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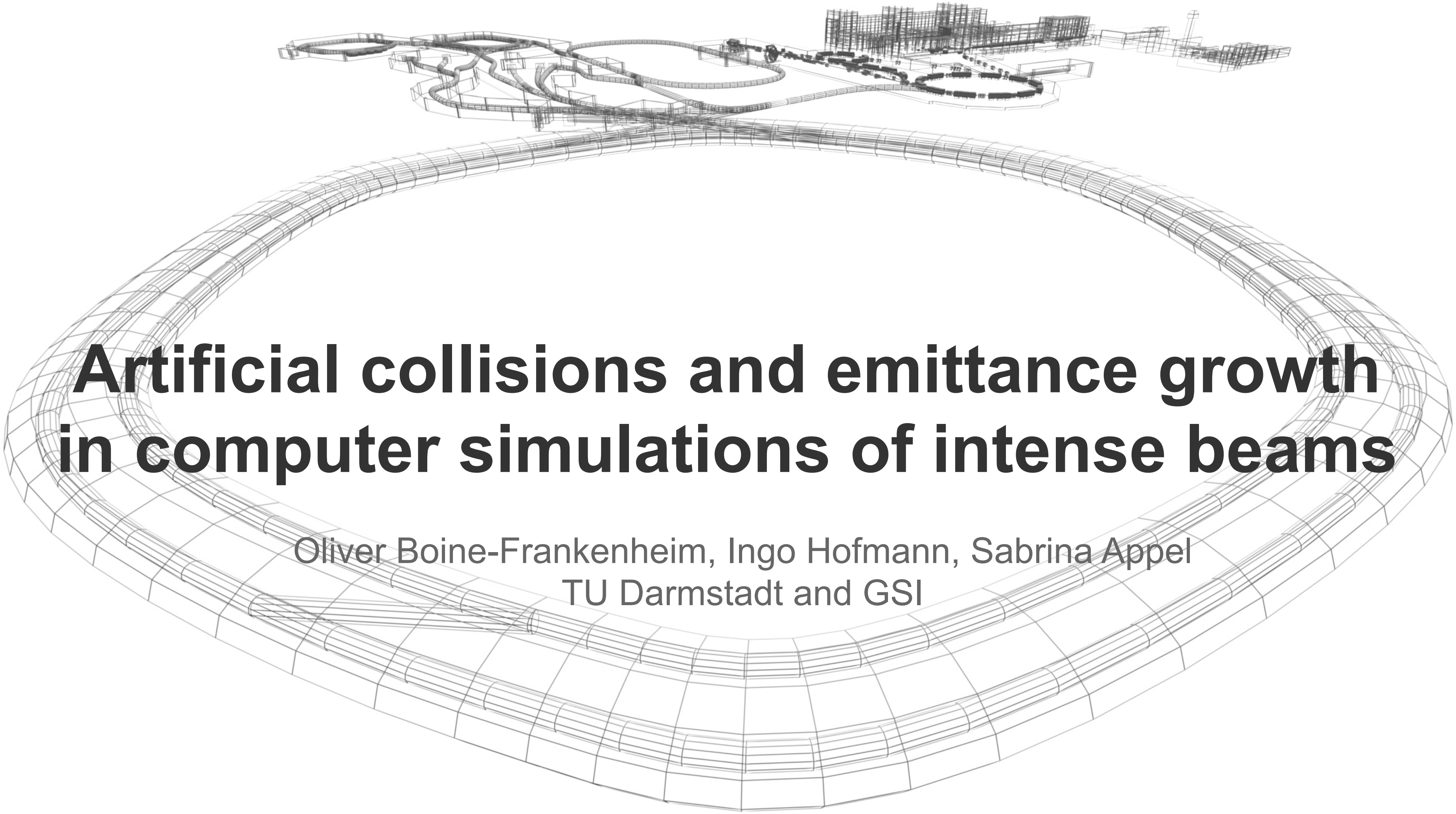

Artificial collisions and emittance growth in computer simulations of intense beams

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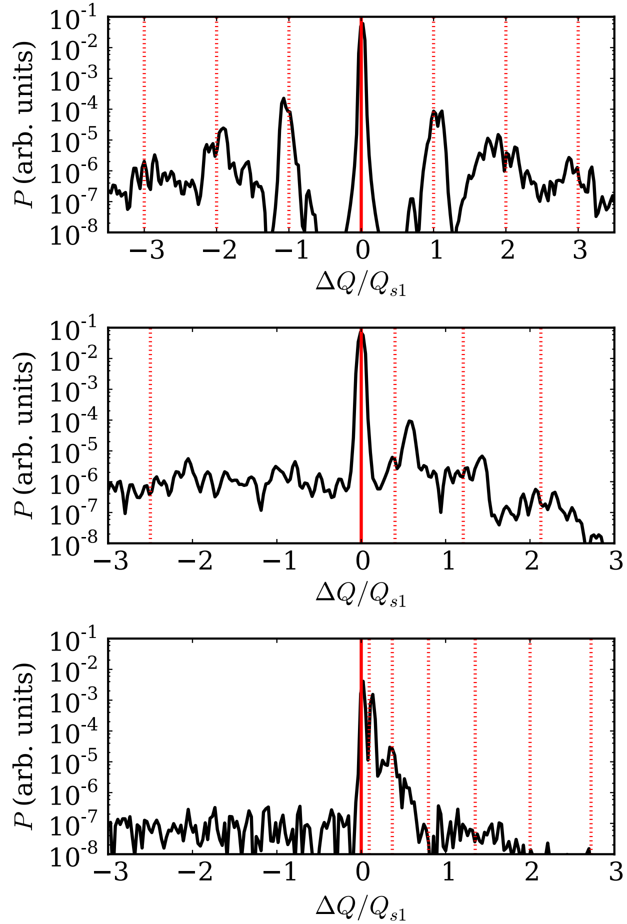
TU Darmstadt and GSI

Artificial Schottky noise in computer beams

- Intrinsic feature of Particle-In-Cell (PIC) simulations using M macro-particles, grids and Poisson or Maxwell solvers.
- Because the real particle number N>>M the noise in computer beams is much stronger than the Schottky noise in real beams.
- The noise in computer beams has been **used to predict the Schottky noise spectra in real beams** (see example).
- **Like in real beams:** Schottky noise <-> IBS and diffusion.
- It would be useful to have **scaling laws** for the resulting emittance growth due to **'artificial' IBS** (Intrabeam Scattering) as a function of:
  macro particles M, real particles N, grid spacing,….

