Cluster algebras from 2d gauge theories

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Introduction

2d gauge theories with $\mathcal{N} = (2,2)$ SUSY $\xleftrightarrow{\text{Seiberg-like dualities}}$ Cluster Algebras

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2d $\mathcal{N}=(2,2)$ theories have physical and mathematical applications:

- interesting as 2d theories
- 4d $\mathcal{N} = 2$ string compactifications
- Kähler and Calabi-Yau geometry: GLSM $\rightarrow$ NLSM
- gauge theories $\leftrightarrow$ quantum integrable systems [Nekrasov, Shatashvili 09]
- will see more today...

These theories are very much under control

Introduction

Cluster algebra: [Fomin, Zelevinsky 01] to describe coordinate rings of groups and Grassmannians

Provides special class of coordinates, with special properties, on manifolds

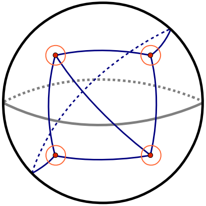
Other contexts:

- Teichmuller theory [Gekhtman, Shapiro, Vainshtein; Fock, Goncharov 03]
- Integrable systems (Y-systems)
- BPS quivers and wall crossing in 4d $\mathcal{N} = 2$ theories
- Amplitudes

Many interesting properties:

- Total positivity
- Laurent phenomenon
- Poisson and symplectic structures
- ...
- (physics $\rightarrow$) sets of Kähler metrics

Example: (decorated) Teichmüller theory



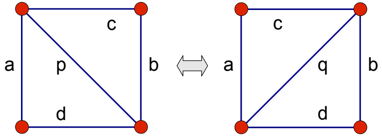
Teichmüller space $\mathcal{T}_g^s$:

$$\downarrow$$

Decorated Teichmüller space $\widetilde{\mathcal{T}}_g^s$

Penner coordinates: $f(e) = e^{\ell(e)}$

- Flip an edge:



Ptolemy relation

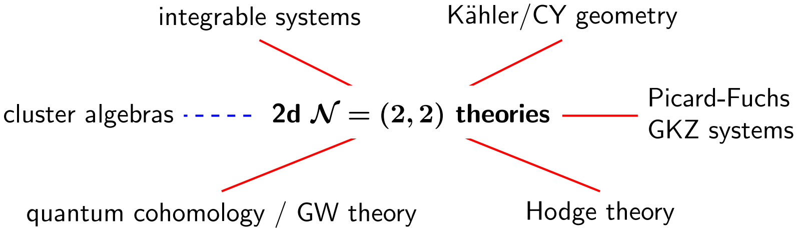
$$f(q) = \frac{f(a)f(b) + f(c)f(d)}{f(p)}$$

- Projection $\widetilde{\mathcal{T}}_g^s \to \mathcal{T}_g^s$: Thurston shear coordinates $f(e) \to g(e)$

$$e^{g(p)} = f(a)f(b)f(c)^{-1}f(d)^{-1}$$

Introduction

Main motivations:

1. 

2. 2d $\mathcal{N} = (2,2)$ theories: $\begin{cases} \text{chiral ring} \\ \text{twisted chiral ring} \\ tt^* \text{ equations} \\ S^2 \text{ partition function} \\ \text{elliptic genus} \\ \dots \end{cases}$ $\rightarrow$ cluster algebras

Outline

- Seiberg-like dualities of 2d $\mathcal{N} = (2,2)$ gauge theories
- S^2 partition function
- Cluster algebras
- From dualities to cluster algebras
- Some applications

2d Seiberg-like dualities

2d $\mathcal{N} = (2,2)$ gauge theories

The class of gauge theories we consider includes:

- Vector multiplet: $V = (A_\mu, \lambda, \sigma, D)$

  $\rightarrow$ (field strength) twisted chiral multiplet : $\Sigma = (\sigma, \lambda, F_{12} + iD)$

