# Squeezed coherent states and a measure of entanglement

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# Outline

# 1. Introduction

1. A beam splitter is an optical device which can generate quantum entangled states. The beam splitter 50:50 allows to split the incident intensity to equal reflected and transmitted intensities.
2. Usual CS of the harmonic oscillator used as the input state do not generate entanglement while the corresponding SCS do.
3. Entangled states have potential applications in quantum cryptography and quantum teleportation.
4. Different CS are constructed as finite sum and infinite superpositions of Fock states and the linear entropy is computed.

# 2. Generalized squeezed coherent states and linear entropy

## 2.1. Construction of generalized squeezed coherent states

SCS can be constructed using different methods.
They are superposition of usual Fock states and may be constructed as the solution of the eigenvalue equation

$$(A+\gamma A^\dagger)\Psi(z,\gamma) = z\ \Psi(z,\gamma), \qquad z,\gamma \in \mathbb{C},$$

where $z$ is the amplitude of coherence and $\gamma$ the squeezing parameter.
$A^\dagger$ and $A$ are $f$-deformed ladder operators defined as:

$$A|n\rangle = \sqrt{n}\ f(n)\ |n-1\rangle, \qquad A^\dagger|n\rangle = \sqrt{n+1}\ f(n+1)\ |n+1\rangle,$$

where the set $\{|n\rangle, n=0,1,...\}$ represents the usual Fock states and $f(n)$ is a positive real function.
Relations with the usual ones $a^\dagger$ and $a$ are given as $A^\dagger = f(N)\ a^\dagger$ and $A = af(N)$ where $N$ is the usual photon number.

For $f(n) = 1$, we call them usual SCS.
In the coherent case ($\gamma = 0$), they are the most semi-classical quantum states. It is well-known that when they are used as input states $|\psi\rangle$ in the beam splitter, the output $|out\rangle$ is a direct tensor product of two such coherent states producing thus no entanglement.
When $\gamma \neq 0$, the SCS still minimize the Heisenberg uncertainty relation but the dispersions in the position and momentum fluctuate. SCS ($\gamma \neq 0$) have shown to produce always entanglement.
Deformed or generalized CS have been the object of many papers. Fewer results have been obtained for SCS.

SCS are explicitly given as a superposition of Fock states:

$$\Psi(z,\gamma) = \frac{1}{\sqrt{\mathcal{N}(z,\gamma)}} \sum_{n=0}^{M} \frac{Z_{f(n)}(z,\gamma,n)}{\sqrt{n!}f(n)!} |n\rangle,$$

where $f(n)! = \prod_{i=1}^{n} f(i)$, $f(0)! = 1$ and $\mathcal{N}(z,\gamma)$ is the normalization factor.

The function $Z_{f(n)}(z,\gamma,n)$ satisfies $(n = 1, 2, \ldots)$

$$Z_{f(n)}(z,\gamma,n+1) - zZ_{f(n)}(z,\gamma,n) + \gamma\, n\,(f(n))^2 Z_{f(n)}(z,\gamma,n-1) = 0,$$

with $Z_{f(n)}(z,\gamma,0) = 1$ and $Z_{f(n)}(z,\gamma,1) = z$.

Note that $M \to \infty$.

In the usual case ( $f(n) = 1$), we have:
for $\gamma = 0$, $Z_o(z, 0, n) = z^n$;
for $\gamma \neq 0$,

$$Z_o(z, \gamma, n) = \left(\frac{\gamma}{2}\right)^{\frac{n}{2}} \mathcal{H}\left(n, \frac{z}{\sqrt{2\gamma}}\right),$$

where $\mathcal{H}(n, w)$ are the Hermite polynomials in a complex variable $w$.
The states $\Psi(z, \gamma)$ are normalizable for all $z \in \mathbb{C}$ and $|\gamma| < 1$.