Head-tail modes with space charge seen in the computer noise spectrum (R. Singh et al., PRST-AB 2013)



PIC simulation scheme (for beams)

$q:$ beam ion charge

$Q = q\frac{N}{M}:$ macro particle charge

$N:$ number of beam ions

$M \ll N:$ number of macro-particles

$$x_i'' - \kappa(s)x_i - \frac{qE_x(x_i,y_i,s)}{\gamma_0 m v_0^2} = 0$$

$$y_i'' + \kappa(s)y_i - \frac{qE_y(x_i,y_i,s)}{\gamma_0 m v_0^2} = 0$$

-> 'artificial' collisions of macro-particles Q and beam particles q.

Integrate equation of motion, assign new coordinates:
$F_i \rightarrow v_i \rightarrow x_i$

Weighting:
$x_i \rightarrow \rho_p$

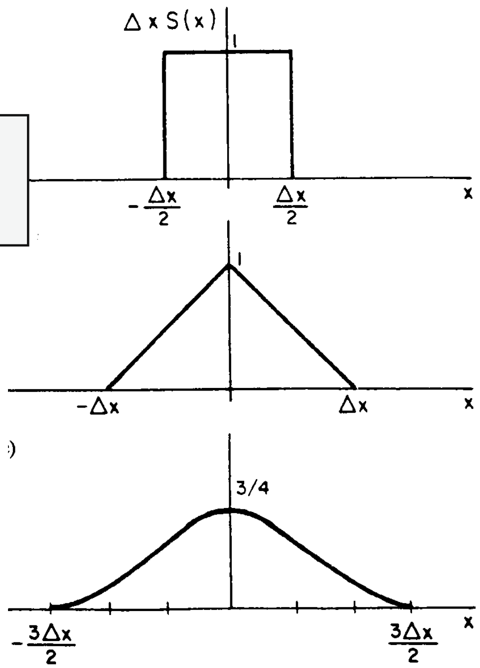
Δs

$$\epsilon_0 \nabla \cdot \vec{E} = \rho(x,y,s)$$

Integrate Poissons's equation on the grid: $\rho_p \rightarrow E_p$

Weighting:
$E_p \rightarrow F_i$

$$\rho(x,y,s) = Q\sum_{i=0}^{M} S(\vec{x} - \vec{x}_i)$$



$$\vec{F}_i = \sum_{j=0}^{M} \frac{qQS(\vec{x}_i - \vec{x}_j)(\vec{x}_i - \vec{x}_j)}{4\pi\epsilon_0\gamma_0 m v_0^2 \left|\vec{x}_i - \vec{x}_j\right|^3}$$

(Coulomb's law)

$q = Ze$ (beam particle)

$Q = Z_p e$ (test particle)

Depending on particle shape S the macro-particles have a finite width: $\Delta \approx \Delta x$

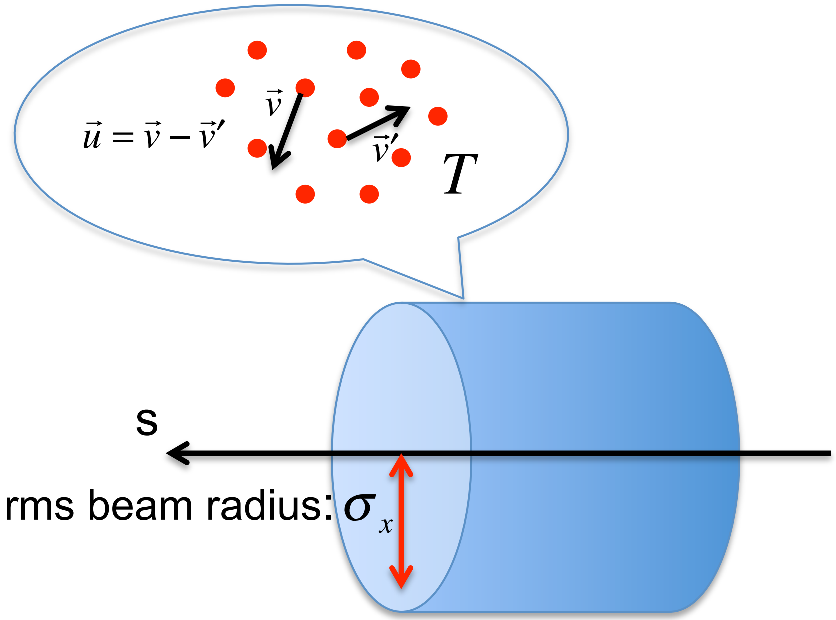
Effect of the grid:
Artificial heating in plasma PIC codes

A. B. Langdon, *Effect of the spatial grid in simulation plasmas*, J. Comput. Phys. (1970)

Temperature increase if the grid is too coarse: $\Delta x \gtrsim \lambda_D$ (Debye length)

How does this relate to (computer) beams ?

