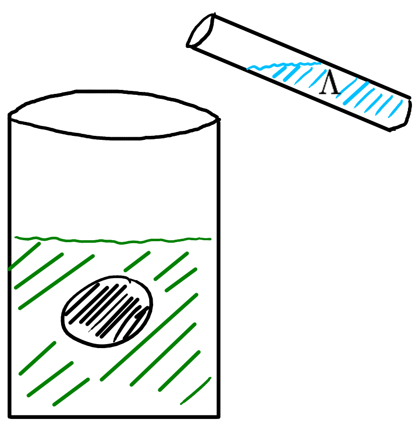
# A Smarr Relation for Lifshitz Black Holes




CAP 2014
Wilson Brenna
June 18, 2014

Work with R. Mann, M. Park, and S. Solodukhin

*Table of Contents*

# Smarr Relations and Thermodynamics

Smarr 1973
PRL 30, 2

First Law of Thermodynamics:

$$dM = T\,dS + \Phi\,dQ$$

↑ Energy !

Smarr Relation:

$$M = 2(TS) + \Phi Q$$

# Smarr Relations and Thermodynamics

## First Law of Thermodynamics:

Electric potential

$$dM = T\,dS + V\,dP + \Phi\,dQ$$

↑ Enthalpy!

Kastor, Ray, Traschen
arXiv:0904.2765

~ Smarr gives us a definition of pressure for which the 1st law is consistent.

## Smarr Relation:

$$M = 2(TS - PV) + \Phi Q$$

$$(D-3)M = (D-2)TS - 2PV + f_{(D)}\Phi Q$$

# Smarr Relations and Thermodynamics

We want to compute thermodynamics and (eventually) universality classes.

Smarr gives the definition of pressure and TD volume:

$$P = \frac{-1}{8\pi}\Lambda \qquad V_{TD} = \frac{4}{3}\pi r_h^3$$

(for Reissner-Nordström Black Holes in 3+1 dimensions)

We can then find the equation of state:

$$P_{(V,T)} = \frac{T}{2r_h} - \frac{1}{8\pi r_h^2} + \frac{Q^2}{8\pi r_h^4}$$

and identify Gibbs free energy and Helmholtz free energy.

# Smarr Relations and Thermodynamics

The 3+1 Dimensional Reissner-Nordström Black Hole:

$$P = \frac{T}{v} - \frac{1}{2\pi v^2} + \frac{2Q^2}{\pi v^4}$$

$$v \equiv 2 l_P^2 r_+$$

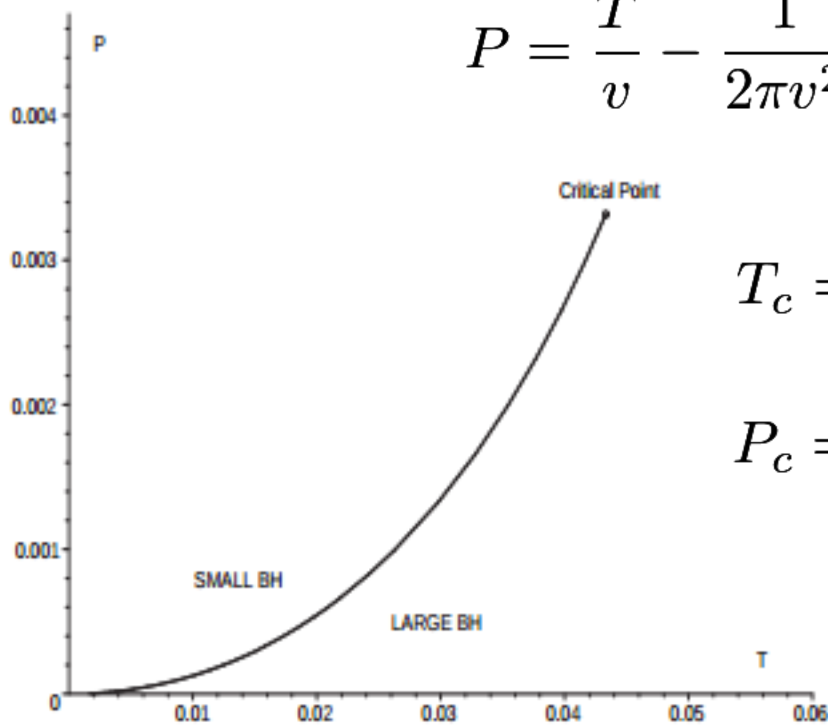



FIG. 9. **Coexistence line of charged AdS black hole.** Fig. displays the coexistence line of small–large black hole phase transition of the charged AdS black hole system in $(P, T)$-plane. The critical point is highlighted by a small circle at the end of the coexistence line.