Now we take:

$$f(n) = \sqrt{v+n}, \quad v \geq 0.$$

We define $J_- = A$, $J_+ = A^\dagger$, $J_0 = N + \frac{1}{2}(v+1)$ ( $N$ isthe usual number operator) and we thus genrate a $su(1,1)$ algebra:

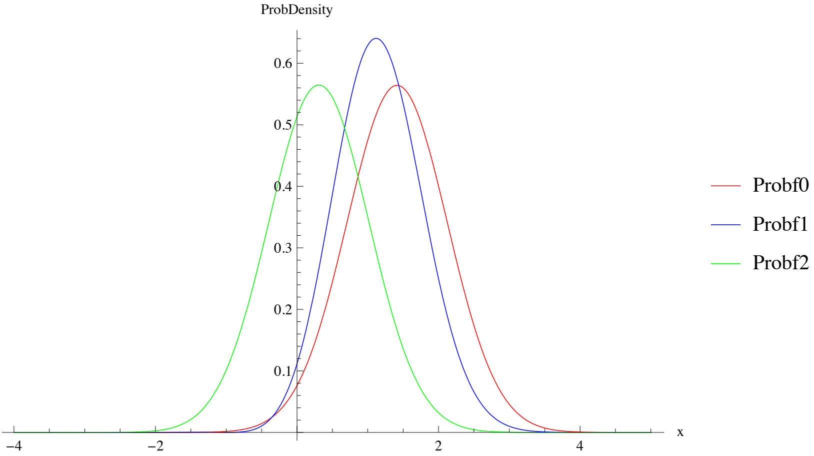
$$[J_0, J_\pm] = \pm J_\pm,\ [J_-, J_+] = 2J_0.$$

Note: the special case where $v = 0$ has been studied in relation with a nonlinear Jaynes-Cummings model known as the Buck-Sukumar model in a kerr medium.

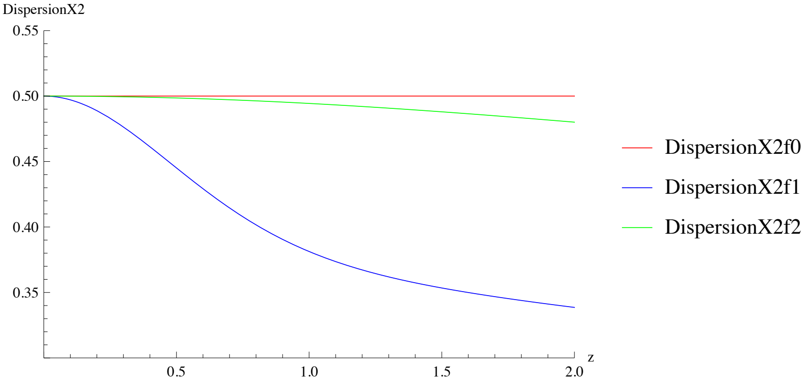
# 2. Generalized squeezed coherent states

## 2.2. Density probability, dispersion in position and momentum and uncertainty

Graph of $|\Psi(z,0)|^2$ (the harmonic oscillator CS) with $f_0(n) = 1,\ f_1(n) = \sqrt{n},\ f_2(n) = \sqrt{20+n}$ and $z = 1$ :

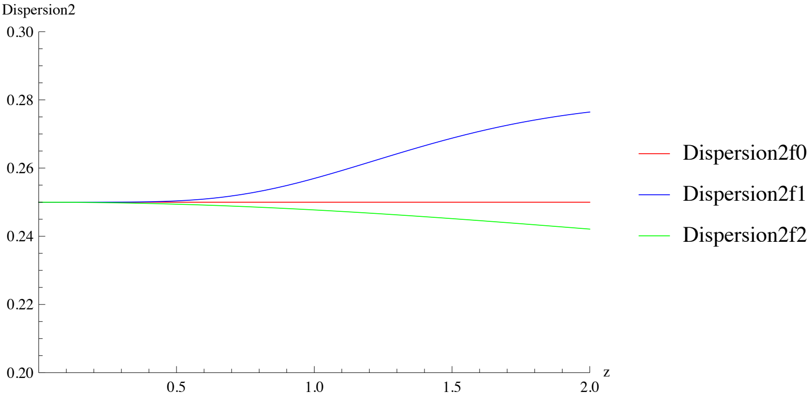


With the position $x$ and momentum $p$ given as $x = \frac{1}{\sqrt{2}}(a^\dagger + a)$ and $p = i\frac{1}{\sqrt{2}}(a^\dagger - a)$, we compute the following dispersions: $(\Delta x)^2$, $(\Delta p)^2$ and $\Delta = (\Delta x)^2(\Delta p)^2$.