Beam temperature: $$k_B T_x = \frac{1}{2} m v^2 = \frac{m \beta_0^2 c^2 \gamma_0 \varepsilon_x^2}{\sigma_x^2}$$



Debye length:

$$\lambda_D = \sqrt{\frac{\epsilon_0 \gamma_0^2 k_B T_x}{q^2 n}} \approx \sqrt{\left(\frac{L}{\Delta\mu}\right) \varepsilon_x}$$

Space charge induced phase advance shift (per length L):

$$\frac{\Delta\mu_x}{L} = -\frac{e^2 Z Z' N}{2\pi\epsilon_0 m v_0^2 \gamma_0^3 \varepsilon_x}$$

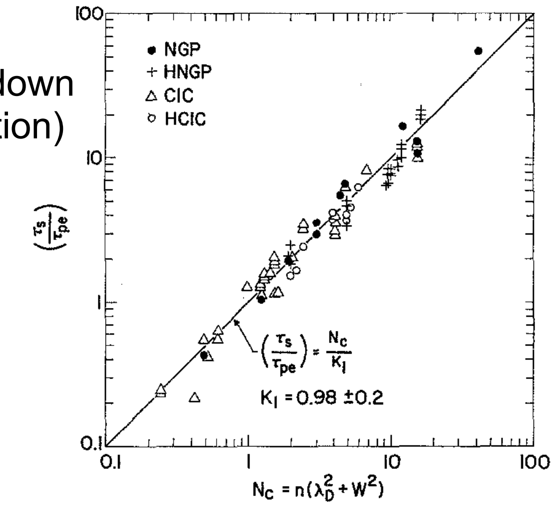
$\lambda = qZ'N$ (line charge density)

Space charge dominated beams: $\sigma_x \gg \lambda_D$

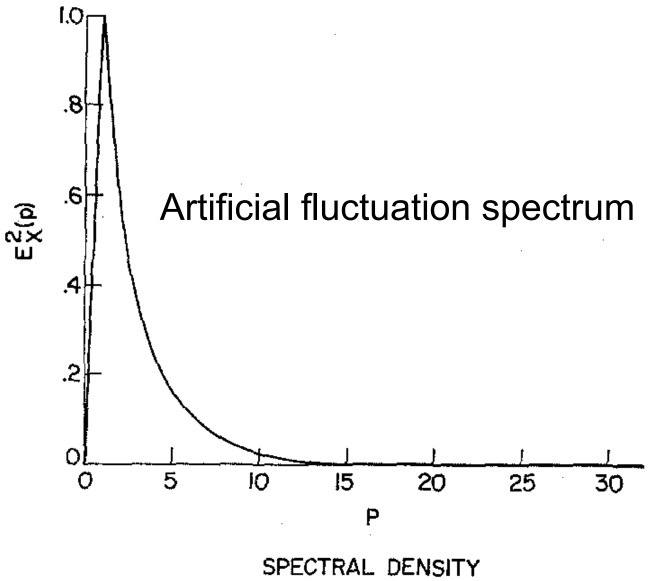
Effect of the finite number of particles: Artifical collisions and fluctuations in PIC codes

R.W. Hockney, *Measurement of the Collision and Heating Times in a 2D Thermal Computer Plasma,* J. Comput. Phys. (1971)

slowing-down time (friction)



macro-particle number M



A.B.Langdon, C.K. Birdsall, *Theory of Plasma Simulation using Finite-Size Particles*, Phys. of Fluids (1970)

$$D(\vec{v}) = \frac{q^2}{2m^2} \int^{\infty} \frac{d\vec{k}}{(2\pi)^3} S(\vec{k}) \left| \vec{E}(\vec{k}, \omega = \vec{k}\vec{v}) \right|^2$$

(Diffusion) (Fluctuations/Noise spectrum)

For computer beams: Work by J. Struckmeier

Artificial collisions and emittance growth in computer beams

J. Struckmeier, Stochastic effects in real and simulated charged particle beams, PRST-AB 2000

$$D \approx \nu \frac{k_B T}{m}$$ (Einstein relation)

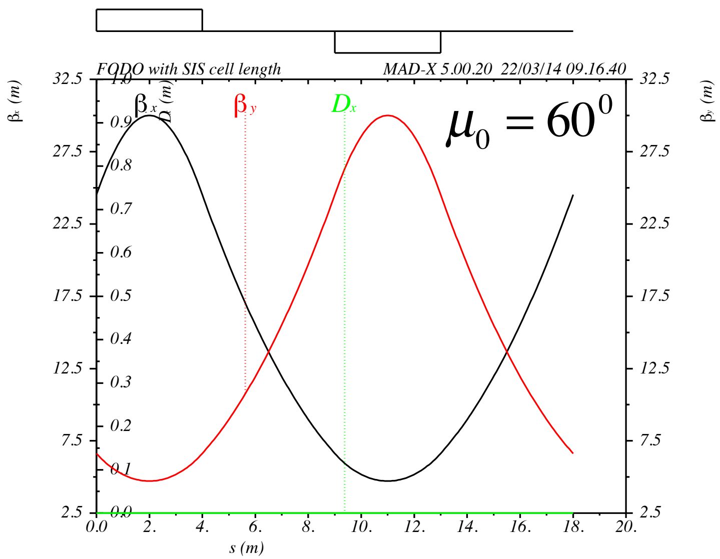
Entropy/Emittance growth only for anisotroptic (beam) temperatures: $T_x \neq T_y$

$$\frac{dS}{dt} = \frac{1}{2} k_B \nu \frac{(T_x - T_y)^2}{T_x T_y}$$

(2D beam)

$$\frac{1}{\varepsilon}\frac{d\varepsilon}{dt} = \frac{dS}{dt}, \quad \varepsilon = \varepsilon_x \varepsilon_y$$

(4D emittance)



For weak space charge:

$$\hat{\beta}_x \approx \sigma_x^2 \varepsilon_x$$ (beta-function)

Emittance growth along a transport channel (length L):

$$\frac{\Delta\varepsilon}{\varepsilon_0} \approx \beta_0 c L \nu A \qquad A = \left\langle \frac{\varepsilon_x}{\varepsilon_y}\frac{\hat{\beta}_y}{\hat{\beta}_x} + \frac{\varepsilon_y}{\varepsilon_x}\frac{\hat{\beta}_x}{\hat{\beta}_y} - 2 \right\rangle$$

(anisotropy factor for a cell)

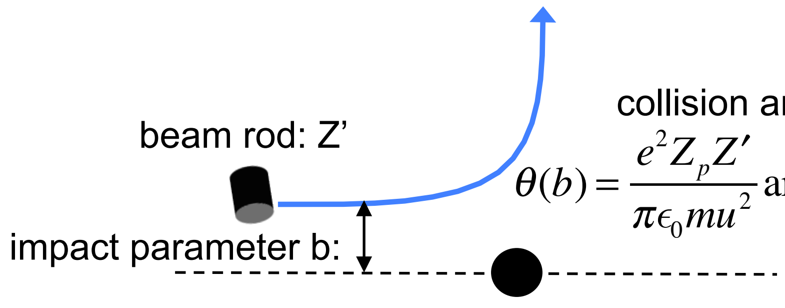
Collision frequency (for 2D and 2.5D computer beams) ?

