Exact Solutions of 2d Supersymmetric gauge theories

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w. Sergei Gukov and Pavel Putrov

2d (0,2) gauge theories

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- 4d N=1 : 4d N=2
- 2d (0,2) : 2d (2,2)

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    - IR fixed points
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  - Gauge invariant d.o.f.: Fermi multiplet Λ

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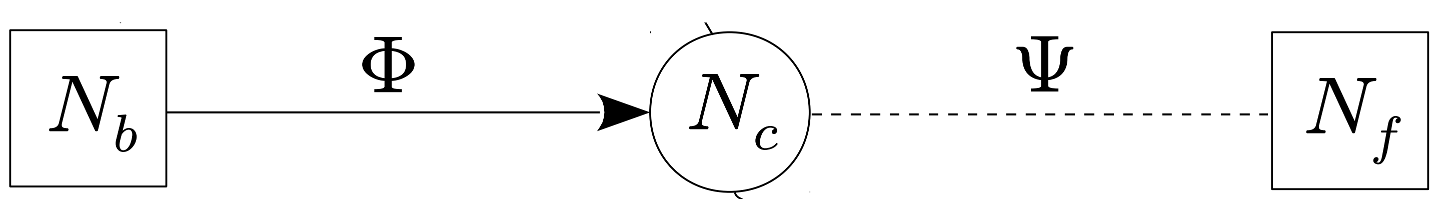

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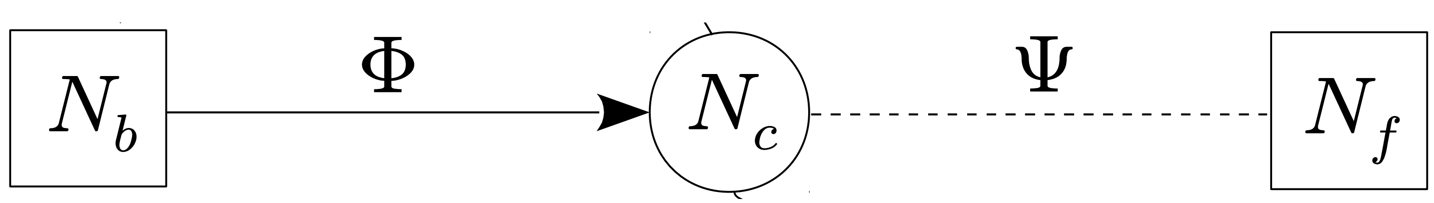

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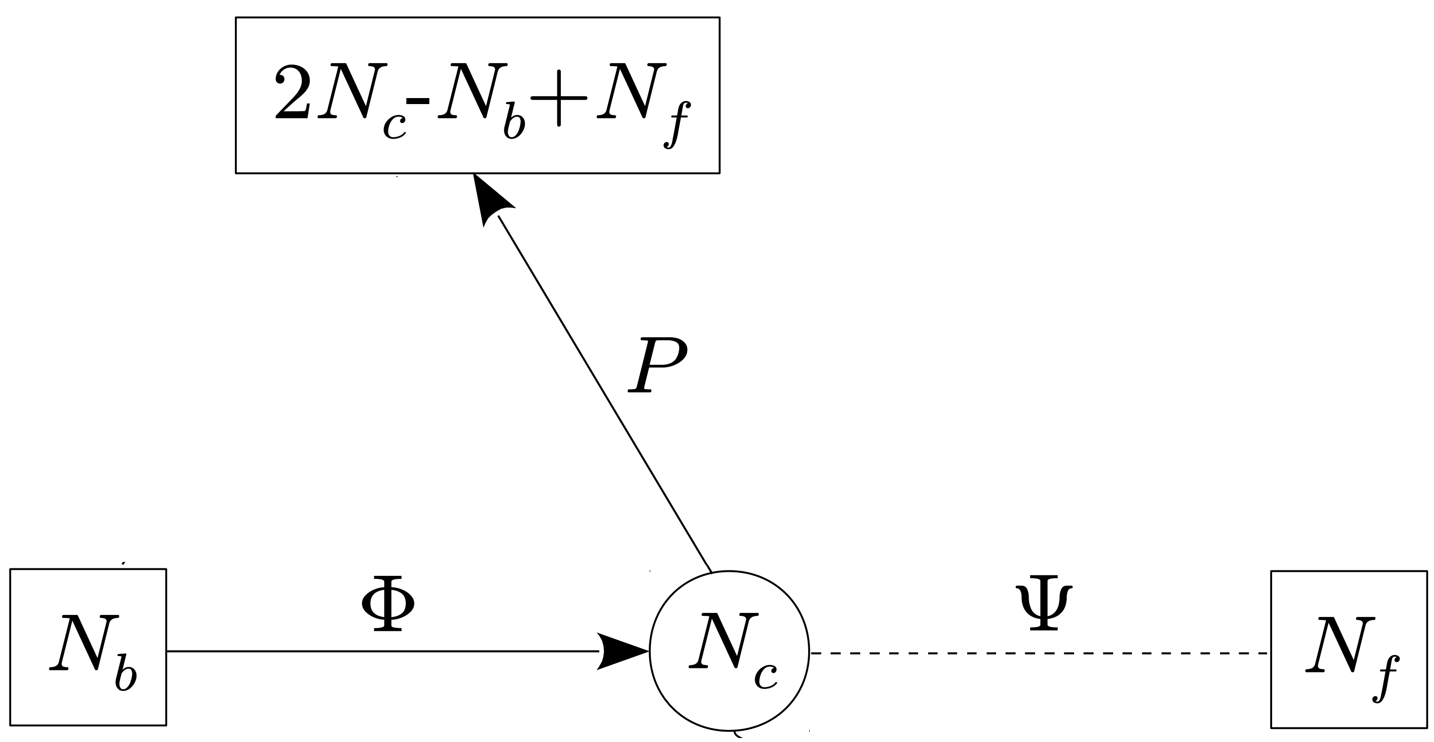

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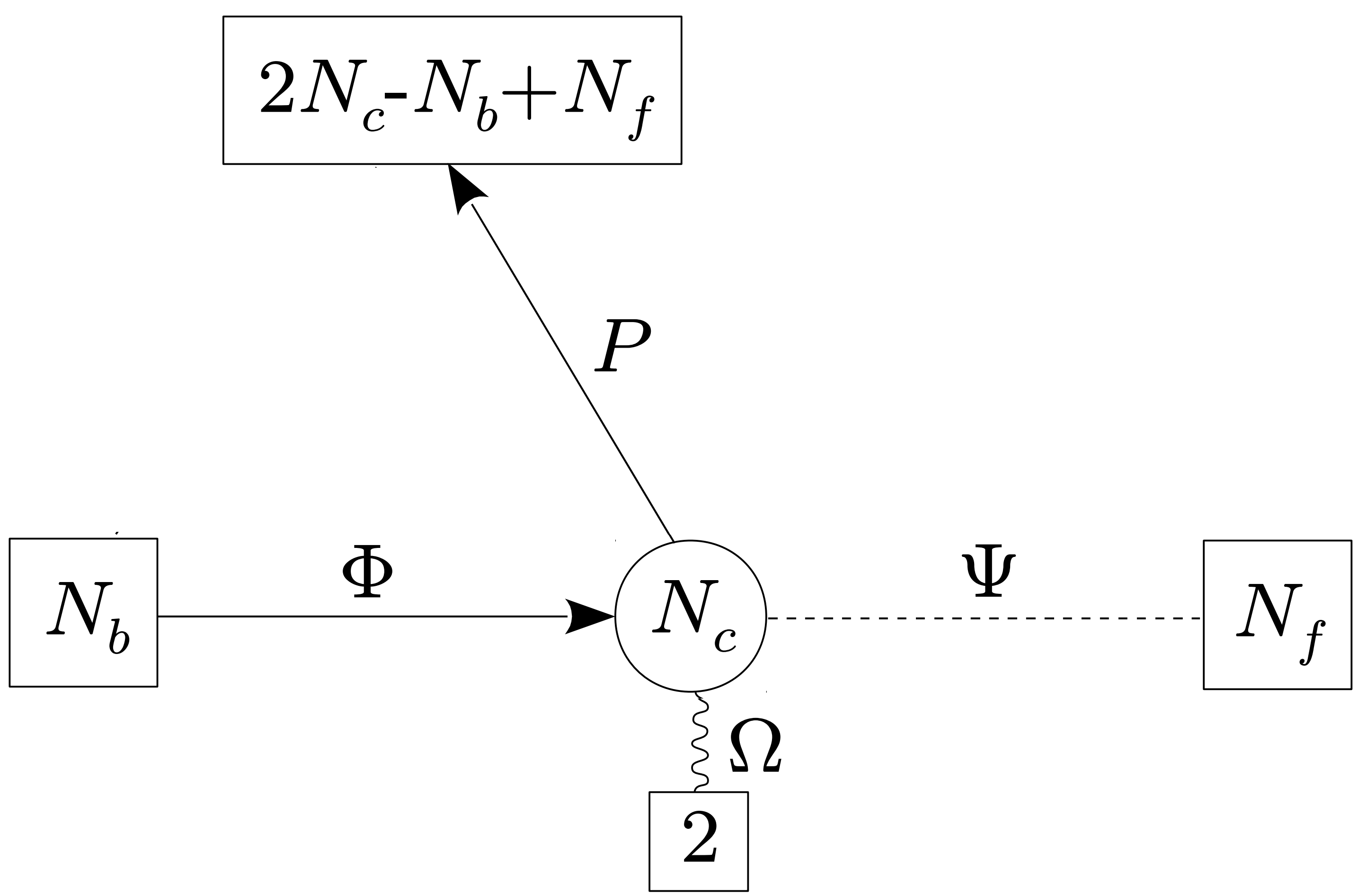