# 2. Generalized squeezed coherent states and linear entropy

## 2.2. Density probability, dispersion in position and momentum and uncertainty



# 3. Beam splitter and linear entropy

## 3.1. Beam splitting transformation

A beam splitter can be described by an unitary operator $\hat{B}(\theta)$ connecting the input state with the output one

$$|out\rangle = \hat{B}(\theta)\,|in\rangle = \exp\left[\frac{\theta}{2}(a^\dagger b e^{i\phi} - a b^\dagger e^{-i\phi})\right]\,|in\rangle,$$

where the input state is given as

$$|in\rangle = |\psi\rangle \otimes |0\rangle.$$

$a^\dagger, a$ and $b^\dagger, b$ independently act on the bipartite fields and satisfy $[a, a^\dagger] = [b, b^\dagger] = 1$.
$\phi$ is the phase difference between the reflected and transmitted fields ($\phi = 0$ in our approach).
If $|in = |n\rangle \otimes |0\rangle$, a usual Fock state, we get ($su(2)$ coherent state):

$$\hat{B}(\theta)\big(|n\rangle \otimes |0\rangle\big) = \sum_{q=0}^{n} \binom{n}{q}^{\frac{1}{2}} T^q R^{(n-q)}\,|q\rangle \otimes |n-q\rangle.$$

# 3. Beam splitter and linear entropy

## 3.1. Beam splitting transformation

$$\hat{B}(\theta)\big(|n\rangle \otimes |0\rangle\big) = \sum_{q=0}^{n} \binom{n}{q}^{\frac{1}{2}} T^q R^{(n-q)} \; |q\rangle \otimes |n-q\rangle.$$

Note that:

1. $T$ and $R$ are the transmissibility and reflectivity of the beam splitter and $|T|^2 + |R|^2 = 1$;
2. $T = \cos(\theta/2)$, $R = -e^{-i\phi}\sin(\theta/2)$ with $\theta$ the angle of the beam splitter ;
3. We use a symmetric 50:50 beam splitter ( $\theta = \pi/2$). Some comments for the non-symmetric case at the end of the talk.

Starting with the density operator $\rho_{ab} = |out\rangle\langle out|$, the linear entropy $S$ is defined as

$$S = 1 - Tr(\rho_a^2),$$

where $\rho_a$ is the reduced density operator of the system $a$ obtained by performing a partial trace over the system $b$ of $\rho_{ab}$.

$S \in [0, 1[$: 0 means non entanglement (case of a separable state) and 1 would indicate a maximum of entanglement.

Note that the linear entropy is an upper born of the von Neumann entropy.

If the input state is chosen such that $|\psi\rangle$ is a SCS , we get:

$$\begin{aligned} |out\rangle &= \hat{B}(\theta)\big(\Psi(z,\gamma)\otimes|0\rangle\big) \\ &= \frac{1}{\sqrt{\mathcal{N}}}\sum_{q=0}^{M}\sum_{m=0}^{M-q}\frac{Z_{f(n)}(z,\gamma,m+q)}{\sqrt{q!m!f(m+q)!}}T^q R^m|q\rangle\otimes|m\rangle. \end{aligned}$$

The linear entropy takes the form $(|T|^2=|R|^2=\frac{1}{2})$

$$\begin{aligned} S &= 1-\frac{1}{\mathcal{N}^2}\sum_{q=0}^{M}\sum_{j=0}^{M}\sum_{m=0}^{M-max(q,j)}\sum_{n=0}^{M-max(q,j)}2^{-(q+j+m+n)} \\ &\quad \frac{Z_{f(n)}(m+q)\overline{Z_{f(n)}(m+j)}Z_{f(n)}(n+j)\overline{Z_{f(n)}(n+q)}}{q!j!m!n!f(m+q)!f(m+j)!f(n+j)!f(n+q)!}. \end{aligned}$$

Note: we see that when $M$ is finite, some of the sums must go until $M-max(q,j)$.

