorphic in q

- Charge conjugation symmetry:

$$\mathcal{E}_{\rm disc}(\tau; \pm\xi_1, \pm\xi_2, \pm z) = \mathcal{E}_{\rm disc}(\tau; \xi_1, \xi_2, z)$$

- Fourier expansion of the elliptic genus in the limit $g \to \infty$???

$$\varphi(\tau; \xi_i, z) = \frac{\vartheta_1(\tau, \xi_1 + z)\vartheta_1(\tau, \xi_1 - z)}{\vartheta_1(\tau, \xi_1 + \xi_2)\vartheta_1(\tau, \xi_1 - \xi_2)}$$

Spectrum of Taub-NUT CFT

Separation of discrete states: $\mathcal{E} = \mathcal{E}_{\rm disc} + \mathcal{E}_{\rm rest}$

When all the chemical potentials are real,

$$\mathcal{E}_{\rm disc} = \sum_p e^{2\pi i p \xi_1} \int_{\gamma_0}^{\gamma_0+1} d\gamma \; e^{-2\pi i \gamma p} \frac{\vartheta_1(\tau,\gamma+z)\vartheta_1(\tau,\gamma-z)}{\vartheta_1(\tau,\gamma+\xi_2)\vartheta_1(\tau,\gamma-\xi_2)}$$

$$\text{“}=\text{”} \frac{\vartheta_1(\tau,\xi_1+z)\vartheta_1(\tau,\xi_1-z)}{\vartheta_1(\tau,\xi_1+\xi_2)\vartheta_1(\tau,\xi_1-\xi_2)}$$

Fourier transf. of meromorphic function is **ambiguous**

- Contour is determined by both charge p and modulus g

$$w\tau_2 < \mathfrak{Im}(\gamma_0) < (w+1)\tau_2 \qquad w \in I(p,g^2) \equiv \{ w \in \mathbb{Z} \big| -1 - \frac{p}{g^2} < w < -\frac{p}{g^2} \}$$

- Discrete bound states of TN CFT depends on the size parameter g ?

Spectrum of Taub-NUT CFT

Let's expand each of Fourier coefficients in powers of q:

$$\mathcal{E}_{\rm disc}(\tau;\xi_1,\xi_2,z) = \sum_p e^{2\pi i p\xi}\mathcal{E}^p_{\rm disc}(\tau;\xi_2,z)$$

- **p=0** : not ambiguous because $\varphi(\tau;\xi_i,z)$ is a Jacobi form of **index 0** w.r.t. ξ_1

$$\mathcal{E}^{p=0}_{\rm disc}(q,\xi_2,z) = 1 + q\left[\chi_{\mathbf{1}} - 2\tilde{\chi}_{\frac{\mathbf{1}}{\mathbf{2}}}\chi_{\frac{\mathbf{1}}{\mathbf{2}}} + \tilde{\chi}_{\mathbf{1}} + 2\right] + \mathcal{O}(q^2)$$

- **p≠0** : depends on p and g^2

$$\mathcal{E}^{p>0}_{\rm disc} = q^{|pw|}\left[\left(\chi_{\frac{\mathbf{p}}{\mathbf{2}}} - \tilde{\chi}_{\frac{\mathbf{1}}{\mathbf{2}}}\chi_{\frac{\mathbf{p-1}}{\mathbf{2}}} + \chi_{\frac{\mathbf{p}}{\mathbf{2}}-\mathbf{1}}\right) + \mathcal{O}(q)\right], \text{ where } \quad -g^2 < p + g^2 w < 0$$

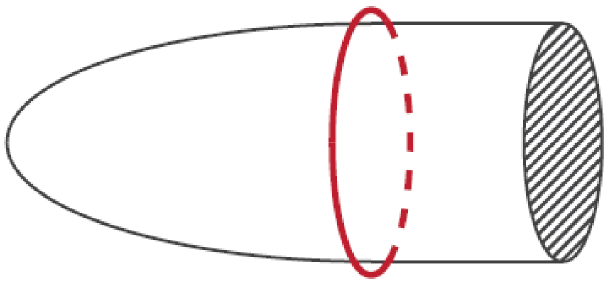
$$\mathcal{E}^{p<0}_{\rm disc} = q^{|pw|}\left[\left(\chi_{\frac{|\mathbf{p}|}{\mathbf{2}}} - \tilde{\chi}_{\frac{\mathbf{1}}{\mathbf{2}}}\chi_{\frac{|\mathbf{p}|-\mathbf{1}}{\mathbf{2}}} + \chi_{\frac{|\mathbf{p}|}{\mathbf{2}}-\mathbf{1}}\right) + \mathcal{O}(q)\right], \text{ where } \quad 0 < p + g^2 w < g^2$$