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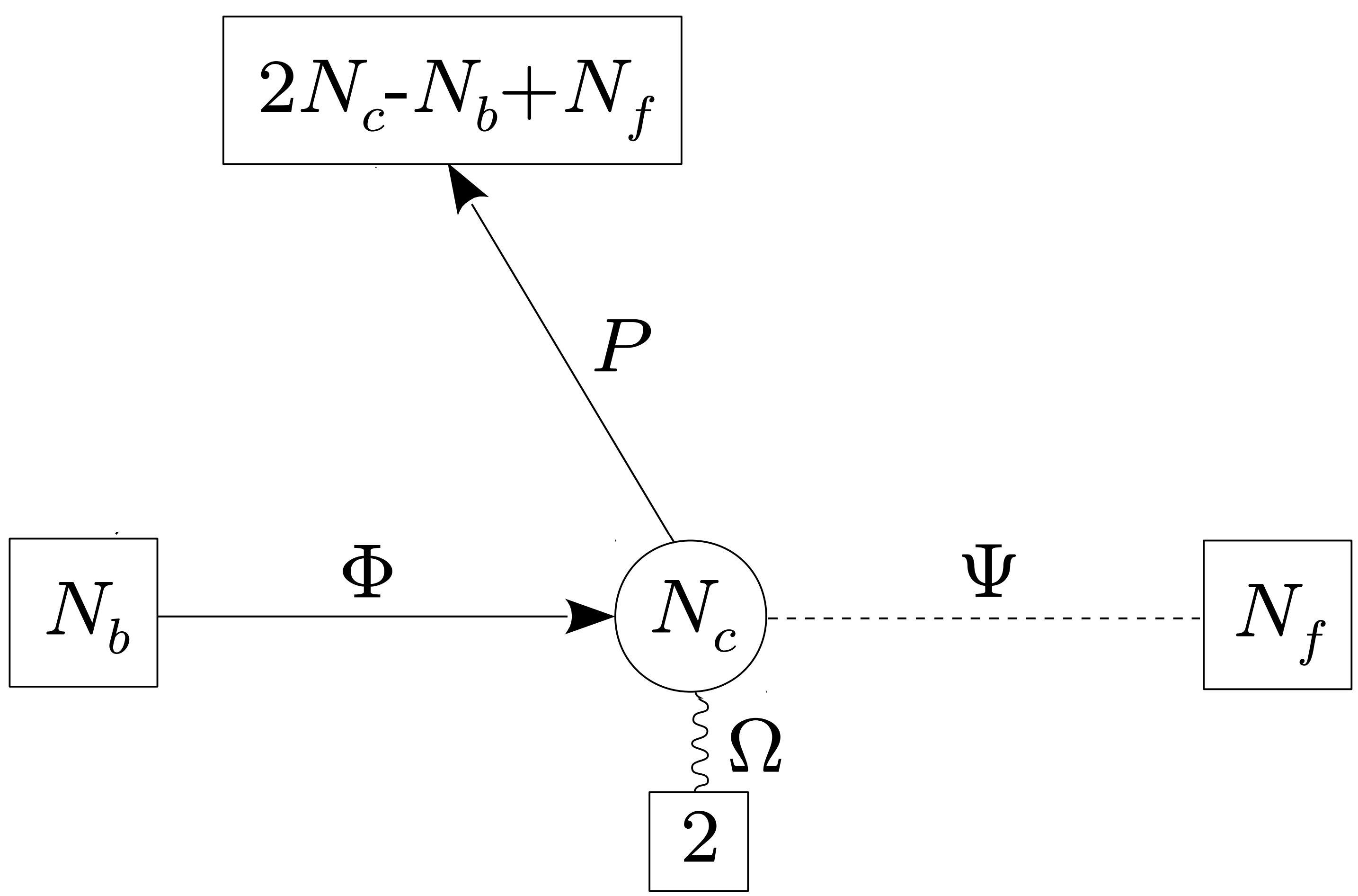

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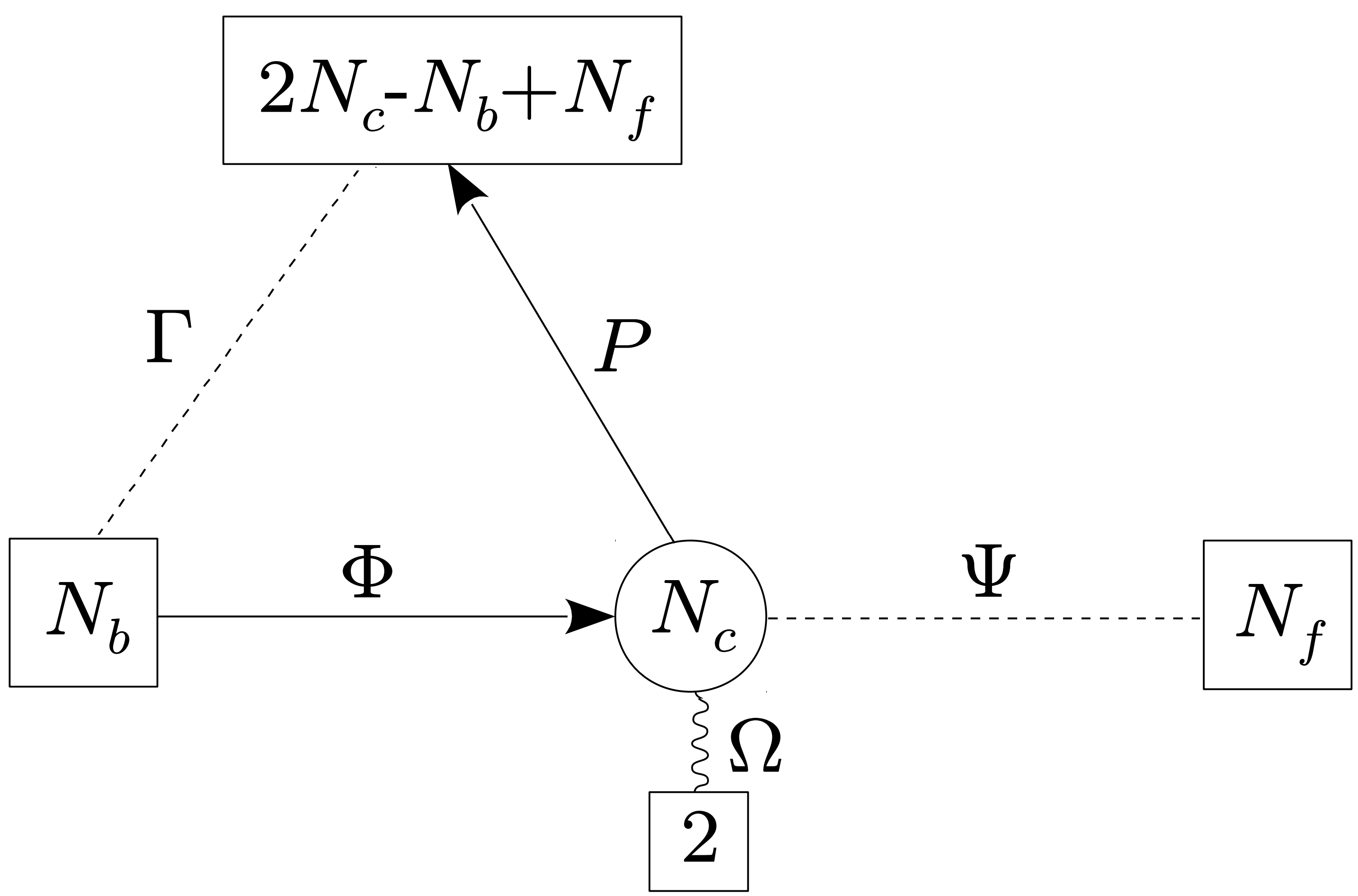