# 4. Measure of entanglement in the SCS

## 4.1. Generalized coherent states

When $\gamma = 0$, $Z_{f(n)}(z, \gamma, n) = z^n$ for all $f(n)$. We can write

$$S_{\gamma=0}(x, M, f(n)) = 1 - \frac{\sigma(x, M, f(n))}{\mathcal{N}^2(x, M, f(n))}.$$

with

$$\sigma(x, M, f(n)) = \sum_{q=0}^{M} \sum_{j=0}^{M} \sum_{m=0}^{M-max(q,j)} \sum_{n=0}^{M-max(q,j)} \left(\frac{x}{2}\right)^{q+j+m+n} \times \frac{1}{q!j!m!n!f(m+q)!f(m+j)!f(n+q)!f(n+j)!}$$

and

$$\mathcal{N}^2(x, M, f(n)) = \left(\sum_{n=0}^{M} \frac{x^n}{n!(f(n)!)^2}\right)^2.$$

# 4. Measure of entanglement in the SCS

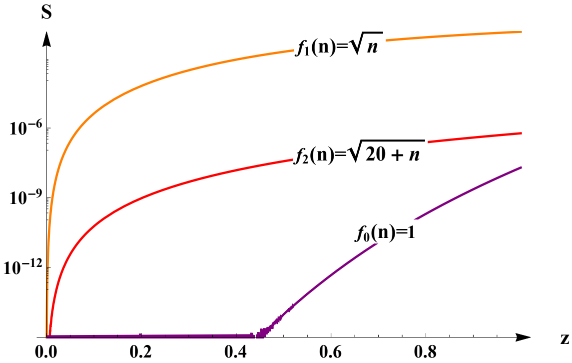
## 4.1. Generalized coherent states

For $f(n) = 1$ and $\gamma = 0$, the usual CS are such that $S_{\gamma=0}(x, M, 1) = 0$.
For the calculations, we take $M$ finite. It means that the set of values of $x$ where $S_{\gamma=0} \to 0$ in these usual coherent states could depend on $M$. This will give us a interval of allowed values of $x$ with respect to the choice of $M$ .
For example, we have shown that for $x \in [0, 1]$, it is enough to take $M = 10$ in our calculations of $M$.

# 4. Measure of entanglement in the SCS

## 4.1. Generalized coherent states

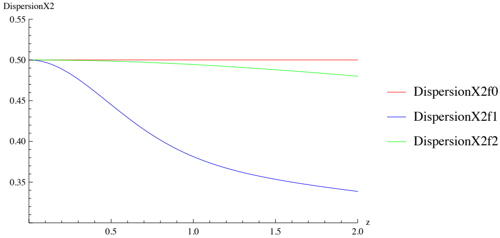
Let us now compare with the other choices of $f(n)$.

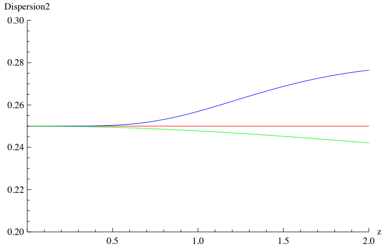



Note that the axis of the graph is logarithmic. This better reflect the different orders of magnitude of the linear entropy.

# 4. Measure of entanglement in the SCS

## 4.1. Generalized coherent states



For the squeezed vacuum, we take $z = 0$. We get

$$Z_{f(n)}(0, \gamma, n+1) + \gamma \ n \ (f(n))^2 Z_{f(n)}(0, \gamma, n-1) = 0, \ \ n = 1, 2, ...$$

with $Z_{f(n)}(0, \gamma, 0) = 1$ and $Z_{f(n)}(0, \gamma, 1) = 0$. It gives

$$Z_{f(n)}(0, \gamma, 2n) = \frac{(2n)!}{n!} \prod_{i=0}^{n-1} \left(f(2i+1)\right)^2 \left(-\frac{\gamma}{2}\right)^n,$$

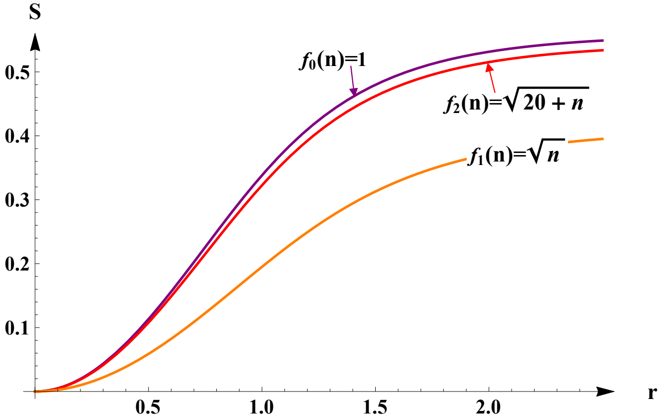
$$Z_{f(n)}(0, \gamma, 2n+1) = 0.$$

For $f(n) = 1$, we know that the entanglement is increasing with the values of $\gamma$ ($\gamma < 1$ and is taken to be real for our calculations).