$$T_c = \frac{\sqrt{6}}{18\pi Q}$$

$$P_c = \frac{1}{96\pi Q^2}$$

| | |
|---|---|
| $\alpha$ | 0 |
| $\beta$ | $1/2$ |
| $\gamma$ | 1 |
| $\delta$ | 3 |

Kubiznak, Mann
arXiv:1205.0559

# Smarr Relations and Thermodynamics

The Van der Waals Fluid: $$\left(P + \frac{a}{v^2}\right)(v - b^2) = kT$$

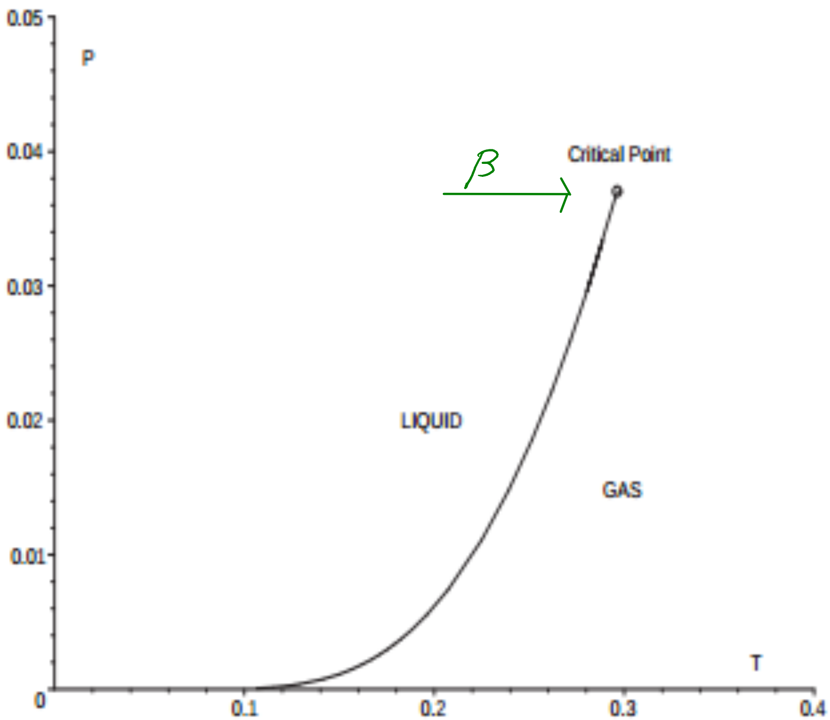



FIG. 5. **Coexistence line of liquid–gas phase.** Fig. displays the coexistence line of liquid and gas phases of the Van der Waals fluid in $(P, T)$-plane. The critical point is highlighted by a small circle at the end of the coexistence curve.

| | |
|---|---|
| $\alpha$ | $0$ |
| $\beta$ | $1/2$ |
| $\gamma$ | $1$ |
| $\delta$ | $3$ |

# Lifshitz-Symmetric Spacetimes

## Anti de Sitter Spacetime

spherically symmetric hypersurface

$$ds^2 = -\left(\frac{r^2}{L^2}\right)dt^2 + L^2\frac{dr^2}{r^2} + r^2 d\Omega^2$$

## Lifshitz ~ Introduce an Anisotropy in Time

$t \to \lambda^z t$ while $x \to \lambda x$

$$ds^2 = -\left(\frac{r^{2z}}{L^{2z}}\right)dt^2 + L^2\frac{dr^2}{r^2} + r^2 d\Omega^2$$