$\chi_{\mathbf{R}}(z)$: $SU(2)_3$ character, $\tilde{\chi}_{\mathbf{R}}(\xi_2)$: diag[$SU(2)_1$ x $SU(2)_2$] character

Spinning String that Winds TN Circle

COM of a string winding TN circle :

- Scherk-Schwarz reduction:



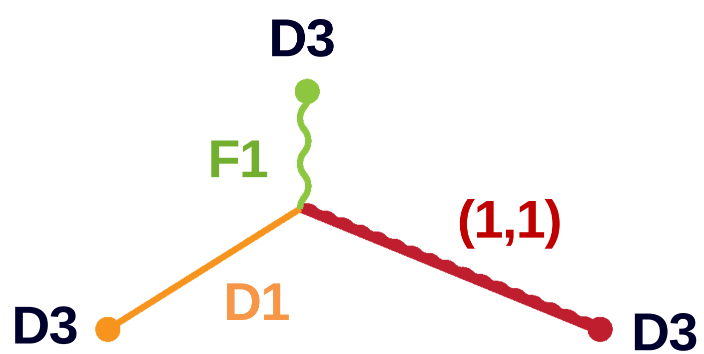
$$\partial_1 \vec{r} = 0 \qquad \partial_1 \kappa = w$$

- Quantum Mechanics:

$$\mathcal{L}^b_{\rm QM} = \frac{H(r)}{2}\frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt} + \frac{H(r)^{-1}}{2}\left(\frac{d\kappa}{dt} + \frac{\vec{w}}{2}\cdot\frac{d\vec{r}}{dt}\right)^2 \underbrace{- \frac{1}{2}H(r)^{-1}w^2}_{\text{string tension}}$$

- The above QM system describe the dynamics a pair of distinct monopoles in 4d N=4 SU(3) SYM

¼ BPS dyon :



Spinning String that Winds TN Circle

SUSY Taub-NUT Quantum Mechanics [D.Bak,K.Lee,P.YI] and many others…

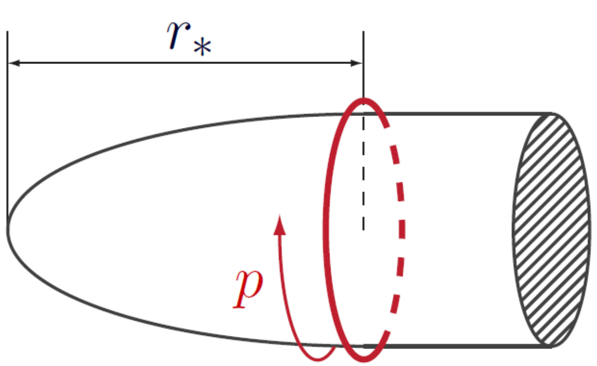
- BPS bound states : p is momentum along the TN circle

$$|p| \geq 2 \ \textbf{:} \ \ j = \frac{|p|}{2}, \frac{|p|-1}{2}, \frac{|p|-1}{2}, \frac{|p|}{2} - 1$$

$$|p| = 1 \ \textbf{:} \ \ j = \frac{1}{2}, 0, 0$$

$|p| = 0$ **:** unique threshold bound state

- What are these states ?: spinning string that winds TN circle



$$V_{\text{eff}}(r) = \frac{1}{2} H(r) p^2 + \frac{1}{2} H(r)^{-1} w^2$$

$$2r_* = \frac{g^2|p|}{g^2|w| - |p|}$$

$$E = |Z| = V_{\text{eff}}(r_*) = |pw|$$

Spinning String that Winds TN Circle

Marginal stability and Wall-crossing

$$2r_* = \frac{g^2|p|}{g^2|w| - |p|}$$

Bound states become unstable when $g^2|w| - |p| \leq 0$

- ¼ BPS to 2 x ½ BPS :