# 4. Measure of entanglement in the SCS

## 4.2. Generalized squeezed vacuum

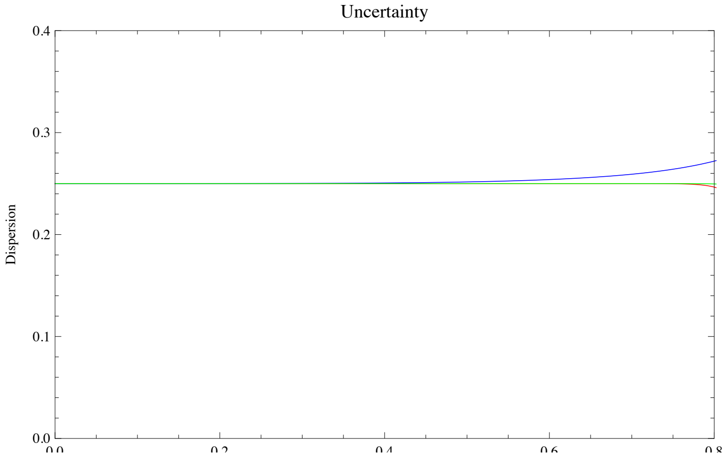
We compare the behavior of the linear entropy $S$ for different choices of $f(n)$. It behaves differently than in the coherent states. We chose to draw the graph in function of $r$ ($\gamma = \tanh r$).



# 4. Measure of entanglement in the SCS

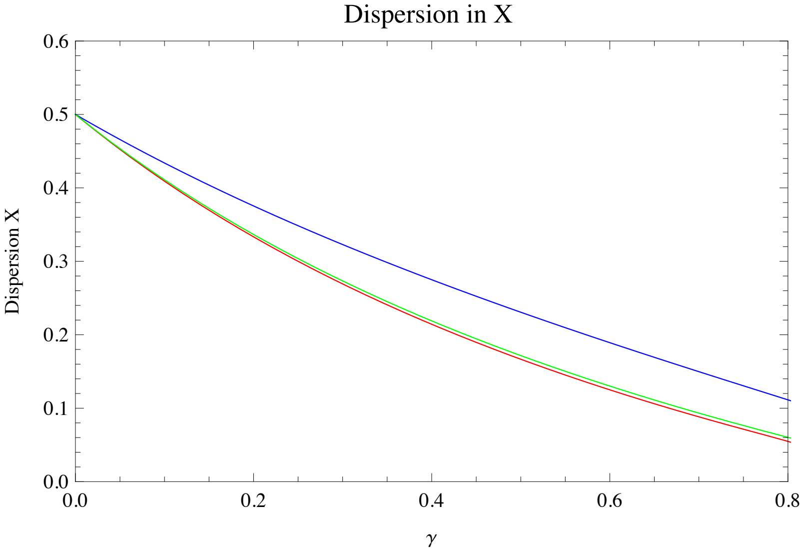
## 4.2. Generalized squeezed vacuum

The following figures describe the dispersions and localisation in x for $f_0(n) = 1$(red), $f_1(n) = \sqrt{n}$(blue),$f_2(n) = \sqrt{(}20 + n)$(red).