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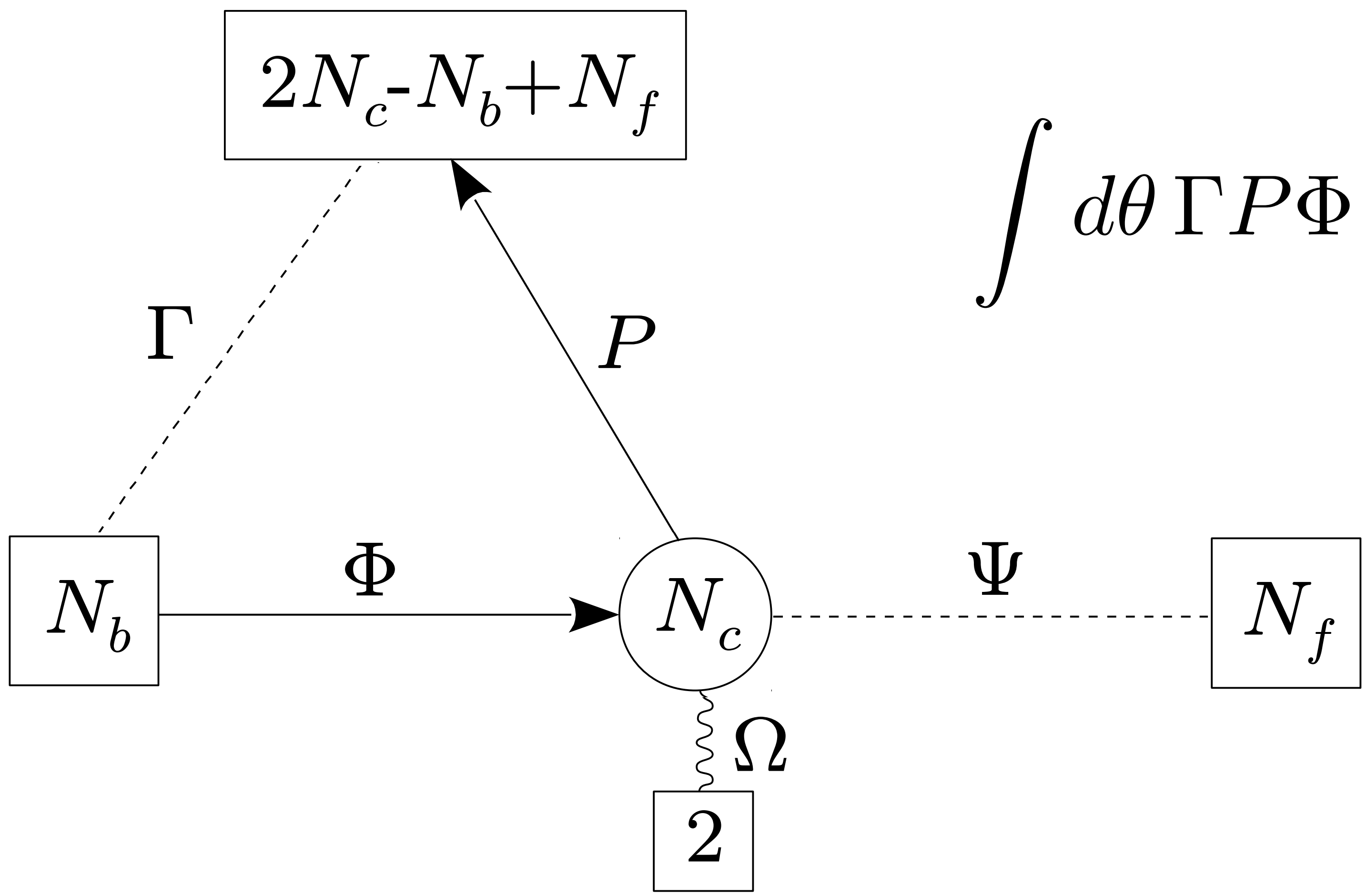
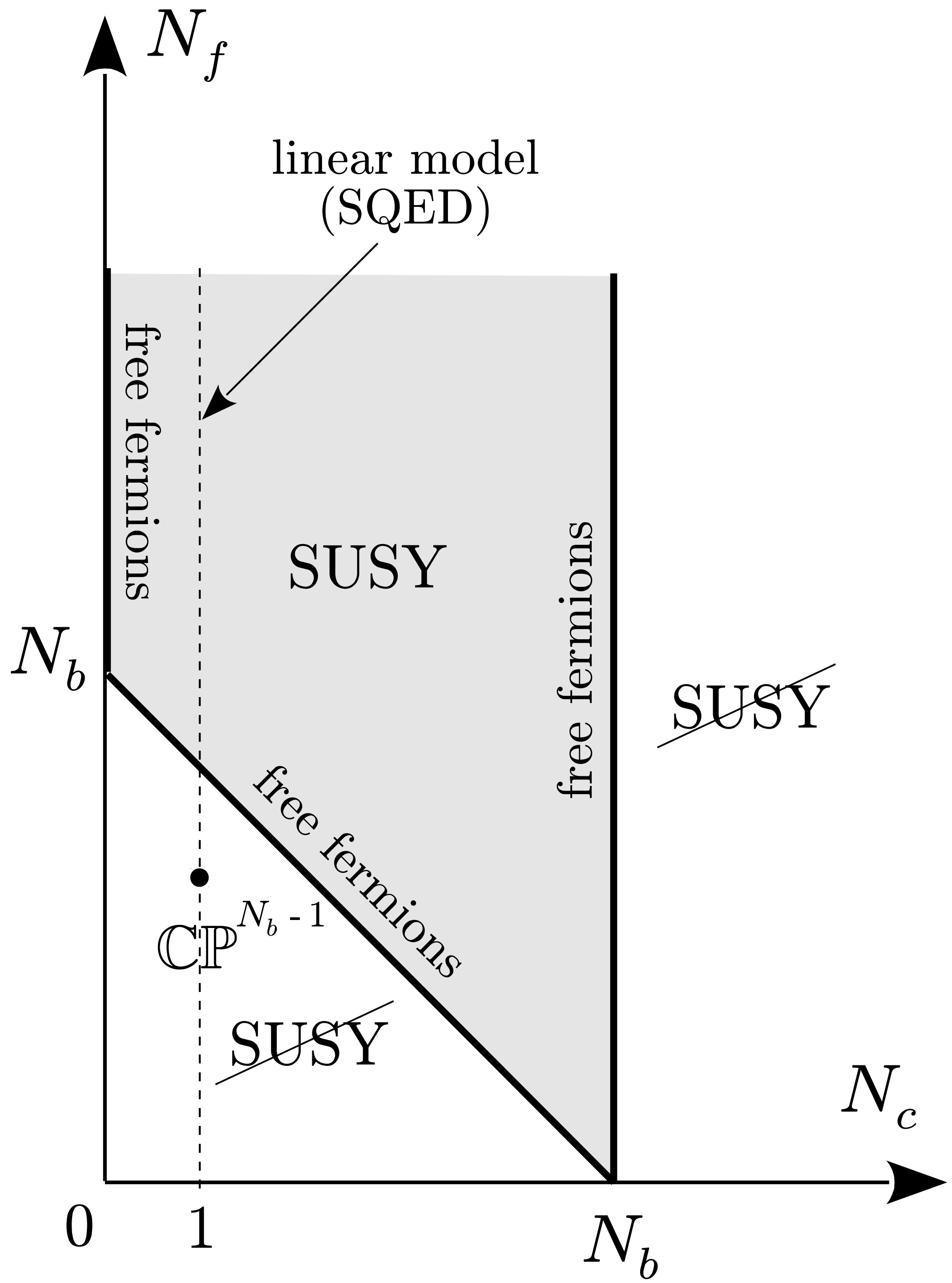


$$\int d\theta\, \Gamma P \Phi + \int d\theta\, \Lambda$$

Phase diagram

Phase diagram



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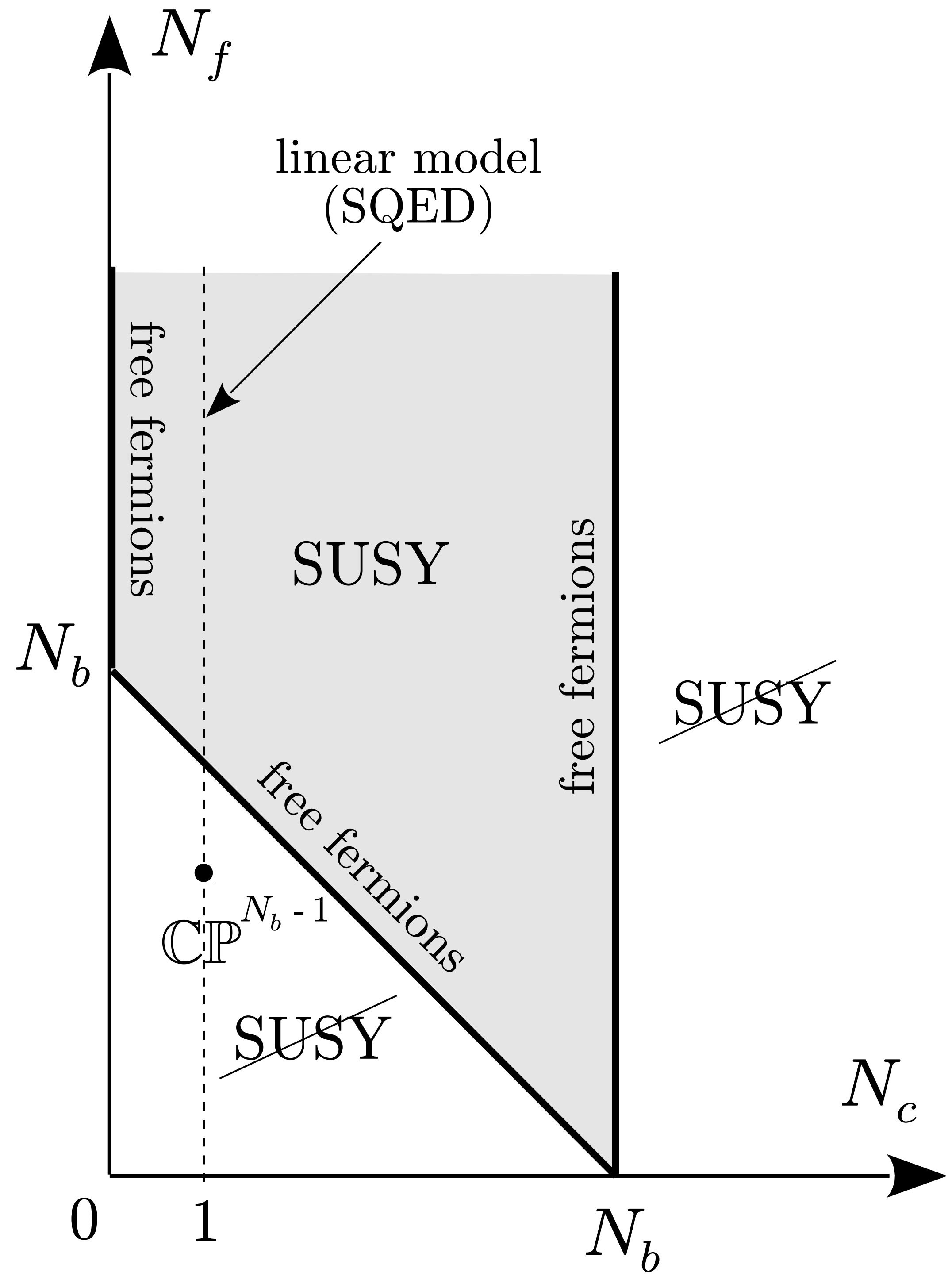
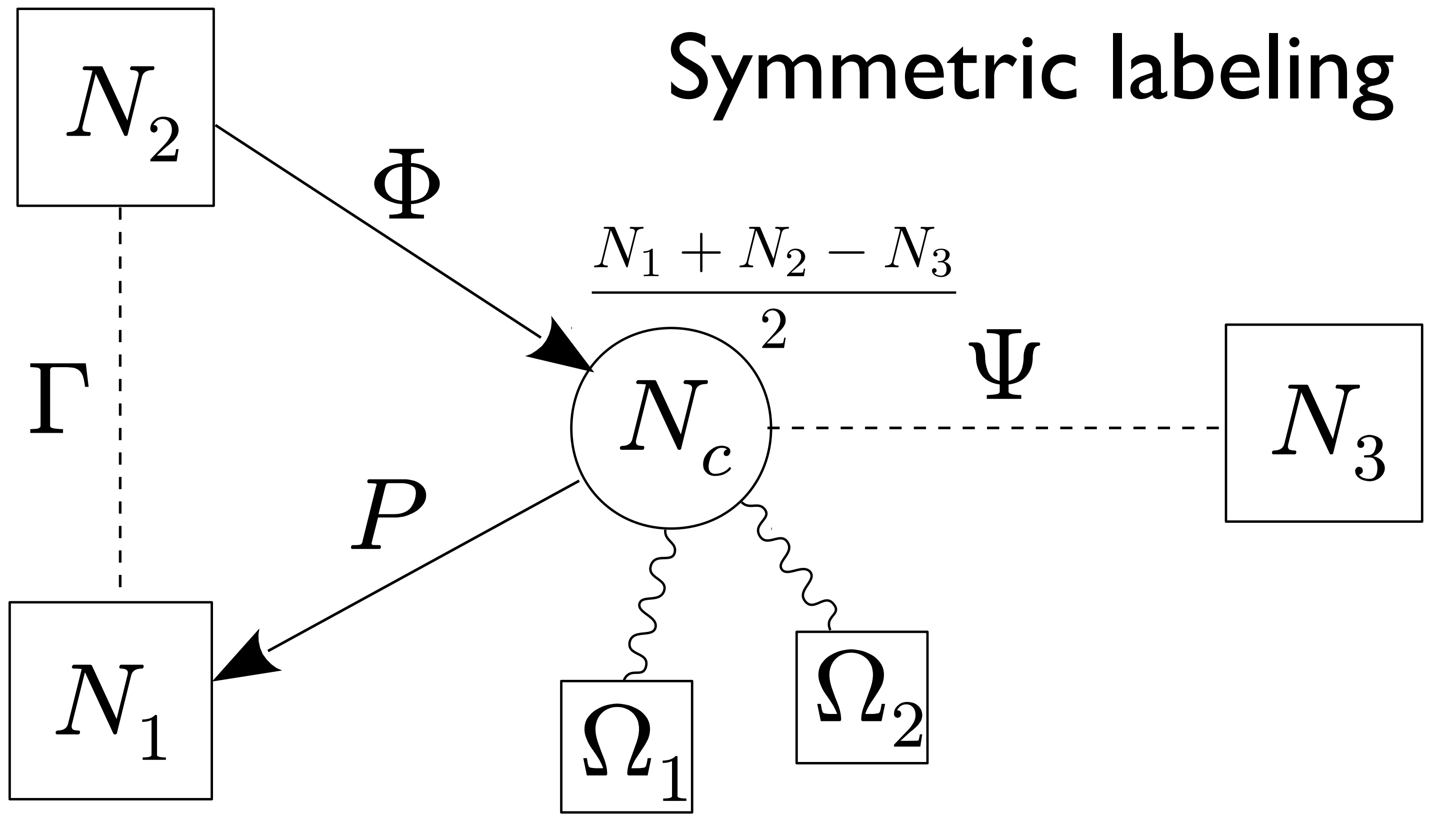