- Chiral multiplet: $\Phi = (\phi, \psi, F)$

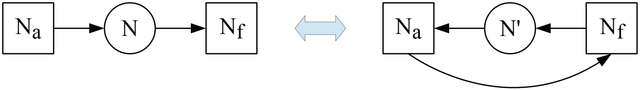
- Interactions:
  - gauge coupling g
  - complexified FI term: $\mathcal{L}_{FI} = \xi D + \frac{\theta}{2\pi} F_{12}$, $t = 2\pi\xi + i\theta$
  - twisted superpotential $\widetilde{W}(\Sigma) = t\Sigma + \ldots$
  - superpotential $W(\Phi)$
  - twisted masses V_{ext}, $\langle \sigma_{\text{ext}} \rangle$

2d Seiberg-like dualities [Jockers, Kumar, Lapan, Morrison, Romo 12; FB, Cremonesi 12]

A: $U(N)$ with N_f fundamentals, N_a antifundamentals

$\updownarrow$

B: $U\big(\max(N_f, N_a) - N\big)$ with N_a fundamentals, N_f antifundamentals, $N_f N_a$ gauge singlets, superpotential $W_{\text{dual}} = \tilde{q} M q$



Comments:

- cfr. with 4d: no gauge anomaly $\rightarrow$ any N_f, N_a
- similar to Hori-Tong duality – but $U(N)$ instead of $SU(N)$
- $N_f = N_a$: flow to IR CFT (otherwise gapped)
- map of parameters: complexified FI term, twisted masses

$$t = 2\pi\xi + i\theta \,, \quad m_j \,, \quad \tilde{m}_f \qquad \Rightarrow \qquad z = (-1)^{\text{out - colors}} e^{-t} \,, \qquad \boxed{z' = \frac{1}{z}}$$

Geometric interpretation [Witten 82; Jia, Sharpe, Wu 14]

Only fundamentals. Large positive FI: geometric realization

$$\mathsf{Gr}(N, N_f) \qquad = \qquad \mathsf{Gr}(N_f - N, N_f)$$

With antifundamentals (assume $N_f \geq N_a$).

- Theory A: each antifundamental gives a copy of tautological bundle S

Equality of bundles:

$$S^{N_a} \to \mathsf{Gr}(N, N_f) \qquad = \qquad (Q^*)^{N_a} \to \mathsf{Gr}(N_f - N, N_f)$$

Universal quotient bundle: $\qquad 0 \to S \to \mathcal{O}^{N_f} \to Q \to 0$

- Theory B: mesons give $(\mathcal{O}^{N_f})^{N_a}$, superpotential imposes short exact sequence.
- Flip fundamentals $\leftrightarrow$ antifundamentals $\quad \Rightarrow \quad$ FI parameter $\xi \leftrightarrow -\xi$

Chiral ring

Chiral operators modulo F-term relations

Theory A: $\tilde{Q}_f Q_j$ (no baryons)

Theory B: $\tilde{q}_j q_f$, M_{fj}, but W imposes $\tilde{q}_j q_f = 0$

Map: $\tilde{Q}_f Q_j = M_{fj}$

- Add superpotential $W = f(\tilde{Q}_f Q_j) = f(M_{fj})$

Twisted chiral ring

Generators: $\mathrm{Tr}\,\sigma^k$ $k = 1, \dots, N$ or symm. polynomials in $\sigma = \mathrm{diag}(\sigma_1, \dots, \sigma_N)$

$$Q(x) = \det(x - \sigma) = x^N - x^{N-1}\,\mathrm{Tr}\,\sigma + \dots$$

Relations: effective superpotential on the Coulomb branch $\qquad t = 2\pi\xi + i\theta$

$$\widetilde{W}_{\text{eff}} = -t\sum_a \sigma_a - \sum_j \sum_{\rho\in R_j} \big(\rho(\sigma) - m_j\big)\big[\log\big(\rho(\sigma) - m_j\big) - 1\big]$$