Coulomb collisions in 2D (and 2.5D)

In a 2D beam the beam macro-particles are rods: **Collision angle independent on b !**

All particles with relative velocities less than

$v_\perp^2 = \frac{Z_p Z' e^2}{2\pi\epsilon_0 m}$ are deflected by angles > 90^0.



collision angle: $\theta(b) = \frac{e^2 Z_p Z'}{\pi\epsilon_0 m u^2} \arccos\frac{b}{\lambda_D}$

$$F_p(\vec{v}) = m\nu\vec{v} = m\int d^2v\, db\, u f(\vec{v})\Delta\vec{v}$$ (2D friction force)

$$F = \frac{Z' Z_p e^2}{2\pi\varepsilon_0 r}$$

(2D 'Coulomb' force)

$$\Rightarrow \quad \nu \approx \left(\frac{v_\perp}{v}\right)^4 n v \lambda_D = N\left(\frac{e^2 Z_p Z'}{\pi\epsilon_0 m}\right)^2 \frac{\sigma_x \lambda_D}{v_0^3 \varepsilon_x^3} \propto N^{1/2} Z^4$$

(2D collision rate)

2D computer beam: $Z_p = \frac{N}{M} Z \quad \Rightarrow \quad v_\perp^2 \rightarrow \frac{N}{M} v_\perp^2 \qquad n \rightarrow \frac{M}{N} n \qquad \nu \rightarrow \frac{N}{M}\nu \propto \frac{N^{3/2}}{M}$

Collision frequency for finite sized macro-particles: $\nu \propto \frac{N^{3/2}}{M}\left(1 - \frac{\Delta}{\lambda_D}\right) \qquad \Delta \approx \Delta x$

2.5D ≈ 2D: $\Delta x, \Delta y \ll \Delta z \quad \Rightarrow \quad v_{x,y} \gg v_z$ Collisions are 2D in 2.5D codes

Example case:
FODO channel with 2D space charge

$$\mu_0 = 60^o, \quad \Delta\mu_x = -20^o$$

RMS envelope equations
(used for matching):

$$a_x'' + \kappa(s) a_x - \frac{\epsilon_x^2}{a_x^3} - \frac{2K}{a_x + a_y} = 0$$

$$a_y'' + \kappa(s) a_y - \frac{\epsilon_y^2}{a_y^3} - \frac{2K}{a_x + a_y} = 0$$

Perveance:

$$K = \frac{qI}{2\pi\epsilon_0 mc^3\beta^3\gamma^3} \propto Z^2 N$$

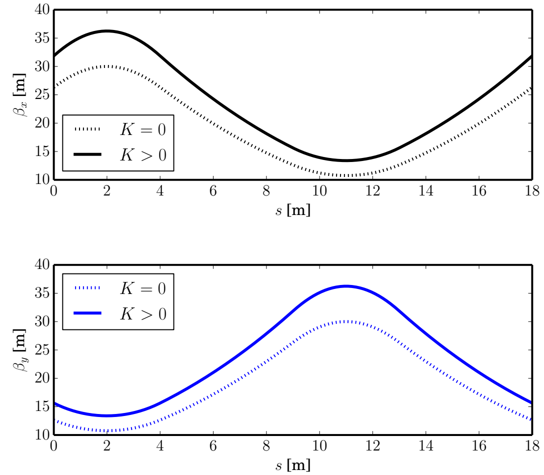
space charge induced
phase advance shift:

$$\Delta\mu_x = \mu_x - \mu_{x0} \propto Z^2 N$$

FODO cell:

$$\kappa_x = -\kappa_y, \quad \varepsilon_x = \varepsilon_y$$

$$\Rightarrow \quad A > 0$$

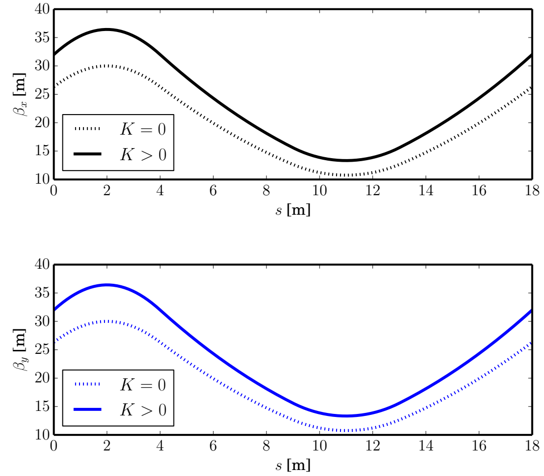


FODOxx cell:

$$\kappa_x = -\kappa_y, \quad \varepsilon_x = \varepsilon_y$$

$$\Rightarrow \quad A = 0$$

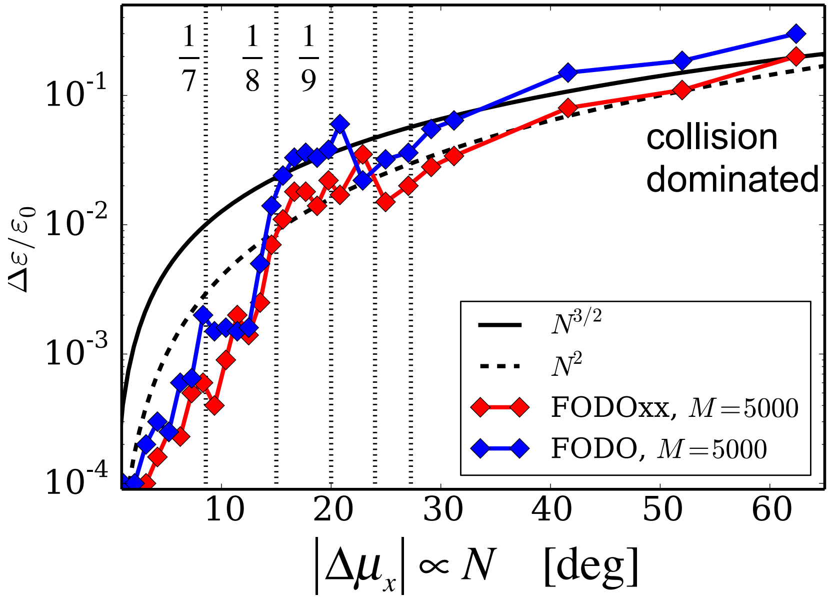
No emittance growth
expected.