- Elliptic genus should count these bound states with $\bar{L}_0 \propto (E+Z) = 0$, i.e., $pw < 0$

$$\mathcal{I}_{w>0}(\tau;\xi_1,\xi_2,0) = \sum_{\underbrace{-g^2w<p<0}} q^{|pw|} e^{2\pi i p \xi_1} \left(\chi_{\frac{|\mathbf{p}|}{\mathbf{2}}} - 2\chi_{\frac{|\mathbf{p}|-\mathbf{1}}{\mathbf{2}}} + \chi_{\frac{|\mathbf{p}|}{\mathbf{2}}-\mathbf{1}} \right) + \cdots$$

region where bound states are stable

Spectrum of Taub-NUT CFT

Collecting all the result, one expect that the discrete part of the elliptic genus should contain the contribution from spinning string that winds TN circle:

$$\mathcal{E}_{\rm disc}(\tau;\xi_1,\xi_2,0) = 1 + \sum_{p>0} e^{2\pi i p\xi_1}\left[\sum_{-g^2<p+g^2w<0} + \sum_{p+g^2w<-g^2}\right] q^{|pw|}\left(\chi_{\frac{|\mathbf{p}|}{\mathbf{2}}} - 2\chi_{\frac{|\mathbf{p}|-\mathbf{1}}{\mathbf{2}}} + \chi_{\frac{|\mathbf{p}|}{\mathbf{2}}-\mathbf{1}}\right)$$
$$+ \sum_{p<0} e^{2\pi i p\xi_1}\left[\sum_{0<p+g^2w<g^2} + \sum_{g^2<p+g^2w}\right] q^{|pw|}\left(\chi_{\frac{|\mathbf{p}|}{\mathbf{2}}} - 2\chi_{\frac{|\mathbf{p}|-\mathbf{1}}{\mathbf{2}}} + \chi_{\frac{|\mathbf{p}|}{\mathbf{2}}-\mathbf{1}}\right) + \cdots$$

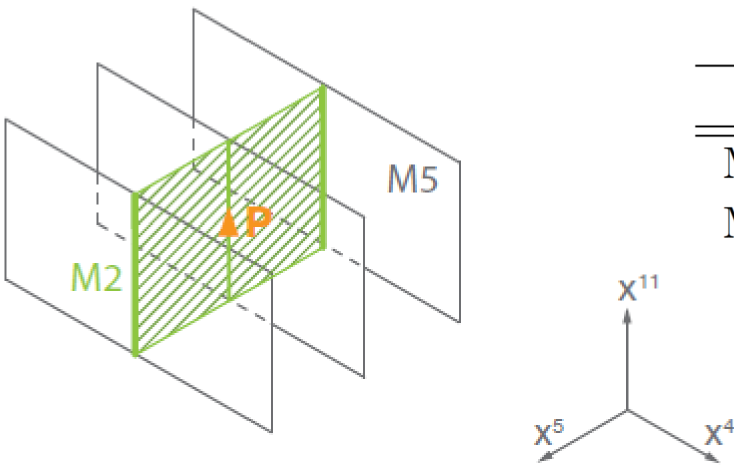
[1] Perfect agreement with each Fourier coefficient

[2] Terms in red boxes that are mixed by world-sheet oscillator modes contribution

[3] Contour is chosen from world-sheet approach, consistent to the wall-crossing

SU(3) Monopole String

SU(3) self-dual string: a M2 between two parallel M5s (not adjacent)

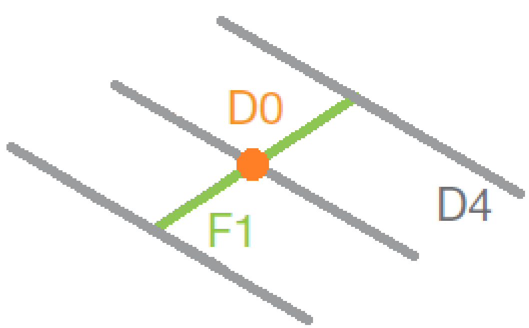


	0	1	2	3	4	5	6	7	8	9	11
M5	×	×	×	×	×						×
M2	×					×					×

$SO(4)_{1234} =$ $SU(2)_1 \times SU(2)_2$

$SO(4)_{6789} =$ $SU(2)_3 \times SU(2)_4$

How to study (BPS) spectrum of the self-dual string ? : **IIA reduction along x^{11}**



¼ BPS states of electric charge **(1,0,-1)** and instanton charge **k** in the Coulomb phase of 5d N=2 U(3) SYM

SU(3) Monopole String

SUSY index of ¼ BPS states: 5d Nekrasov instanton partition function

- Single particle index of ¼ BPS states of electric charge (1,0,-1)
  [E.Koh,H.Kim,S.Kim,K.Lee,**S.L.**]

$$i_{rel} = \int_0^1 d\gamma \; \frac{\vartheta_1(\tau,\gamma+z)\vartheta_1(\tau,\gamma-z)}{\vartheta_1(\tau,\gamma+y_1)\vartheta_1(\tau,\gamma-y_2)}$$