- Produced by superconformal index
- SUSY preserved in the shaded region
- SUSY broken outside
- Gapless theory at all points
- We will focus on the shaded region

(0,2) SQCD

Symmetric labeling



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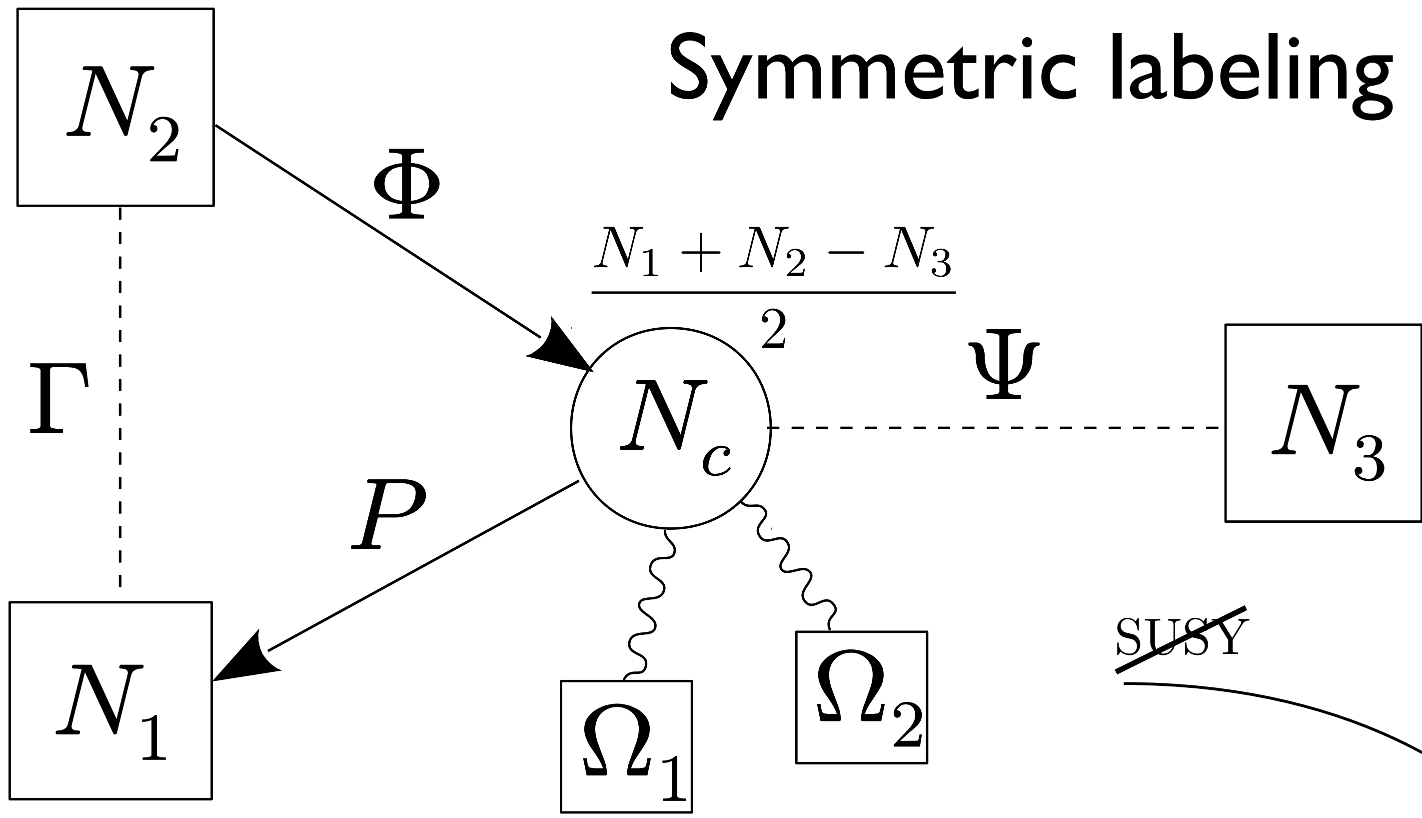