~ Conjectured dual to certain condensed matter systems

# Smarr Relations and Thermodynamics

Smarr obeys Eulerian Scaling:

$$(D-3)M = (D-2)TS - 2PV$$

Euler scaling
$$f(\alpha^p x, \alpha^q y) = \alpha^r f(x,y)$$

implies
$$r f(x,y) = p\frac{\partial f}{\partial x}x + q\frac{\partial f}{\partial y}y$$

$$\left(\text{take } \frac{\partial}{\partial \alpha} : \; r\alpha^{r-1} f(x,y) = p\alpha^{p-1} x \frac{\partial f}{\partial x} + q\alpha^{q-1} y \frac{\partial f}{\partial y}\right)$$

then let $\alpha = 1$

# Smarr Relations and Thermodynamics

Smarr obeys Eulerian Scaling:

$$(D-3)M = (D-2)TS - 2PV$$

through the first law:

$$dM = TdS + VdP \rightsquigarrow \frac{\partial M}{\partial S} = T \;, \quad \frac{\partial M}{\partial P} = V$$

We have $M(S,P) \sim L^{D-3}$ * conjecture

$S \sim L^{D-2}$ (Wald / Iyer ~> BH entropy ~ area)

$P \sim L^{-2}$ (Kastor, Traschen)

# Smarr Relations and Thermodynamics

We obtain a nonlinear set of equations:

Equations: ① Smarr $(D-3)M = (D-2)TS - 2PV$

② 1st Law $dM = TdS + VdP$

$\hookrightarrow \quad \frac{\partial M}{\partial r_+} = T\frac{\partial S}{\partial r_+} \quad ; \quad \frac{\partial M}{\partial \ell} = V\frac{\partial P}{\partial \ell}$

③ Eulerian $M \sim L^{D-3} = \ell^{\alpha} r_+^{\beta} \; ;$

$\alpha + \beta = D - 3$

# Smarr Relations and Thermodynamics

Is there an exact solution?

# Smarr Relations and Thermodynamics

Is there an exact solution?

Yes!

$$M = r_+ \frac{T}{\beta} \frac{\partial S}{\partial r_+}$$

$$V = \frac{r_+}{l} \frac{\alpha T}{\beta dP} \frac{\partial S}{\partial r_+}$$

# Smarr Relations and Thermodynamics

Is there an exact solution?

For the Lifshitz spacetime given, $T = \left(\frac{r}{\ell}\right)^{z+1} \frac{1}{4\pi} f'(r) \Big|_{r=r_+}$

and so we can obtain $\alpha, \beta$ from this form (in linear superposition)!

$$\boxed{M : \text{known} \qquad V : \text{known}}$$

# A Lifshitz Smarr Relation

Can we define a mass via other methods?

# A Lifshitz Smarr Relation

Can we define a mass via other methods?

In some cases, yes!

e.g. Gim, Kim, Li; arXiv:1403.4704

z=3, D=3

$$ds^2 = -\left(\frac{r^2}{l^2}\right)^z \left(1 - \frac{ml^2}{r^2}\right) dt^2 + \frac{1}{\frac{r^2}{l^2}\left(1 - \frac{ml^2}{r^2}\right)} dr^2 + r^2 d\phi^2$$

$$I = \int d^3x \sqrt{-g} \left[\frac{1}{\kappa}(R + 2\Lambda) + \mathcal{F}(R, R_{\mu\nu}, R_{\mu\nu\alpha\beta})\right]$$

Quasilocal background subtraction $\rightsquigarrow$ $M = \dfrac{r_+^4}{4G\ell^4}$

# A Lifshitz Smarr Relation

Is the mass consistent with the first law?

# A Lifshitz Smarr Relation

Is the mass consistent with the first law?

Yes, in fact, we can write down a Smarr relation!

Quasilocal background subtraction $\rightsquigarrow$ $M = \dfrac{r_+^4}{4G\ell^4}$

$S = \dfrac{2\pi r_+}{G}$, $T_H = \dfrac{r_+^3}{2\pi\ell^4}$, $V = \dfrac{8}{13}\dfrac{r_+^4\pi}{\ell^2}$, $P = \dfrac{\Lambda}{8\pi G}$

↳ then

$$dM = TdS + VdP$$

$$0 = TS - 2VP$$

* prefactors chosen to obey Eulerian scaling; $M \sim L^0$, $P \sim L^{-2}$, $S \sim L^1$