Impose $\partial\widetilde{W}_{\text{eff}}/\partial\sigma_a \in \mathbb{Z}$ and $\sigma_a \neq \sigma_b$: (*quantum equivariant coh*) $\qquad z \simeq e^{-t}$

$$\boxed{\prod_{j=1}^{N_f}(x - m_f) + i^{N_a - N_f}\, z \prod_{f=1}^{N_a}(x - \tilde{m}_f) = C(z)\,Q(x)\,T(x)}$$

$T(x)$ has degree $N' = \max(N_f, N_a) - N$

$$C(z) = \begin{cases} 1 & \text{if } N_f > N_a \\ 1 + z & \text{if } N_f = N_a \\ i^{N_a - N_f}\, z & \text{if } N_f < N_a \end{cases}$$

- Duality: $T(x) = Q'(x) = \det(x - \sigma')$

S^2 partition function

S^2 partition function [FB, Cremonesi 12; Doroud, Gomis, Le Floch, Lee 12]

Any Euclidean 2d $\mathcal{N} = (2,2)$ theory with R_V-symmetry can be placed supersymmetrically on S^2, with *no twist*.

$$Z_{S^2}(\text{param}) = \int \mathcal{D}\phi \, e^{-S(\text{param})}$$

Parameters:

- complexified FI term $\qquad t = 2\pi\xi + i\theta, \qquad z \simeq e^{-t}$
- real twisted masses m_j, $\tilde{m}_f$ and flavor magnetic fluxes $\mathfrak{n}_j$, $\tilde{\mathfrak{n}}_f$
- R-charges R

Localization:

$$Z_{S^2} = \frac{1}{|\mathcal{W}|} \sum_{\mathfrak{m} \in \mathbb{Z}^N} \int d^N\sigma \, e^{i\,\mathbb{R}\mathrm{e}\,\widetilde{W}\left(\sigma + \frac{i}{2}\mathfrak{m}\right)} \, Z_{\text{1-loop}}$$

$$Z^{\text{gauge}}_{\text{1-loop}} = \prod_{\text{roots } \alpha>0} \left(\frac{\alpha(\mathfrak{m})^2}{4} + \alpha(\sigma)^2 \right)$$

$$Z^{\text{chiral}}_{\text{1-loop}} = \prod_{\text{chiral } \Phi} \prod_{\rho \in \mathfrak{R}_j} \frac{\Gamma\left(\frac{R[\Phi]}{2} - i\rho(\sigma) - if^i[\Phi]m_i - \frac{\rho(\mathfrak{m}) + f^i[\Phi]\mathfrak{n}_i}{2}\right)}{\Gamma\left(1 - \frac{R[\Phi]}{2} + i\rho(\sigma) + if^i[\Phi]m_i - \frac{\rho(\mathfrak{m}) + f^i[\Phi]\mathfrak{n}_i}{2}\right)}$$

Quantum Kähler metric

For gauge theories that in the IR flow to NLSMs on Calabi-Yau manifolds, there is a geometric interpretation of Z_{S^2}.

- Geometry of Kähler moduli space:

  Z_{S^2} computes quantum (genus zero) Kähler metric on Kähler moduli space

  $\rightarrow$ (equivariant) GW invariants of CYs

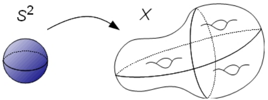
$$Z_{S^2} = \langle \bar{0} | 0 \rangle_{RR} = e^{-K_{\text{Kähler}}}$$

[Jockers, Kumar, Lapan, Morrison, Romo 12]

[Gomis, Lee 12]

[Park, Song 12]

[Bonelli, Sciarappa, Tanzini, Vasko 13]



S^2 partition function

We can prove:

$$Z^{(A)}\left(m_j, \mathfrak{n}_j, \tilde{m}_f, \tilde{\mathfrak{n}}_f; z\right) = f_{\text{contact}}\ \ f_{\text{Kt}}\ \ Z^{(B)}\left(\tilde{m}_f - \tfrac{i}{2}, \tilde{\mathfrak{n}}_f, m_j - \tfrac{i}{2}, \mathfrak{n}_j; z^{-1}\right)$$