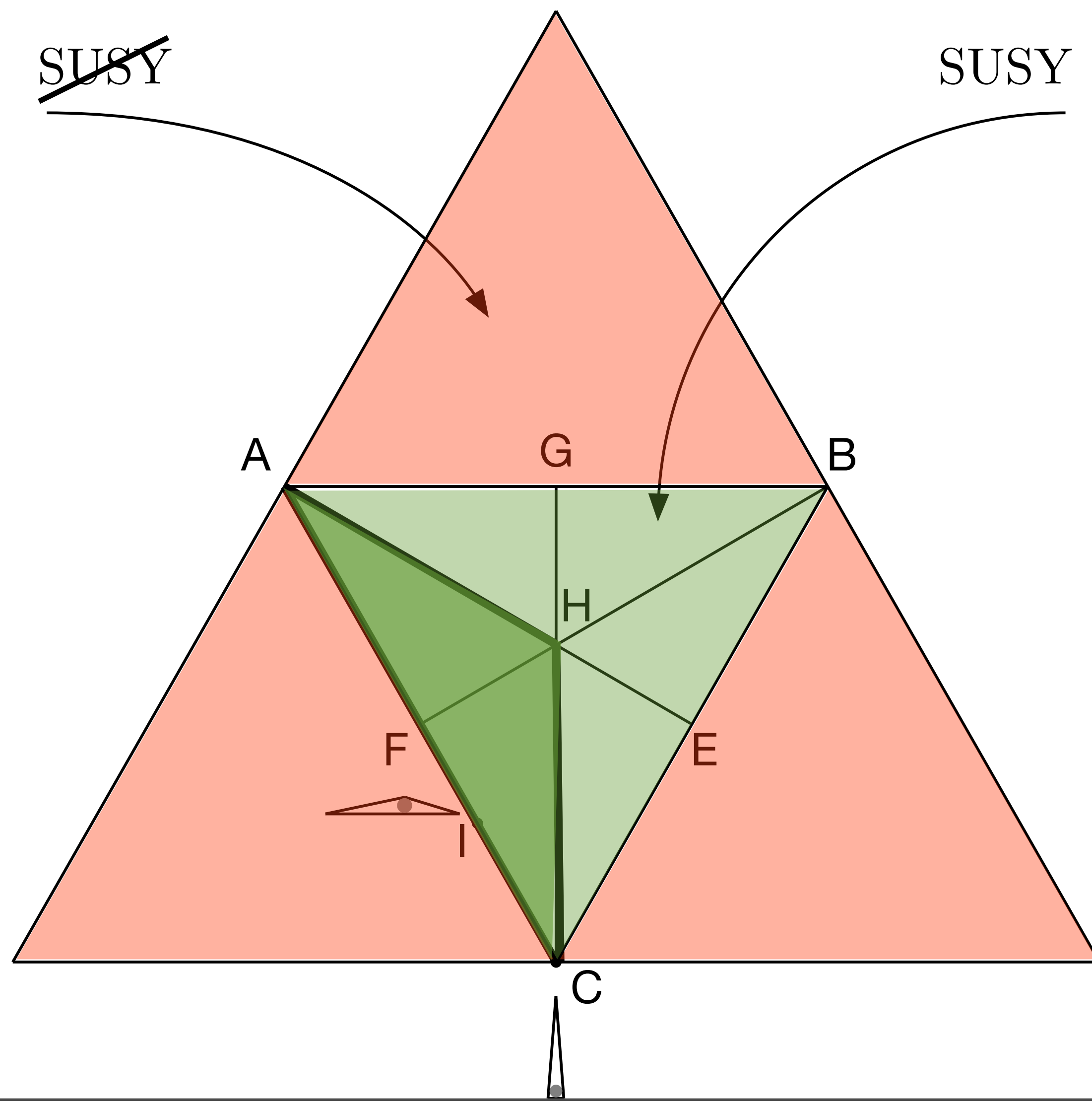

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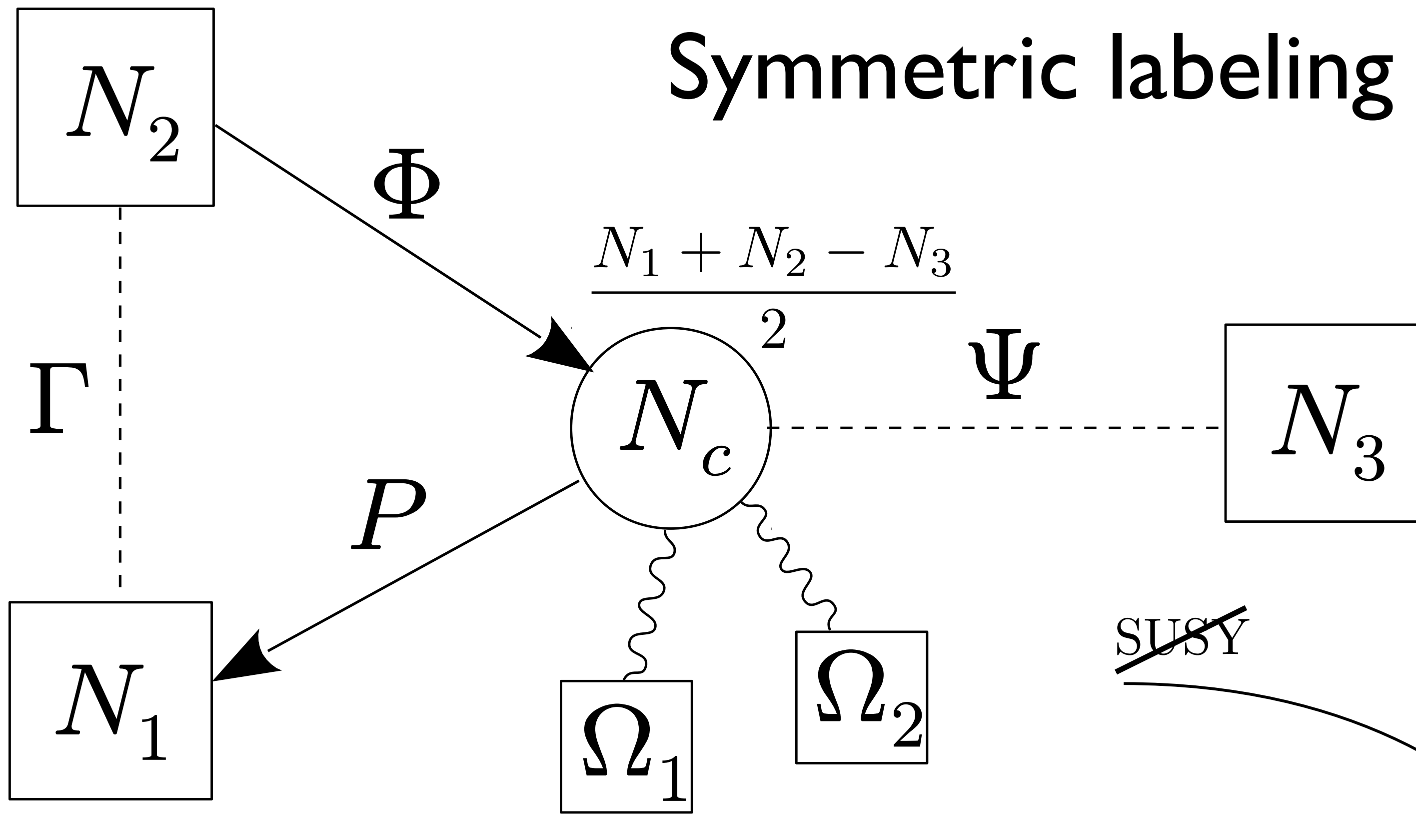
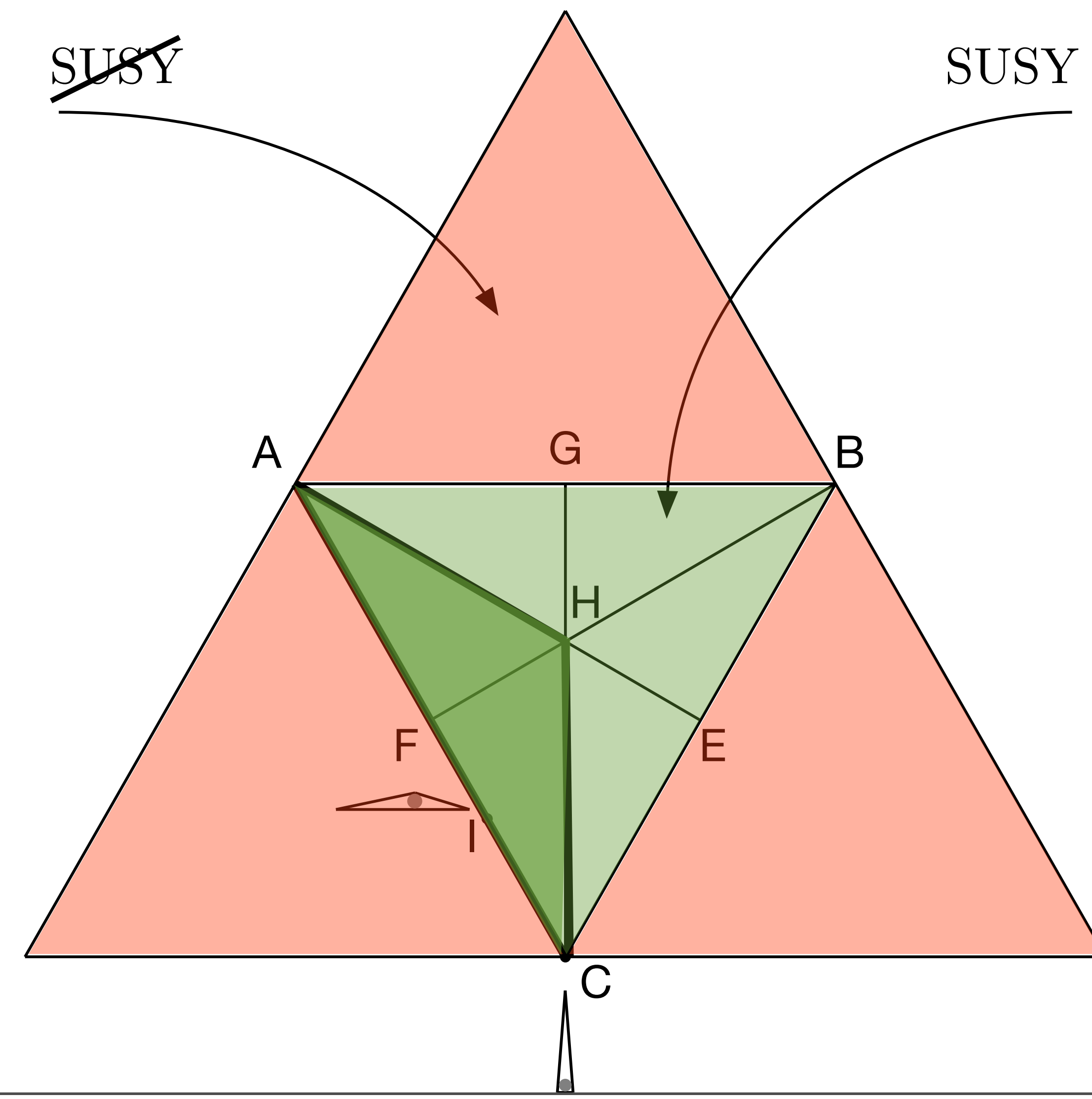


- Symmetric phase diagram
- Triality !



Anomalies

$$\mathrm{Tr}\gamma^3 \mathrm{J}_{\mathrm{SU(N_1)}}\mathrm{J}_{\mathrm{SU(N_1)}} = \frac{N_1+N_2-N_3}{2}T_P(\Box) - N_2 T_\Gamma(\overline{\Box}) = -\frac{1}{4}(-N_1+N_2+N_3)$$

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Abelian symmetries

- Pick a basis so that cross anomalies are zero

	Φ	Ψ	P	Γ	Ω_1	Ω_2
$U(1)_{(1)}$	0	0	1	-1	$-N_1$	0
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$$N \equiv (N_1+N_2+N_3)/2$$

$$n_i \equiv N - N_i$$

Low energy physics

Poincare symmetry

Low energy physics

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Virasoro

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N=2 superconformal

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Affine $\mathfrak{H} \equiv \prod_{i=1}^{3} SU(N_i)_{n_i} \times U(1)_{NN_i}$

Low energy physics

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$$c_{\mathtt{R}} = \frac{3}{4}\frac{(-N_1+N_2+N_3)(N_1-N_2+N_3)(N_1+N_2-N_3)}{N_1+N_2+N_3}$$

$$c_{\mathtt{L}} = c_{\mathtt{R}} - \frac{1}{4}(N_1^2+N_2^2+N_3^2-2N_1N_2-2N_2N_3-2N_3N_1)+2$$

Low energy solution

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Integrable modules

Modules of $\mathfrak{H}$

Modules of N=2

- Sugawara central charge = c_{L}
- Immense simplification: rational CFT
- Modular invariance of the partition function helps fix $\mathcal{H}_{\mathrm{R}}^{\lambda}$

NS-NS partition function

$$Z(\tau,\xi_i;\bar\tau,\bar\eta) := \mathrm{Tr}_{\mathcal{H}}\, e^{2\pi i(\tau L_0+\sum_i \xi_i H_0^i-\bar\tau \bar L_0-\bar\eta\, \bar J_0)}$$

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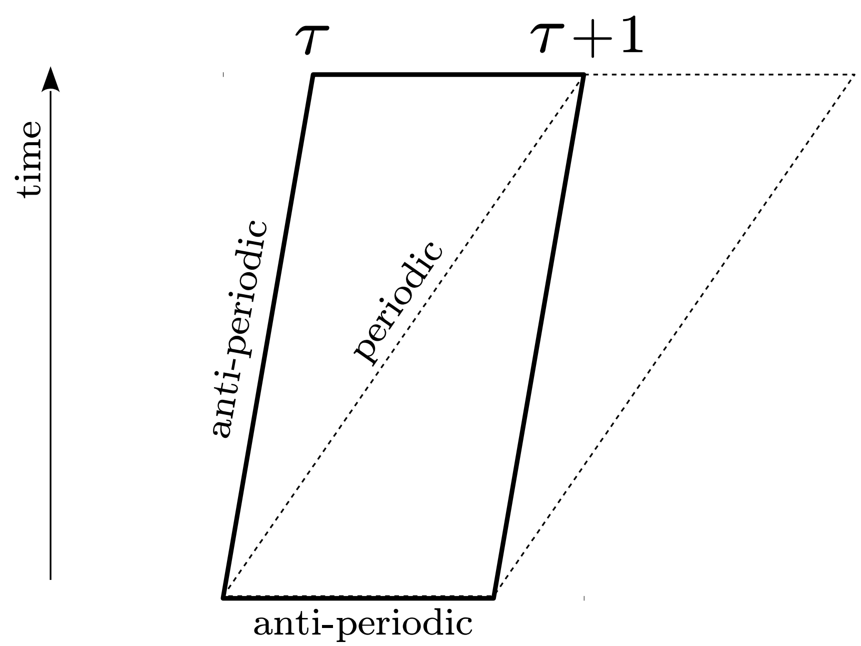
R symmetry Cartan