# A Lifshitz Smarr Relation

Our method:

$$M = \frac{r_+^4}{4Gl^4}$$

Other spacetimes for which our method agrees:

Reissner-Nordström AdS (z=1, D=4)

5D Lifshitz with Higher Curvature terms (z=2, D=5)

3D Lifshitz with Proca Field (z=3, D=3)

4D Lifshitz with Maxwell Charge (z=4, D=4); Pang, arXiv:0901.2777

# A Lifshitz Smarr Relation

Our method:

$$M = \frac{r_+^4}{4Gl^4}$$

Other spacetimes for which our method agrees:

Reissner-Nordström AdS (z=1, D=4) BY ✓

5D Lifshitz with Higher Curvature terms (z=2, D=5) BY ✓

3D Lifshitz with Proca Field (z=3, D=3) BY ✓

4D Lifshitz with Maxwell Charge (z=4, D=4); Pang, arXiv:0901.2777

to be understood

# Conclusion

Anisotropy in Time ~ mass becomes a tricky concept

Lifshitz Smarr Relation ~ appears to give enough information to obtain a TD mass and volume

# Next...

Is a Maxwell field necessary for a finite-T critical point?

What is the universality class?

# Acknowledgments

Robert Mann

Sergey Solodukhin

Miok Park

University of Waterloo

NSERC

Questions?

# Appendix

$$\Delta\rho(T) = \rho_{liq} - \rho_{gas}\Big|_{P=P_c}$$

## Critical Exponents

$$\alpha: \quad C_V \sim |T-T_c|^{-\alpha}$$

$$\gamma: \quad K_T \equiv \frac{-1}{v}\frac{\partial v}{\partial P}\Big|_T \sim |T-T_c|^{-\gamma}$$

$$\delta: \quad |P-P_c| \sim |v-v_c|^{\delta}$$

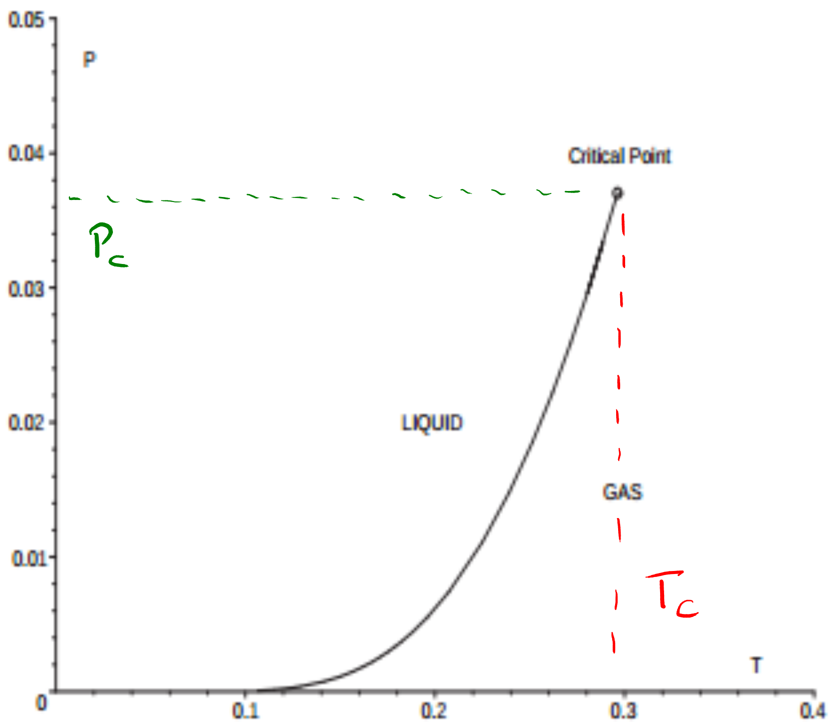
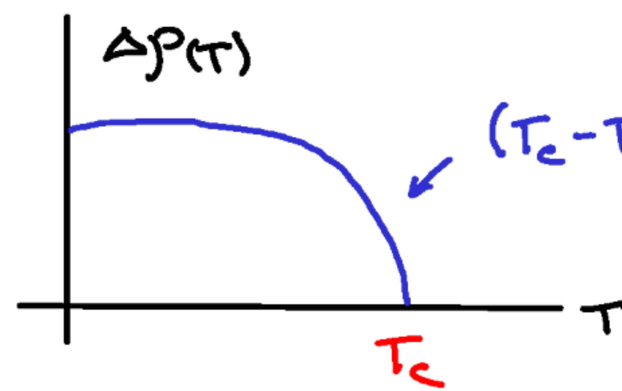