Method: vortex partition function (or Toda [Gomis, Le Floch 14])

$$Z_{S^2} = \sum_{\text{Higgs vacua}} Z_{\text{cl}}\ Z'_{\text{1-loop}}\ Z_{\text{vortex}}\ Z_{\text{antivortex}}$$

- Mass shift: shift of R-charges, compatible with W_{dual}

$$r \equiv R[\tilde{Q}_f Q_j] = R[M_{fj}] \qquad \Rightarrow \qquad R[\tilde{q}_j q_f] = 2 - r$$

- $f_{\text{Kt}} = |f_{\text{hol}}|^2$: real function $\rightarrow$ Kähler transformation [Closset, Cremonesi 14]

FT: local counterterm or improvement transformation [Gomis, Komargodski 14]

Geom: according to [Jockers, Kumar, Lapan, Morrison, Romo 12]

$$Z_{S^2} = e^{-K_{\text{Kähler}}}$$

Kähler transformation (does not affect the metric)

The contact term

- $f_{\text{contact}} = e^{i\,\mathbb{R}\mathrm{e}\,\widetilde{W}_{\text{contact}}}$: phase

Twisted superpotential, only function of parameters (twisted chirals):

$$m_j + \tfrac{i}{2}\mathfrak{n}_j\,, \qquad \tilde{m}_f + \tfrac{i}{2}\tilde{\mathfrak{n}}_f\,, \qquad t$$

Theory A: $\widetilde{W}_A = -t\,\mathrm{Tr}\,\sigma = \log z\,\mathrm{Tr}\,\sigma$

Theory B: result depends on number of flavors

- $N_f > N_a + 1$: $\widetilde{W}_B = \log z^{-1}\,\mathrm{Tr}\,\sigma' + \log z\,\mathrm{Tr}\,m$
- $N_f = N_a + 1$: $\widetilde{W}_B = \log z^{-1}\,\mathrm{Tr}\,\sigma' + \log z\,\mathrm{Tr}\,m + iz$
- $N_f = N_a$: $\widetilde{W}_B = \log z^{-1}\,\mathrm{Tr}\,\sigma' + \log \frac{z}{1+z}\,\mathrm{Tr}\,m + \log(1+z)\,\mathrm{Tr}\,\tilde{m}$

If flavor symmetry is gauged, m, $\tilde{m} \to$ dynamical twisted chiral multiplets

$$\delta\widetilde{W} = \log z_f\,\mathrm{Tr}\,m + \log z_a\,\mathrm{Tr}\,\tilde{m}$$

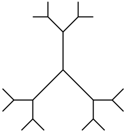
f_{contact} transforms neighboring FIs.

Cluster algebras

Cluster algebra [Fomin, Zelevinsky 01] of rank n

Commutative ring with unit and no zero divisors
with distinguished set of *generators* called cluster variables.

Cluster variables are organized into
overlapping n-subsets called clusters.



- Exchange property (mutations):

for every cluster $\mathbf{X}$ and $x \in \mathbf{X}$,
there is another cluster obtained by substituting $x \to x'$ with rule

$$xx' = M_1 + M_2$$

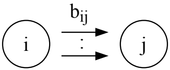
$M_{1,2}$: *monomials* in $n-1$ variables $\mathbf{X} \setminus \{x\}$, with no common divisors.

Any two clusters can be obtained from each other by sequence of *mutations*.

Cluster algebra [Fomin, Zelevinsky 06]

- Seed $\mathbf{X}$:

Antisymmetric $n \times n$ matrix b_{ij} $\rightarrow$ quiver B (no 1-, 2-cycles)



Coefficients $y_i \in \mathbb{P}$ *semifield* $(\cdot, \oplus)$ (*tropical*)

Cluster variables x_i.