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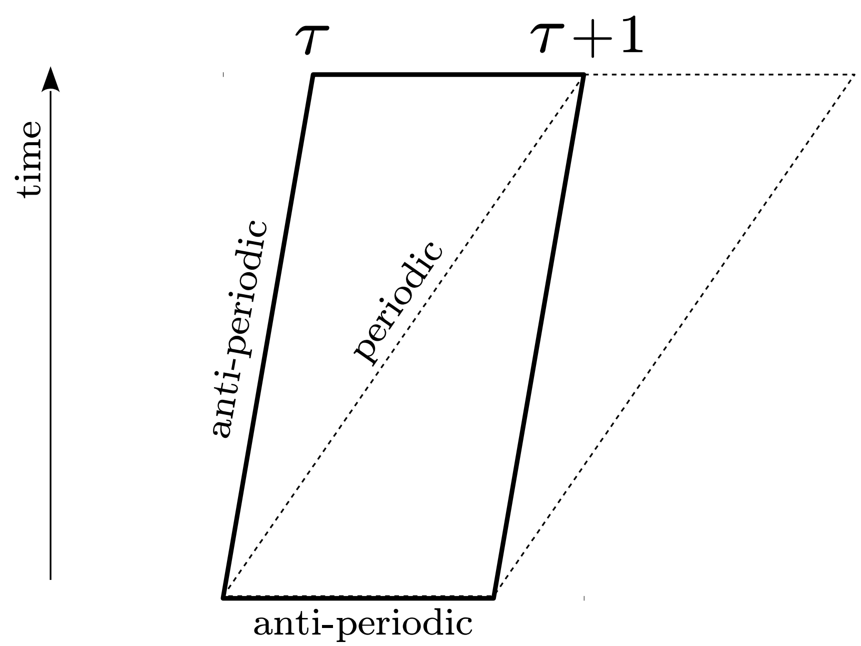

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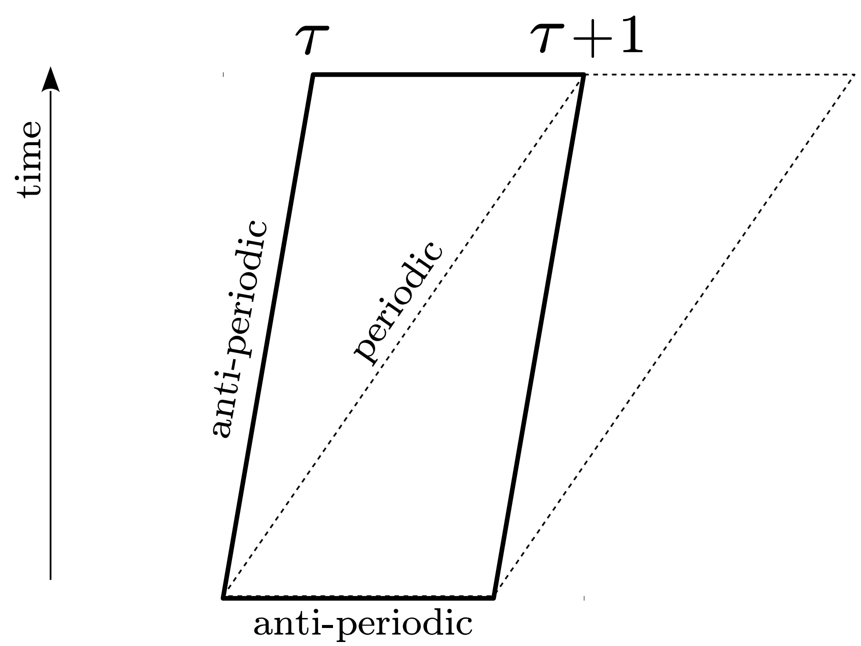

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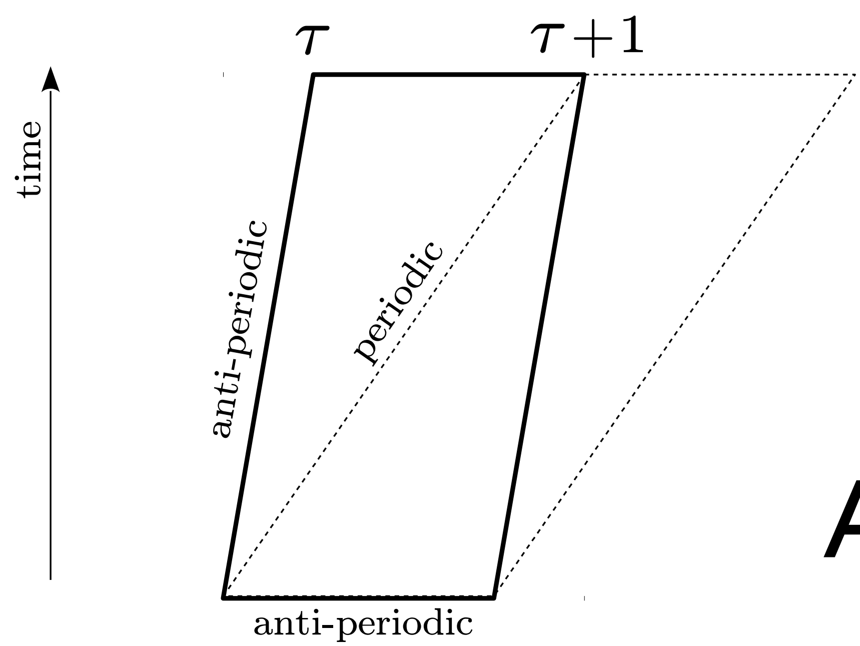

Affine character

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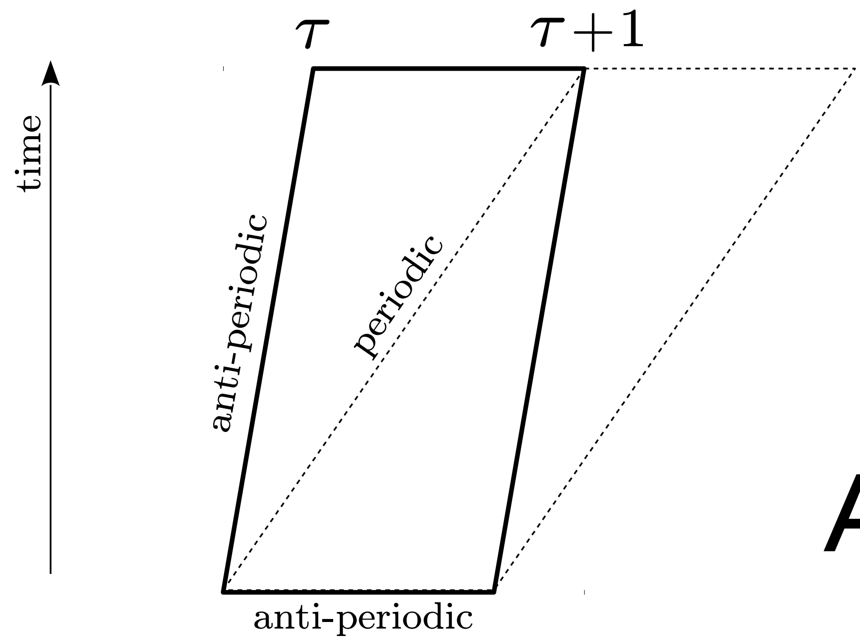

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N=2 character

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- K is NOT the anti-holomorphic affine character of $\mathfrak{H}$
- Note that characters of level-rank dual $\mathfrak{H}^t$ transform with $\bar{S}$

$$\mathfrak{H}^t = \prod SU(n_i)_{N_i} \times U(1)_{N n_i}$$

To summarize

- K is an N=2 character with central charge cR
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- K is a character of the Kazama- Suzuki coset $[\mathfrak{G}]/[\mathfrak{H}^t]$

(For appropriate $\mathfrak{G}$)

Intermission: SUSY WZW

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- Matching the right-moving central charge with that of the coset:

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$U(1)_{N^2}$ module

$$C^{\Lambda,\upsilon}_{\lambda^t}$$

$SO(\dim \mathfrak{G}/\mathfrak{H}^t)$ module

$\mathfrak{H}^t$ module

Solution

- We pick modular invariant combinations (Λ_0, υ_0)

Then $$K_\lambda = \sum_{\lambda^t} L_{\lambda,\lambda^t}\, C_{\lambda^t}^{\Lambda_0,\upsilon_0}$$

has all the desired properties!

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$$\mathcal{H} = \bigoplus_{\lambda,\lambda^t} L_{\lambda,\lambda^t}\, \mathcal{H}^{\lambda}_{\mathrm{L}} \otimes \mathcal{H}^{\lambda^t}_{\mathrm{R}}$$

Module of $\mathfrak{H}$

Module of $[\mathfrak{G}]/[\mathfrak{H}^t]$

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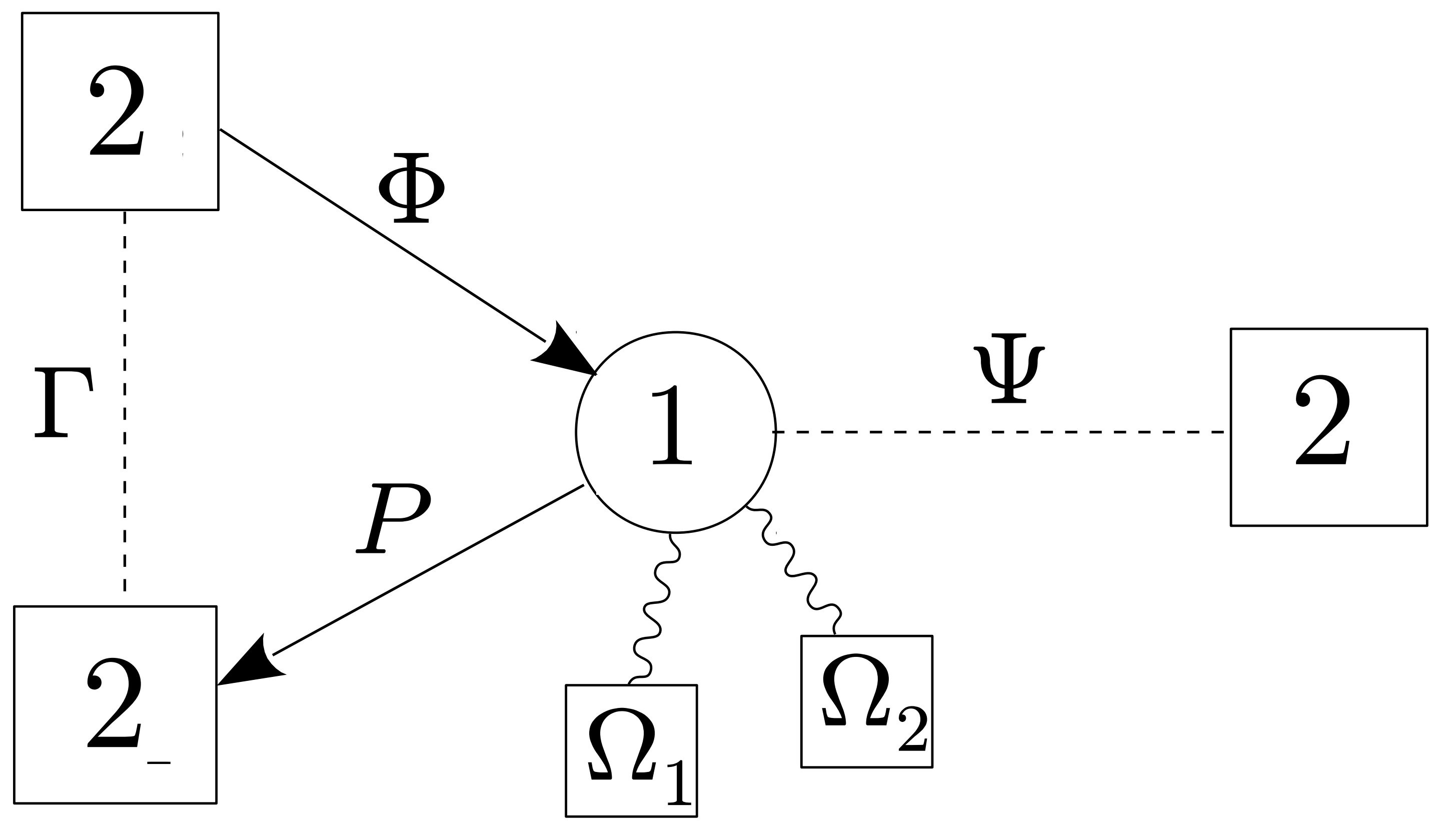
Module of $\mathfrak{H}$