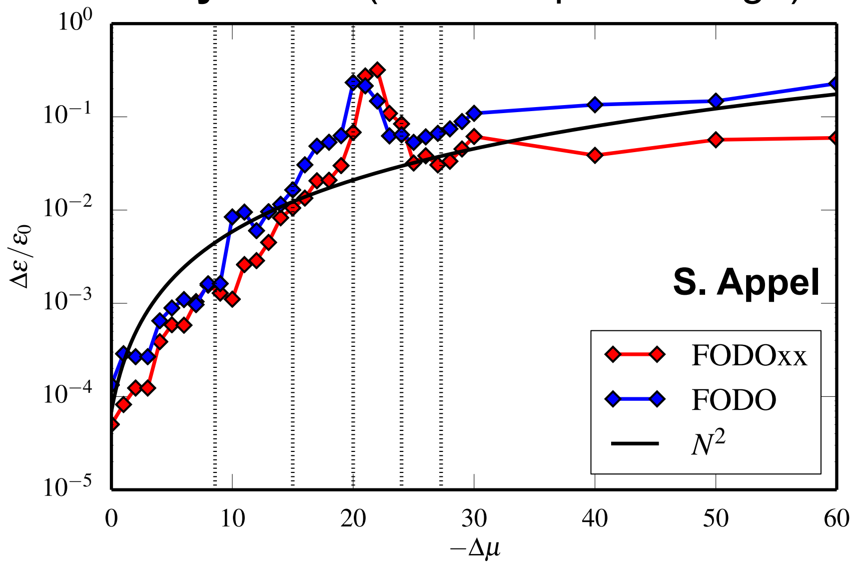
Simulation results for FODO channels

PATRIC (2D): Emittance growth after 1000 cells.

noise/resonance dominated



PyORBIT (with 2D space charge)



$$\nu \propto \left(\frac{\Delta\mu_x}{L}\right)^{3/2} \frac{1}{M} \propto \frac{N^{3/2}}{M}$$

(2D collision rate for a computer beam)

$$\frac{\Delta\varepsilon}{\varepsilon_0} \approx \beta_0 c L \nu (A+G)$$

(emittance growth)

$$G = \frac{\hat{\beta}_{x,\max}}{\langle \hat{\beta}_x \rangle} + \frac{\hat{\beta}_{y,\max}}{\langle \hat{\beta}_y \rangle} - 2$$

(ripple factor)

FODO > FODOxx (only) by factor 2 in emittance growth/collision rate

Artificial 'Schottky' fluctuation spectrum

$$P = |d|^2 + |\Delta\sigma|^2 + \mathcal{O}(\Delta\langle x^3\rangle, \cdots)$$

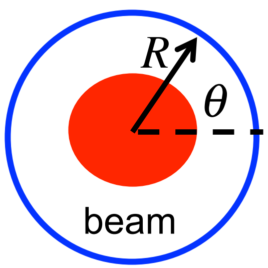
k=1	k=2	k>2
dipolar	quadrupolar	higher order

$$D \propto P = \sum_k P_k$$

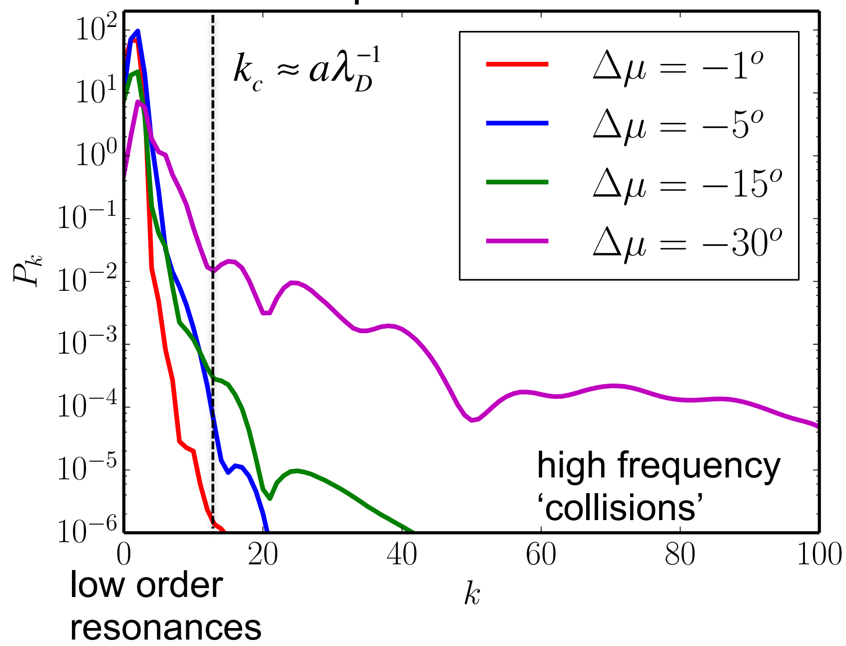
Diffusion — Electric field fluctuations

'Harmonic decomposition' of the electric field fluctuations

$$E_k = \int_0^{2\pi} E_r(R,\theta)\exp(ik\theta)d\theta$$



Fluctuation spectrum after 1000 cells.