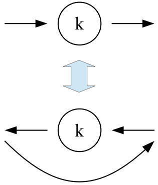
- *Semifield* $(\cdot, \oplus)$: auxiliary addition, not subtraction nor zero.
  - Eg: $\mathbb{R}_+$.
  - Tropical semifield $\mathbb{P}_1$ $(\cdot, \oplus)$: $\mathbb{P}_1 = \{u^a \mid a \in F\}$ $F = \mathbb{Z}$ or $\mathbb{Q}$ or $\mathbb{R}$

    Operations: $u^a \cdot u^b = u^{a+b}$, $u^a \oplus u^b = u^{\max(a,b)}$

    u is formal variable
  - Eg: $\mathbb{P}_0 = \{1\}$, $1 \cdot 1 = 1 \oplus 1 = 1$.

Cluster algebra [Fomin, Zelevinsky 06]

- Mutation (at node k):



$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + \operatorname{sign}(b_{ik})\,[b_{ik}b_{kj}]_+ & \text{otherwise} \end{cases}$$

$$y'_j = \begin{cases} y_k^{-1} & \text{if } j = k \\ y_j\, y_k^{[b_{kj}]_+} (y_k \oplus 1)^{-b_{kj}} & \text{otherwise} \end{cases}$$

$$x'_j = \begin{cases} \dfrac{1}{x_k}\left(\dfrac{y_k}{y_k \oplus 1} \prod_i x_i^{[b_{ik}]_+} + \dfrac{1}{y_k \oplus 1} \prod_i x_i^{[-b_{ik}]_+}\right) & \text{if } j = k \\ x_j & \text{otherwise .} \end{cases}$$

Hierarchical structure.

- Dual cluster variables: $z_j = \prod_i x_i^{b_{ij}}$

$$z'_j = \begin{cases} z_k^{-1} & \text{if } j = k \\ z_j\, z_k^{[b_{kj}]_+} \left(\dfrac{y_k}{y_k \oplus 1} z_k + \dfrac{1}{y_k \oplus 1}\right)^{-b_{kj}} & \text{otherwise .} \end{cases}$$

Cluster algebra

- Total positivity

  Cluster algebra transformations involve $+$, not $-$.

  Canonical choice of "positive" submanifold of a cluster manifold.

- Laurent phenomenon

  Any cluster variable x_i, viewed as a rational function of the variables in a given cluster $\mathbf{x}'$, is a Laurent polynomial.

  It is conjectured that has *positive* coefficients.

- Poisson and symplectic structures

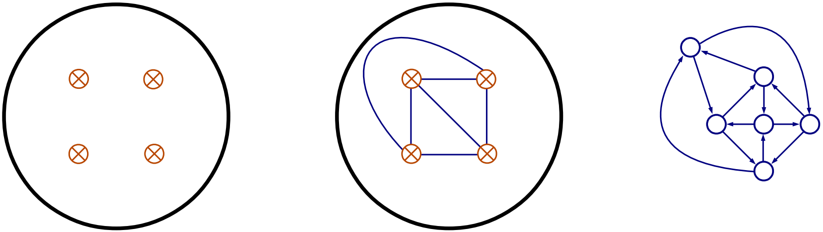
$$\omega = b_{ij} dx_i \wedge dx_j$$

$$\{z_i, z_j\} = b_{ij} z_i z_j \qquad \text{extend by Liebniz} \qquad \text{(log canonical)}$$

They are invariant under mutations.