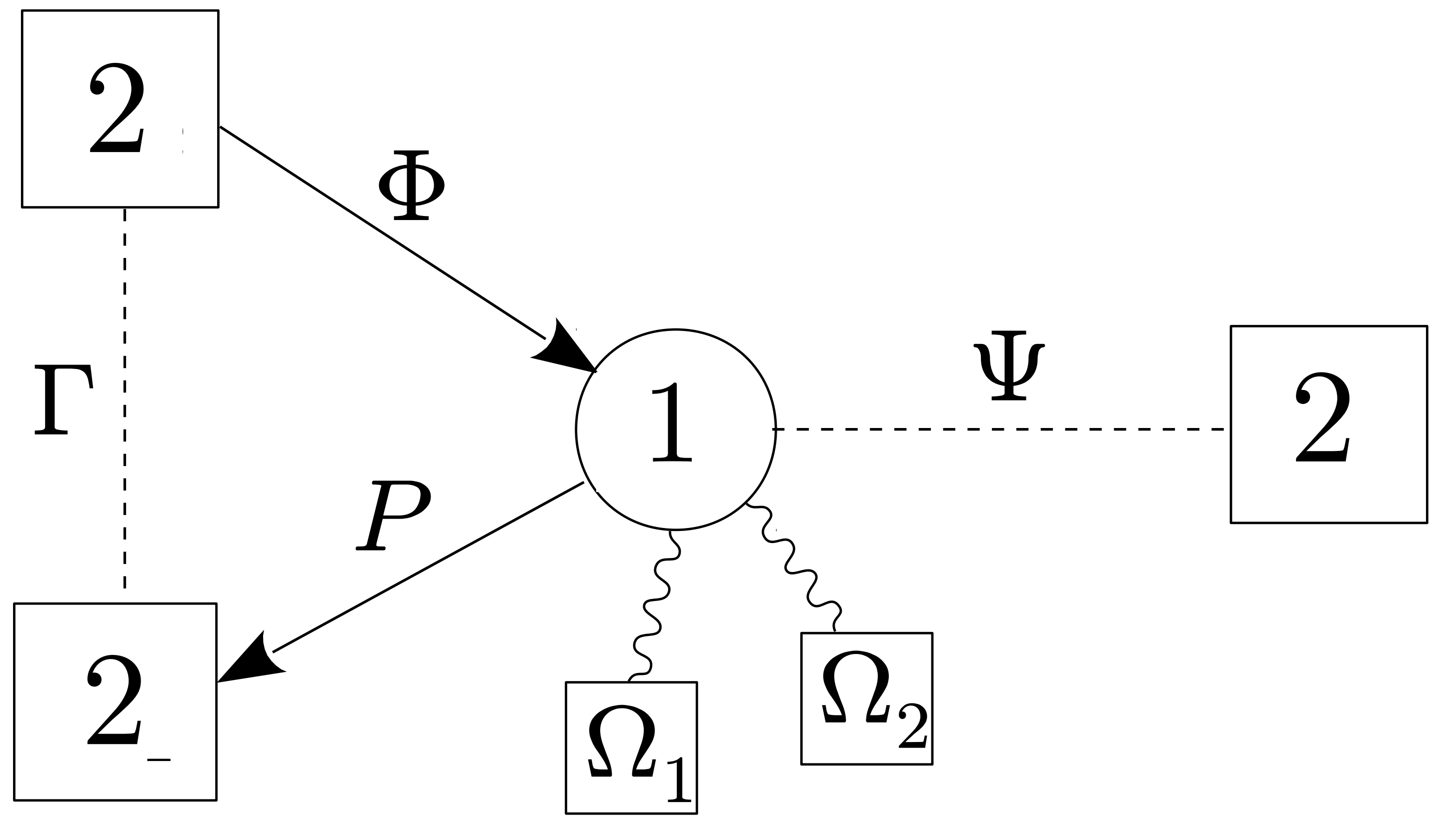
Module of $[\mathfrak{G}]/[\mathfrak{H}^t]$

- Matches with the UV computation of the index

Example



Example



$$\mathfrak{H} = \Big(SU(2)_1 \times U(1)_6 \Big)^3 \qquad [\mathfrak{G}]/[\mathfrak{H}^t] = [U(3)]_3/[U(1)_3]^3$$

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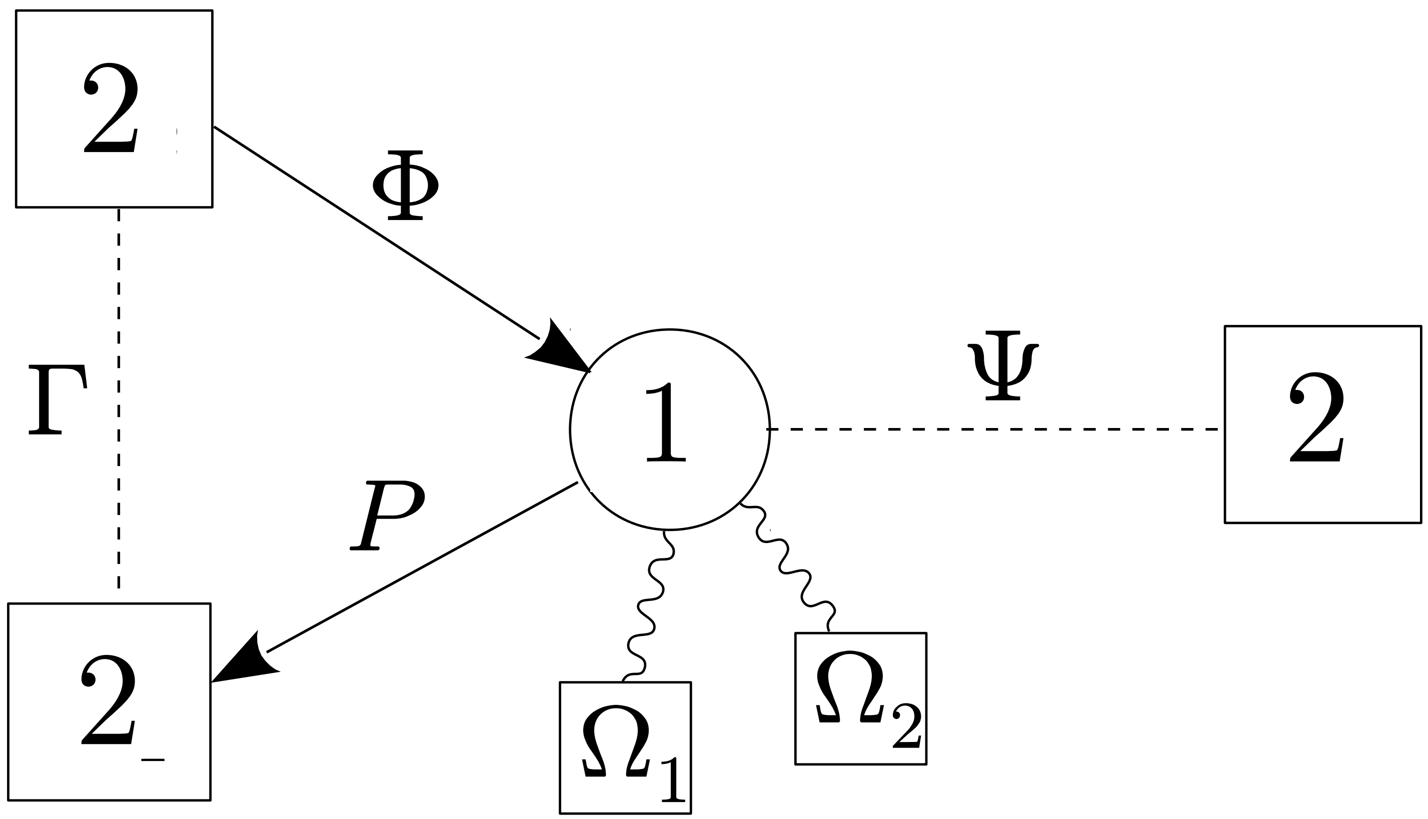

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c=1 minimal model

$$
\begin{aligned}
Z_{\mathcal{T}_{222}} &= \chi^{\mathcal{N}=2}_{(0,0)}(\overline{\tau},\overline{\eta})\Big(\Xi_{0,0,0}(\tau)+\Xi_{1,1,1}(\tau)+\Xi_{-1,-1,-1}(\tau)\Big)\\
&+\chi^{\mathcal{N}=2}_{(\frac{1}{6},\frac{1}{3})}(\overline{\tau},\overline{\eta})\Big(\Xi_{1,0,-1}(\tau)+\Xi_{-1,1,0}(\tau)+\Xi_{0,-1,1}(\tau)\Big)\\
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where