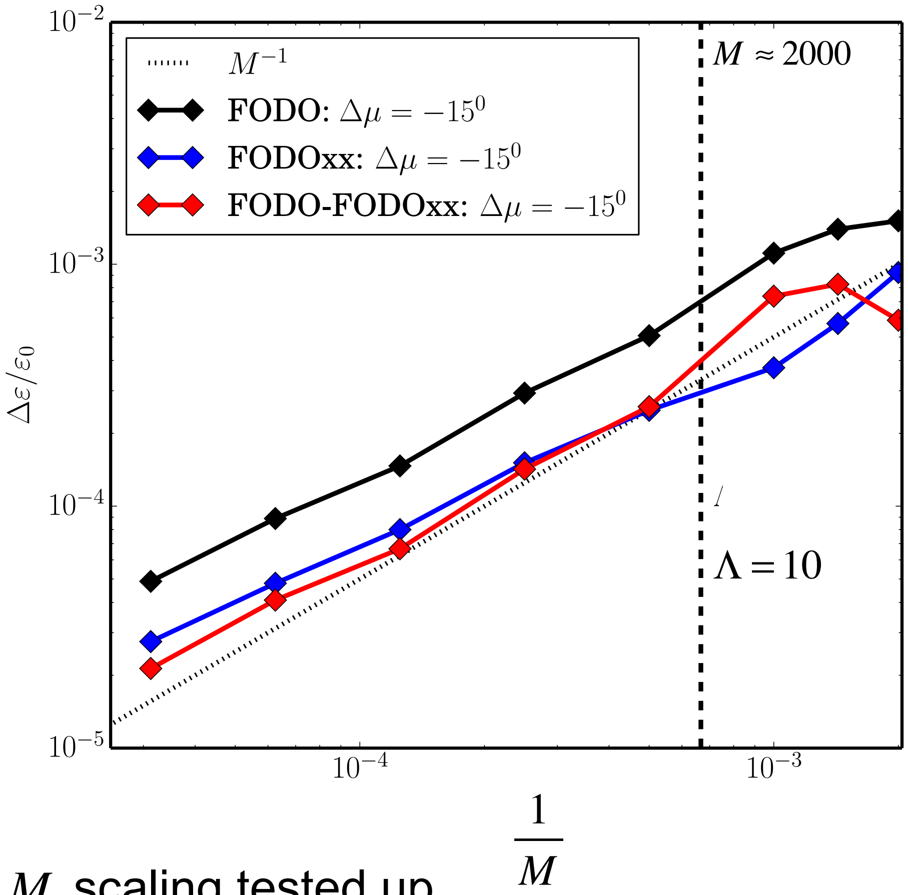


Remark: Saturated 'self-consistent 'fluctuation spectrum does not depend on the initial random macro-particle seed !

Effect of the macro-particle number

'weak coupling':
small angle collisions dominate

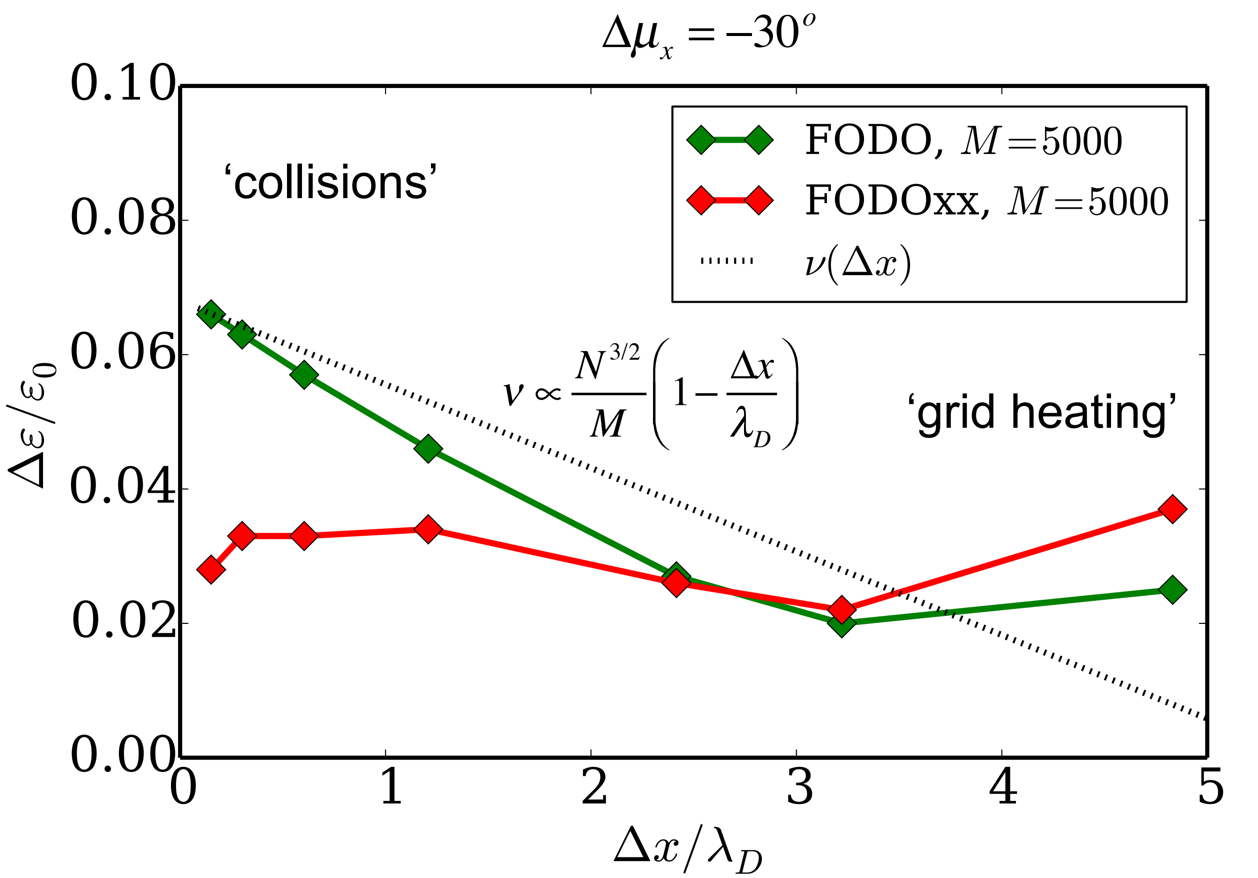
'strong coupling':
large angle collisions dominate



$$\nu \propto \left(\frac{\Delta\mu_x}{L}\right)^{3/2} \frac{1}{M} \propto \frac{N^{3/2}}{M}$$