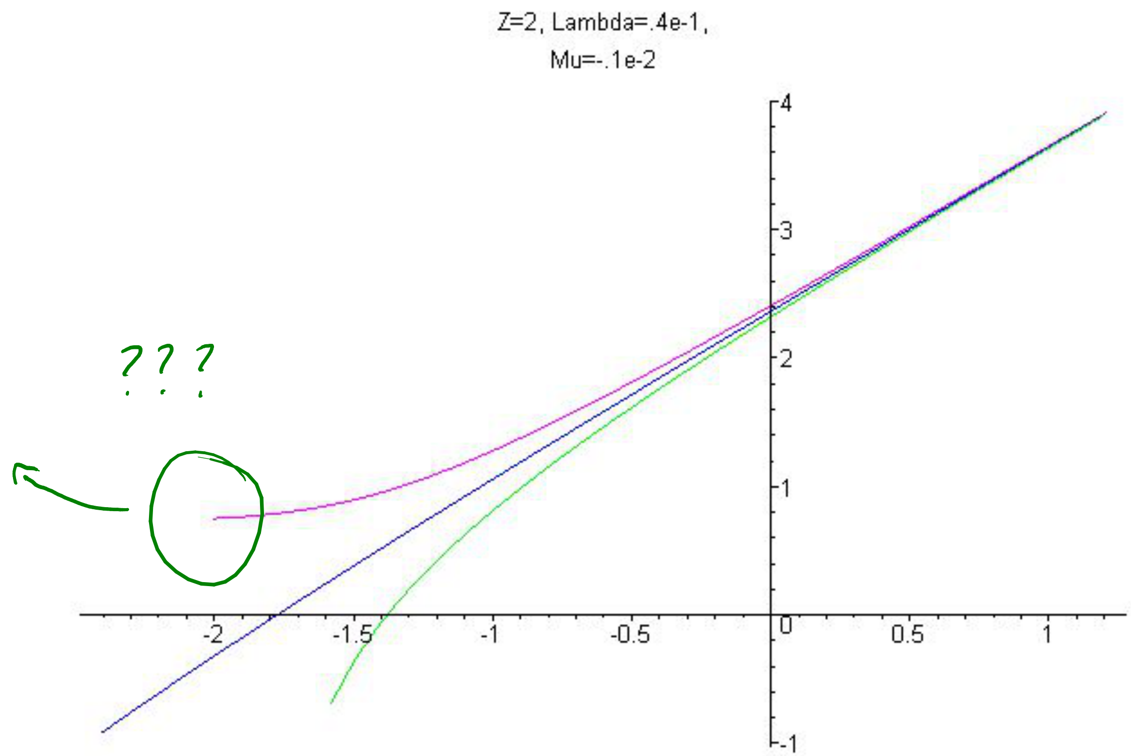

FIG. 5. **Coexistence line of liquid–gas phase.** Fig. displays the coexistence line of liquid and gas phases of the Van der Waals fluid in $(P, T)$-plane. The critical point is highlighted by a small circle at the end of the coexistence curve.

order parameter



# Appendix

Do Lifshitz Phase Transitions exist?



# A Lifshitz Smarr Relation

Spacetime:

$$ds^2 = -\left(\frac{r}{L}\right)^{2z} f_{(r)}^2 \, dt^2 + \frac{L^2 g_{(r)}^2}{r^2} dr^2 + r^2 d\Omega^2$$

Proca f.s. Proca vec. pot.

Action:

$$S = \int d^D x \left( R - 2\Lambda - \frac{1}{4} H^2 - \frac{m^2}{2} B^2 \right)$$

$$\nabla_b H^{ab} = -m^2 B^a$$

Constraints:

$$m^2 = \frac{(D-2)z}{L^2} \qquad \Lambda = \frac{-1}{2L^2}\left[ D^2 + (z-4)D + z^2 - 3z + 4 \right]$$