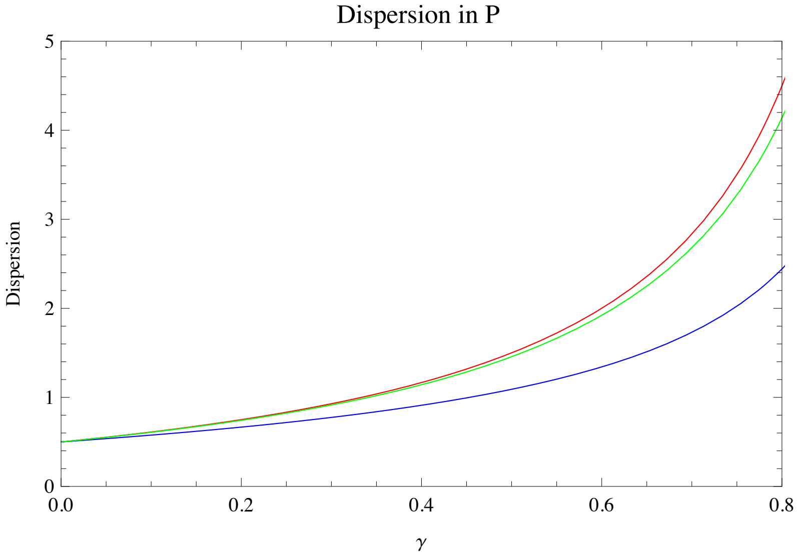
# 4. Measure of entanglement in the SCS

## 4.2. Generalized squeezed vacuum



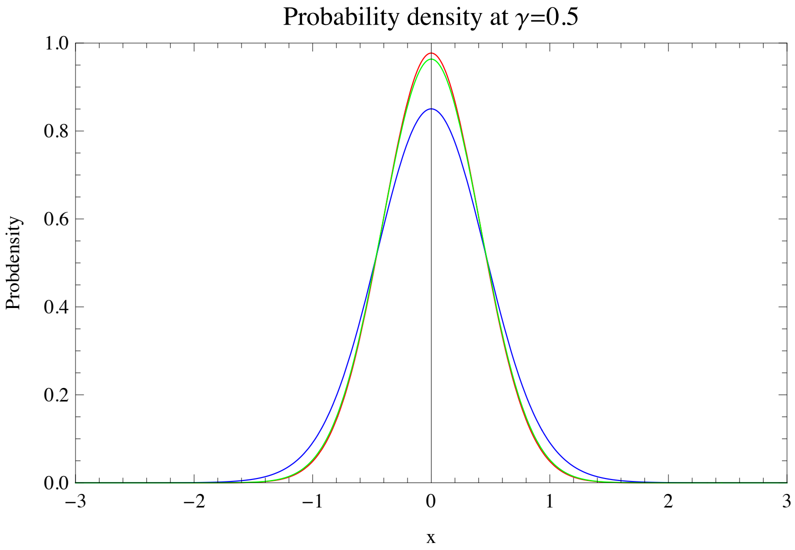
# 4. Measure of entanglement in the SCS

## 4.2. Generalized squeezed vacuum



# 4. Measure of entanglement in the SCS

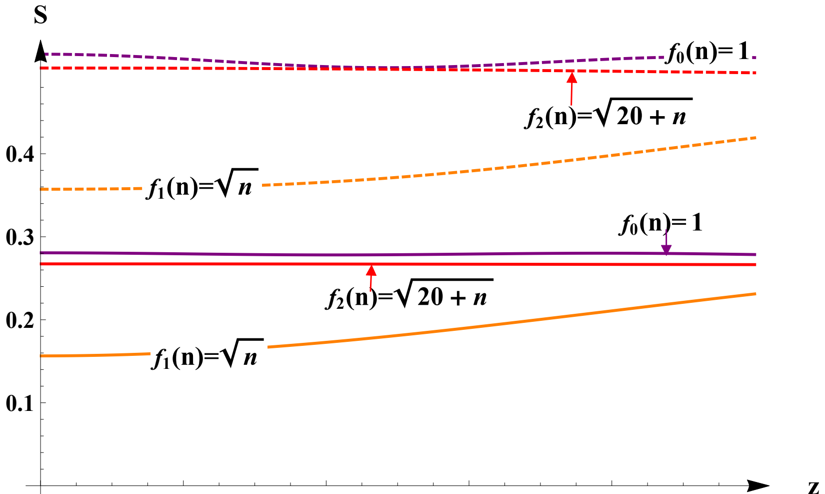
## 4.2. Generalized squeezed vacuum



# 4. Measure of entanglement in the SCS

## 4.3. Generalized squeezed coherent states

Finally, when our states depends on both $z$ and $\gamma$, we get the following figures (for $\gamma = 0.7$ (plain line), $\gamma = 0.95$ (dashed line)) for $S$.



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