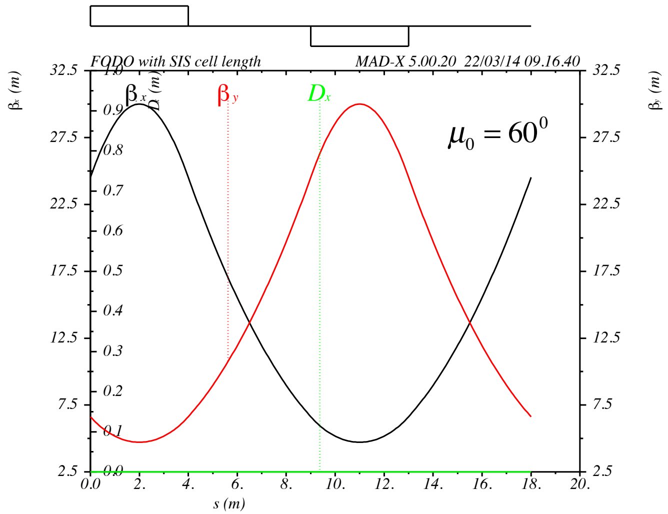
$1/M$ scaling tested up to 50000 macro-particles.

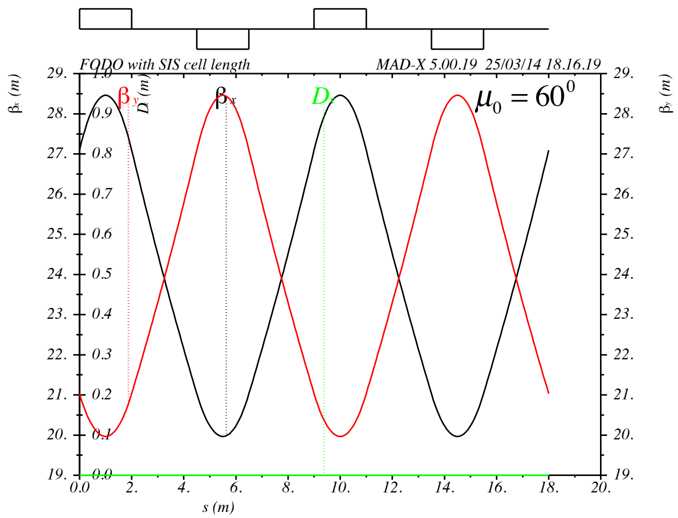
Effect of the grid spacing



in 2D: finite 'particle-particle limit' for $\Delta x \rightarrow 0$

Effect of (periodic) focusing



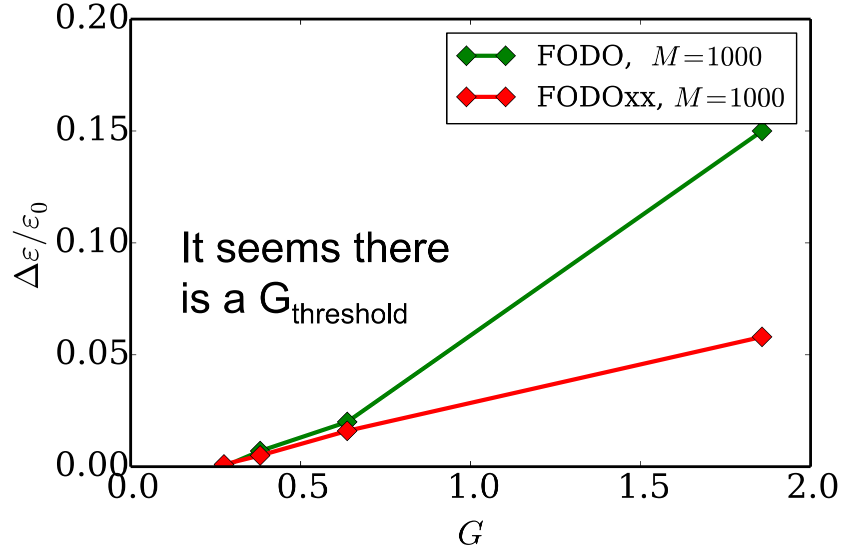


G

$G \to 0$
(limit of constant focusing)

$$\frac{\Delta\varepsilon}{\varepsilon_0} \propto \nu G \qquad G = \frac{\hat{\beta}_{x,\max}}{\langle\hat{\beta}_x\rangle} + \frac{\hat{\beta}_{y,\max}}{\langle\hat{\beta}_y\rangle} - 2$$

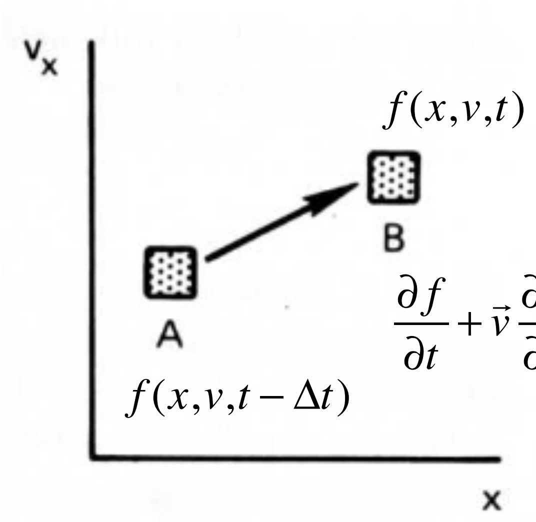
(ripple factor)



Emittance growth decreases strongly with decreasing G and vanishes for constant focusing.