Example: (decorated) Teichmüller theory

Decorated Teichmüller space:



Penner coordinates $f(e)$ on $\widetilde{\mathcal{T}}_g^s$	Cluster variables x_i
Flip on edge	Mutation
Thurston coordinates $e^{g(e)}$ on $\mathcal{T}_g^s$	Dual cluster variables z_i
1	No coefficients – $\mathbb{P}_0 = \{1\}$

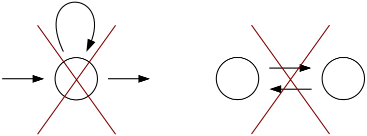
Obs: these are BPS quivers of class $\mathcal{S}$ theories

From dualities to cluster algebras

Quivers and cluster algebras

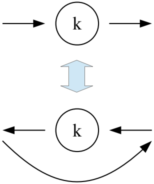
Seiberg-like dualities connect 2d $\mathcal{N} = (2,2)$ quiver gauge theories and cluster algebras.

Class of quiver gauge theories:
no 1-cycles nor 2-cycles.



Apply Seiberg-like duality to a node k

- Quiver: almost CA action



Never generate 1-cycles.

Can generate 2-cycles.

Assume that all 2-cycles are accompanied by *quadratic* superpotential $W \supset X_1 X_2$

$\rightarrow$ integrate them out.

Quivers and cluster algebras

- Coefficients.

Transformation of ranks: $N' = \max(N_f, N_a) - N$

Tropical semifield $\mathbb{P}$ $(\cdot, \oplus)$: $u^a \cdot u^b = u^{a+b}$, $u^a \oplus u^b = u^{\max(a,b)}$

u is formal variable

Ranks: $r_i = u^{N_i}$

$$r'_j = \begin{cases} r_k^{-1}\left(\prod_i r_i^{[b_{ij}]_+} \oplus \prod_i r_i^{[-b_{ij}]_+}\right) & j = k \\ r_j & \text{otherwise} \end{cases}$$

Beta-functions: $y_j \equiv \prod_i r_i^{b_{ij}} \quad \Rightarrow \quad \boxed{y_j = u^{\beta_j}}$

y_j transform as CA coefficients. u interpreted as ratio of energy scales.

Quivers and cluster algebras

- Dual cluster variables. Transformation of FI parameters $z \simeq e^{-t}$:

$$\begin{array}{llll} N_a < N_f : & z_a \to z_a & z \to z^{-1} & z_f \to z_f z \\ N_a = N_f : & z_a \to z_a(1+z) & z \to z^{-1} & z_f \to z_f \frac{z}{1+z} \\ N_a > N_f : & z_a \to z_a z & z \to z^{-1} & z_f \to z_f \end{array}$$

From cluster variables define: $z_j = \prod_i x_i^{b_{ij}}$

$$z'_j = \begin{cases} z_k^{-1} & j = k \\ z_j z_k^{[b_{kj}]_+} \left(\frac{y_k}{y_k \oplus 1} z_k + \frac{1}{y_k \oplus 1} \right)^{-b_{kj}} & \text{otherwise} \end{cases}$$

- Conformal case — $y_i \equiv 1$:

Transformation of FIs exactly reproduces CA

- General case:

The transformation rules are the u^0 term in the expression.
Can be extracted taking $u \to \infty$ limit.

$\to$ Cluster algebra encodes the transformations for every possible choice of ranks.

The superpotential

We assumed that whenever Seiberg-like duality generates a 2-cycle, this is removed by suitable *quadratic* superpotential term W_{quad}.

Highly non-trivial!

Q: given a quiver, there $\exists$ an R-symmetric superpotential W such that $\forall$ sequences of mutations, 2-cycles are always "massive"?

Such W: non-degenerate graded potential

Theory: non-degenerate if it admits such a potential.

- It is easy to produce examples of degenerate theories
- Complete classification of non-degenerate theories is *not* known
- E.g. of non-degenerate quivers:
  - loop-less quivers
  - quivers dual to ideal triangulations of marked Riemann surfaces.

The Q-polynomial

Twisted chiral ring of the quiver: $\quad Q_j(x) = \det(x - \sigma_j)$

$$\prod_i Q_i(x)^{[b_{ji}]_+} + i^{N_a(j)-N_f(j)} z_j \prod_i Q_i(x)^{[-b_{ji}]_+} = C_j(z_j)\, Q_j(x)\, T_j(x) \qquad \forall j$$

Under duality, identify $Q'_k(x) = T_k(x)$.