τ : chemical potential of KK mom. charge

y_1 : chemical potential for Cartan of $SU(2)_1$

y_2 : chemical potential for Cartan of diag[$SU(2)_2 \times SU(2)_4$]

z : chemical potential for Cartan of $SU(2)_3$

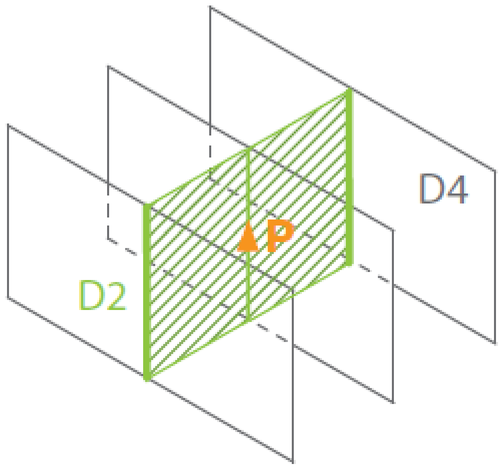
- Prediction on the BPS spectrum of SU(3) self-dual string
  confirmed by [Highighat,Iqbal,Kozca,Lockart,Vafa]

- Confirmation from a different approach ?

SU(3) Monopole String

How to study (BPS) spectrum of the self-dual string?:

- IIA reduction along x^1: SU(3) monopole string



- (4,4) NLSM on TN: low-energy theory on the SU(3) monopole string

M5 on T² requires $\mathcal{E}^{p=0}_{\text{disc}} = i_{rel}$!

- We need to set $y_1 = y_2 = \xi_2$: $SO(4)_{1234}$ broken to $SO(3)_{234}$

$$\mathcal{E}^{p=0}_{\text{disc}} = \int_0^1 d\gamma \, \frac{\vartheta_1(\tau,\gamma+z)\vartheta_1(\tau,\gamma-z)}{\vartheta_1(\tau,\gamma+\xi_2)\vartheta_1(\tau,\gamma-\xi_2)}$$

$$\parallel$$

$$i_{rel} = \int_0^1 d\gamma \, \frac{\vartheta_1(\tau,\gamma+z)\vartheta_1(\tau,\gamma-z)}{\vartheta_1(\tau,\gamma+y_1)\vartheta_1(\tau,\gamma-y_2)}$$

Part II

Quantum Higgs Theory

arXiv:1406.6000

[Harvey,Kutasov,**S.L.**]

Conjecture

2d N=(4,4) U(1) gauge theory coupled to a single charged hypermultiplet has no classical Higgs branch.

Quantum mechanically, the theory is believed to have [Witten]

[1] isolated Higgs vacuum (no massless vector multiplet)

[2] with mass gap (c=0)

Similarly, (4,4) U(N) gauge theory with N fundamental hypermultiplets is conjectured to have such isolated quantum Higgs vacuum with mass gap

Motivation

IIA matrix string theory with F=1 fivebranes

- 2d (4,4) U(N) gauge theory + 1 adjoint + F fundamental hypermultiplets

$$\mathcal{M}_{\text{Coulomb}} \quad \oplus \quad \mathcal{M}_{\text{Higgs}}$$

IIA matrix string theory: graviton	Intrinsic d.o.f. on F fivebranes

- When N=F=1, no classical Higgs branch is contradict to a single 5-brane theory

- There must exist a quantum Higgs vacuum with no massless vector multiplets

Motivation

2d (4,4) U(1) gauge theory + F hypermultiplets of charge 1

- One-loop exact metric on the Coulomb branch

$$ds^2 = \left(\frac{1}{g^2} + \frac{F}{|\vec{\phi}|^2}\right) d\vec{\phi} \cdot d\vec{\phi}$$

- Throat at the origin: two separated theories associated with Coulomb + Higgs (F>1)

- These theories are expected to be conformal with $\hat{c}_c = \mathrm{rk}[G] = 1$

$$\hat{c}_h = n_h - n_v = F - 1$$

- When F=1, the quantum Higgs vacuum is an isolated vacuum of an infrared-trivial theory with a mass gap (c_h=0)