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Remarkably

$$
Z_{\mathcal{T}_{222}} = \chi^{\mathcal{N}=2}_{(0,0)}(\overline{\tau},\overline{\eta})\chi^{(\mathrm{E}_6)_1}_{\bullet}(\tau,\xi_i)+\chi^{\mathcal{N}=2}_{(\frac{1}{6},\frac{1}{3})}(\overline{\tau},\overline{\eta})\chi^{(\mathrm{E}_6)_1}_{\square}(\tau,\xi_i)+\chi^{\mathcal{N}=2}_{(\frac{1}{6},-\frac{1}{3})}(\overline{\tau},\overline{\eta})\chi^{(\mathrm{E}_6)_1}_{\overline{\square}}(\tau,\xi_i)
$$

Triality and enhancement

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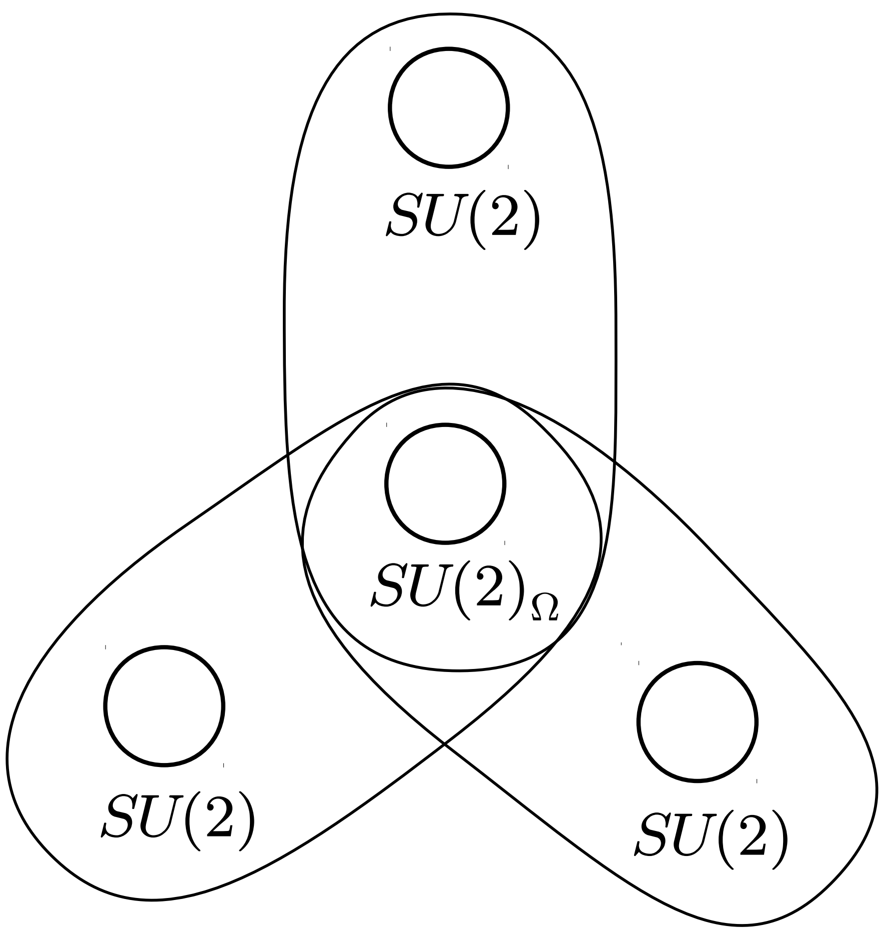

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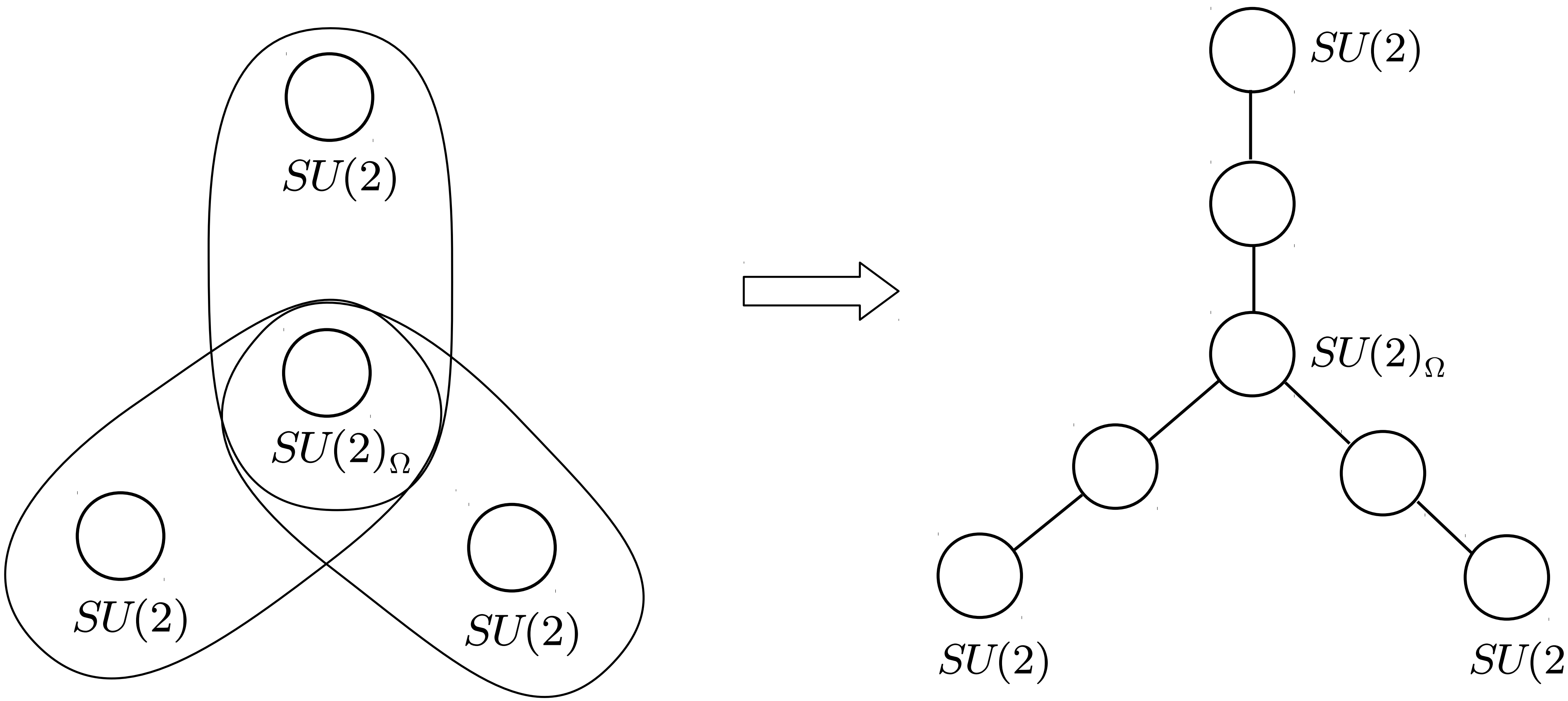
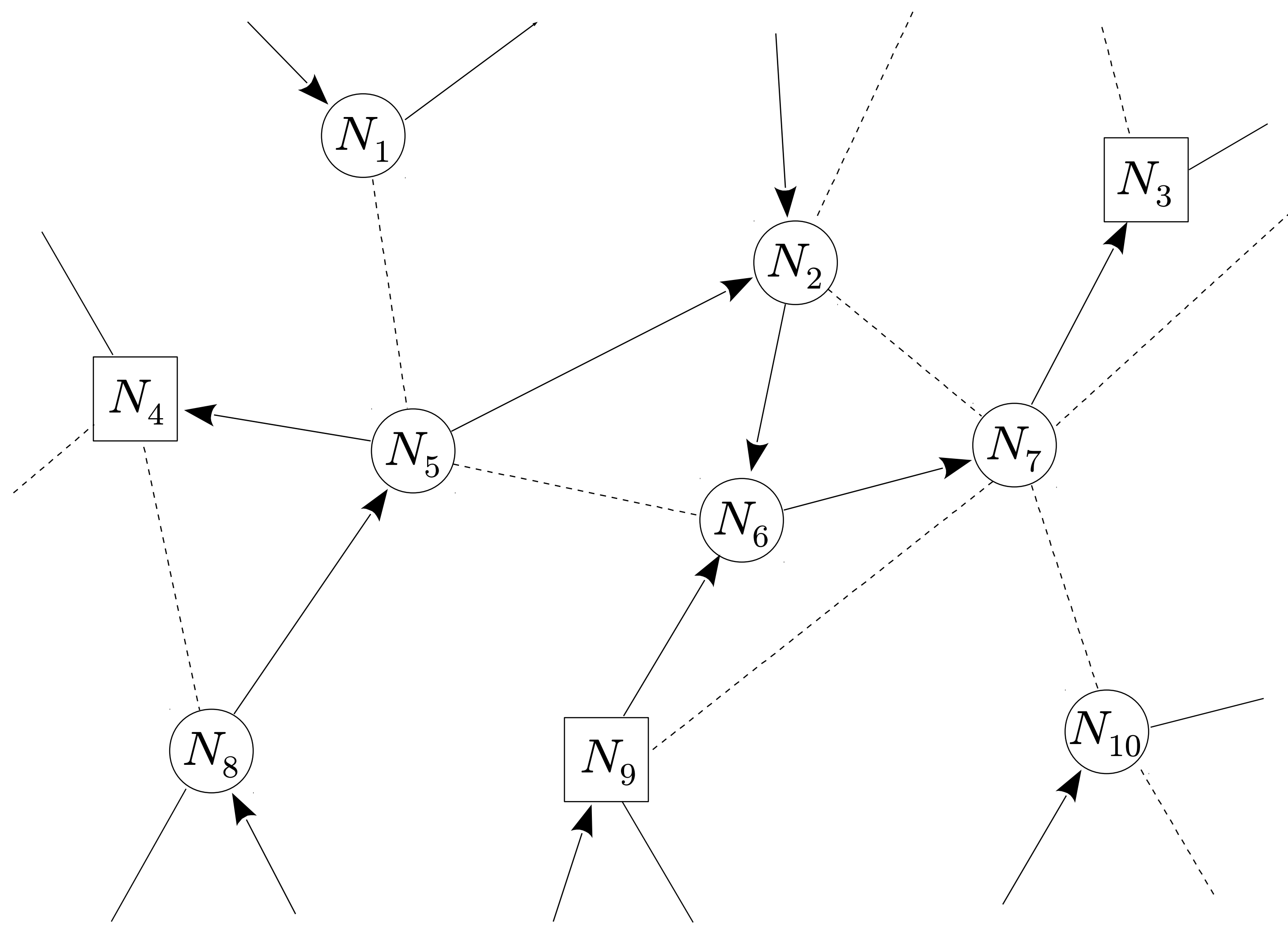


Solution to a general quiver



Thank you!