Low Noise or Noise Free Schemes

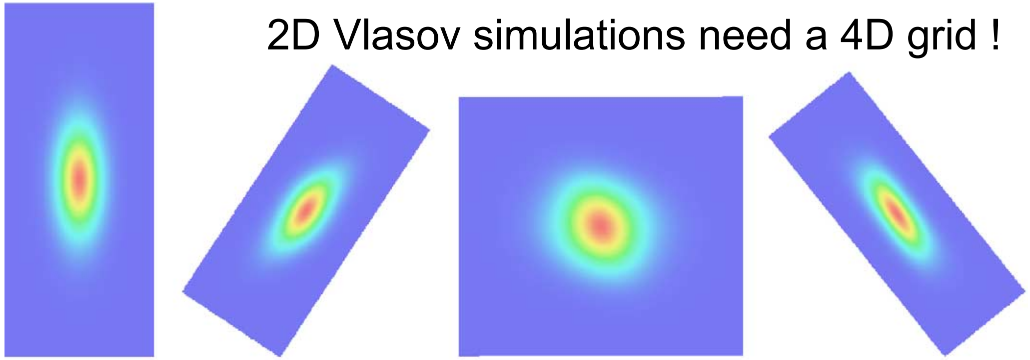
Noise free: Direct Vlasov solvers



$$\frac{\partial f}{\partial t}+\vec{v}\frac{\partial f}{\partial \vec{x}}-\frac{e\vec{E}}{m_e}\frac{\partial f}{\partial \vec{v}}=0$$

Vlasov simulation: 2D beam profile in a FODO channel

2D Vlasov simulations need a 4D grid !



Sonnendrücker, Vay, et al., CPC (2004)
Al-Khateeb, Boine-F., et al. PRST- AB (2003)

Pro: Noise free
Con: Grid induced dispersion, 4D (6D) adaptive grid needed

Low noise: δF-PIC scheme

$$f(\vec{x},\vec{p},s)=f_0(\vec{x},\vec{p})+\delta f(\vec{x},\vec{p},s) \Rightarrow \rho(x,y,s)=Q\sum_{i=0}^{M} w_i S(\vec{x}-\vec{x}_i) \qquad \frac{dw}{dt}=-(1-w)\frac{d\ln f_0}{dt}$$

(known matched distribution) (unknown: halo, …..)

Pro: Noise only from the 'halo', not from the beam core.
Con: weight equation

(additional equation for particle weights)

δF-PIC:
Standard scheme for PIC codes used in magnetic fusion !

Aydemir, PoP (1994)
Qin, Davidson, et al., PRST-AB (2000)
Sonnendrücker, et al, (2013)

Conclusions and Outlook

The **'numerical IBS' induced** emittance growth for an initially rms matched beam distribution with 2D space charge and periodic focusing has been studied using two different codes (PATRIC and pyORBIT).

The topic is a bit 'academic' as **the emittance growth can be controlled by using more macro-particles on modern computers** (+ digital filters). **Still:**

- Scaling laws with M, current, grid spacing are useful to determine the required M.
- The artificial Schottky noise can be used as valuable diagnostics for computer beams.

We found a very approximate (!) scaling law for the numerical emittance growth:

$$\frac{\Delta\varepsilon}{\varepsilon_0} \approx t\nu(A+G) \qquad \nu \propto \frac{N^{3/2}}{M}\left(1-\frac{\Delta}{\lambda_D}\right)$$ (2D and 2.5D collision rate)

- G: 'ripple' (dominant contribution), A: anisotropy (adds a factor 2, roughly)
- for weak space charge: resonances + fluctuations dominate
- for strong space charge: artificial collisions dominate
- Open question: Exact origin of the emittance growth for A=0 ?

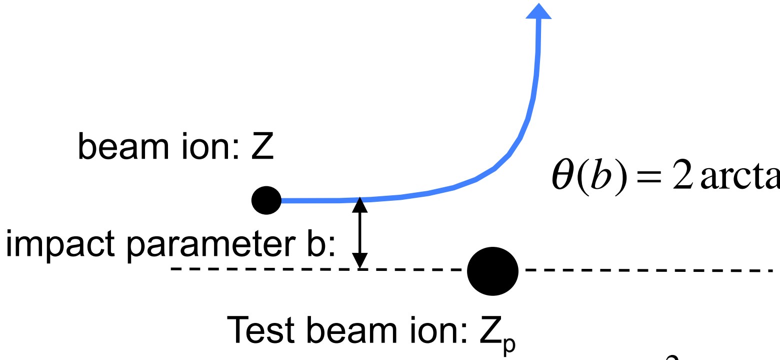
- **Cures:** Larger M + digital filters, δf-PIC (very attractive !)
- **3D:** please wait for Ingo's presentation !

Coulomb collisions in 3D

$$b_\perp = \frac{ZZ_p e^2}{4\pi\varepsilon_0 m v^2}$$

$$\sigma \approx \pi b_\perp^2$$

(cross section)



$$\theta(b) = 2\arctan\frac{e^2 Z_p Z}{4\pi\epsilon_0 m u^2 b}$$

$$\nu = n v \sigma \approx n_i \left(\frac{ZZ_p e^2}{4\pi\varepsilon_0 m v^{3/2}}\right)^2 \propto NZ^4$$

(collision rate)

$$F_{pi} = \frac{ZZ_p e^2}{4\pi\varepsilon_0 r^2}$$

(Coulomb force)

$$F_p(\vec{v}) = m\beta_f \vec{v} = m\int d^3 v d\Omega u f(\vec{v}) \frac{d\sigma}{d\Omega}\Delta\vec{v}$$

(friction force)

$$L_C \approx \ln\frac{\lambda_D}{b_\perp}$$

(Coulomb log)

$$\Lambda = \frac{\lambda_D}{b_\perp} \gg 1$$

(coupling parameter)

Computer beam: $Z_p = \frac{N}{M} Z \Rightarrow b_\perp \to \frac{N}{M} b_\perp \qquad n \to \frac{M}{N} n \qquad \nu \to \frac{N}{M}\nu \propto \frac{N^2}{M}$

$$\Lambda \to \frac{M}{N}\Lambda$$

(close) collisions are more important in a computer beam !

$$\nu \to \frac{N}{M}\nu \propto \frac{N^2}{M}\ln\left(\frac{\lambda_D}{\Delta}\right)$$