- Conformal cases: $\quad y_i = 1$.

Take $z_i \in \mathbb{R}_+$: they transform as coefficients.

Q-polynomials transform as cluster variables!

$$Q'_j(x) = \begin{cases} \dfrac{1}{Q_k(x)} \left(\dfrac{z_k}{z_k+1} \prod_i Q_i(x)^{[b_{ik}]_+} + \dfrac{1}{z_k+1} \prod_i Q_i(x)^{[-b_{ik}]_+} \right) & j = k \\ Q_j(x) & \text{otherwise} \end{cases}$$

Some applications

The quantum Kähler moduli space

"Conformal" quivers flow to IR CFTs: NLSM on (non-compact) Calabi-Yau's

Complexified FI parameters z_i control Kähler moduli

Metric on Kähler moduli space computed by [Jockers, Kumar, Lapan, Morrison, Romo 12]

$$Z_{S^2}(t_i, \bar{t}_i) = e^{-K_{\text{Kähler}}(t_i, \bar{t}_i)}$$

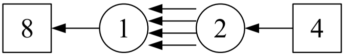
Quantum Kähler moduli space has cluster algebra structure

Also has Poisson structure.

- Compact example: Gulliksen-Negård CY_3

$$\mathcal{X} = \{\phi \in \mathbb{P}^7 \mid \text{rank}(A^a_{4\times 4}\phi_a) \leq 2\} \qquad (h^{1,1}, h^{2,1}) = (2, 34)$$

Can be realized by a $U(1) \times U(2)$ quiver [Jockers, Kumar, Lapan, Morrison, Romo 12]



There is a cubic superpotential that breaks the flavor symmetry.

Quantum integrable systems (spin chains)

Nekrasov-Shatashvili: $\mathcal{N} = (2,2)$ gauge theory $\leftrightarrow$ quantum integrable system

- $\mathcal{N} = (2,2)^*$ SQCD: $\begin{cases} U(N),\ N_f \text{ hypers (fund + antifund), one adjoint} \\ W = \tilde{Q}\Phi Q \end{cases}$

- $SU(2)$ periodic twisted inhomogeneous $\mathrm{XXX}_{\frac{1}{2}}$ spin chain, N_f nodes, sector with $S_z = -\frac{N_f}{2} + N$ (N-particle states)

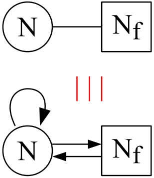
Twisted chiral ring relations $\leftrightarrow$ algebraic Bethe ansatz equations

Quantum integrable systems (spin chains)

Q: Integrable systems for our quivers? Seiberg-like dualities?

Strategy:
- Construct $\mathcal{N} = (2,2)^*$ "quivers"
- Look for a duality
- Limit of parameters $\rightarrow$ $\mathcal{N} = (2,2)$ cycle-free quiver
- Interpret duality in integrable system context

$\mathcal{N} = (2,2)^*$: diagrams with no arrows



"Particle-hole" duality: $U(N) \leftrightarrow U(N_f - N)$

$T^*\mathrm{Gr}(N, N_f) = T^*\mathrm{Gr}(N_f - N, N_f)$

- Limit: It is crucial to expand in the correct vacuum.

Limit is compatible with Seiberg-like duality.

Integrable system: "highly quantum and inhomogeneous limit"

Conclusions

Open directions

- Cluster algebra in systems of Picard-Fuchs equations?

  GKZ-systems, A-systems, ... well-understood only for Abelian theories

- B-side of the story and Hori-Vafa mirror symmetry

- Relation of 2d $\mathcal{N} = (2, 2)$ quivers to other physical systems?

  Teichmüller theory and class $\mathcal{S}$ theories (what is Z_{S^2}? what is $K_{\text{Kähler}}$?)

- Study transformation rules of boundary conditions using Z_{D^2}

Thank you!