(4,4) Duality with Mass Deformation

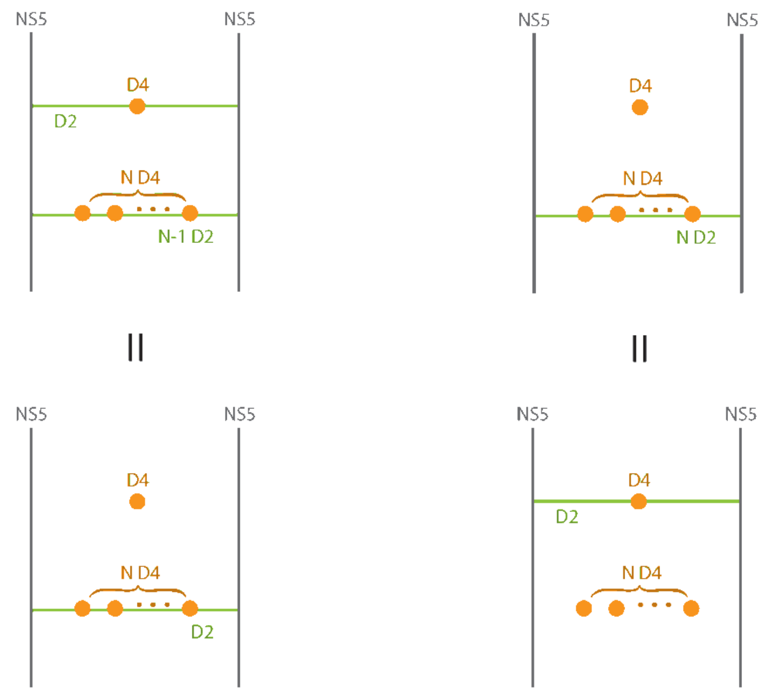
Seiberg-like duality [Benini,Cremonesi][Benini,Eager,Hori,Tachikawa] and others

(4,4) U(N) gauge theory with F fundamental hypermultiplets	=	(4,4) U(F-N) gauge theory with F fundemantal hypermultiplets

To have consistency of the duality map with mass deformation, such quantum Higgs vacuum are required

U(N) with
N+1 hypermultiplets

U(1) with
N+1 hypermultiplets



SUSY Partition Sums

Elliptic genus of (4,4) U(1) gauge theory with a charged hypermuliplet:

$$\mathcal{E}(\tau;\xi_2,z) = \operatorname*{Res}_{u=\zeta_2} Z_{vec}(u)\cdot Z_{hyper}(u) = 1$$

$$Z_{vec}(u) = \frac{i\eta(q)^3}{\vartheta_1(\tau,\xi_2-z)}\frac{\vartheta_1(\tau,2\xi_2)}{\vartheta_1(\tau,\xi_2+z)}$$

$$Z_{hyper}(u) = \frac{\vartheta_1(\tau,u-z)}{\vartheta_1(\tau,u-\xi_2)}\frac{\vartheta_1(\tau,-u-z)}{\vartheta_1(\tau,-u-\xi_2)}$$

- Isolated vacuum of a gapped theory rather than CFT

- Is the isolated vacuum from Coulomb branch? **NO!**

  Elliptic genus of low-energy theory on the Coulomb branch vanishes due to massless U(1) vector multiplet

- Isolated vacuum = Quantum Higgs vacuum with a mass gap

Summary

Elliptic genera of ALE and ALF spaces are computed

- Clarified why and when we expect to have non-holomorphic elliptic genus
- Discrete spectrum of TN CFT exhibits the wall-crossing behavior
- a first example where a contour prescription consistent to wall-crossing is chosen from the world-sheet approach
- Confirmed the prediction of BPS spectrum of SU(3) self-dual string
- Can generalize to the elliptic genus of (0,4) TN CFT

Using the SUSY torus partition function, the conjecture of the quantum Higgs branch can be proven

Separation of Discrete States

One can massage the elliptic genus into the following form

$$\mathcal{E} = -\frac{1}{\pi\eta(q)^6}\int_{-\infty}^{\infty} du \sum_{r_i,s_i,p,w}{}' \frac{(q\bar{q})^{\frac{(p+g^2w)^2}{4g^2}+\frac{u^2}{g^2}}}{2iu+(p+g^2w)}\Big[(q\bar{q})^{iu+\frac{p+g^2w}{2}}-1\Big]q^{-pw}q^{\frac{\sum_i(s_i-1/2)^2}{2}}$$

$$\times(-1)^{s_1+s_2}S_{r_1}(q)S_{r_2}(q)e^{2\pi i\xi_1 p}e^{2\pi i\xi_2(r_1-r_2)}e^{-2\pi i z(s_1-s_2)}\ .$$

$$S_r(q) = \sum_{n=0}^{\infty}(-1)^n q^{\frac{n(n+2r+1)}{2}}$$

- u : momentum conjugate to the radial direction

- Difference between density of states of B and F:

$$\rho_B(u) - \rho_F(u) \propto \frac{1}{2iu+p+g^2w}$$

- Poles = discrete bound states (, roughly speaking)